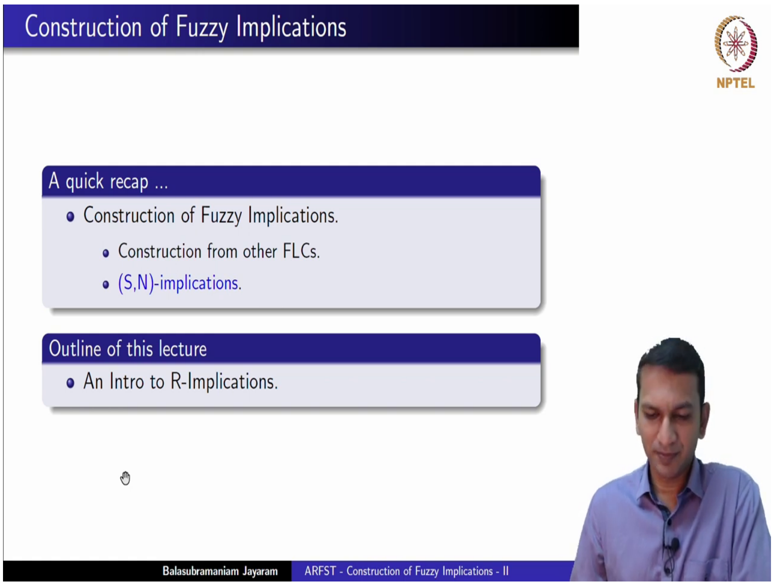
Approximate Reasoning using Fuzzy Set Theory
Prof. Balasubramaniam Jayaram
Department of Mathematics
Indian Institute of Technology, Hyderabad

Lecture – 19
Construction of Fuzzy Implications – II

Hello and welcome to the next of the lectures in this series titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

(Refer Slide Time: 00:30)

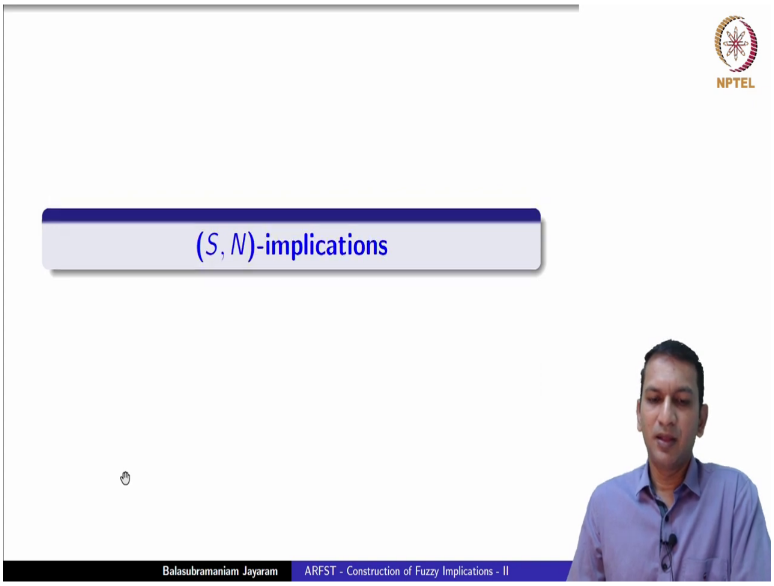


In the last lecture, we began to see how to construct fuzzy implications from other fuzzy logic connectives. Specifically, we have seen how to build S, N-implications, the family of S, N-implications from a given t-conorm and a negation.

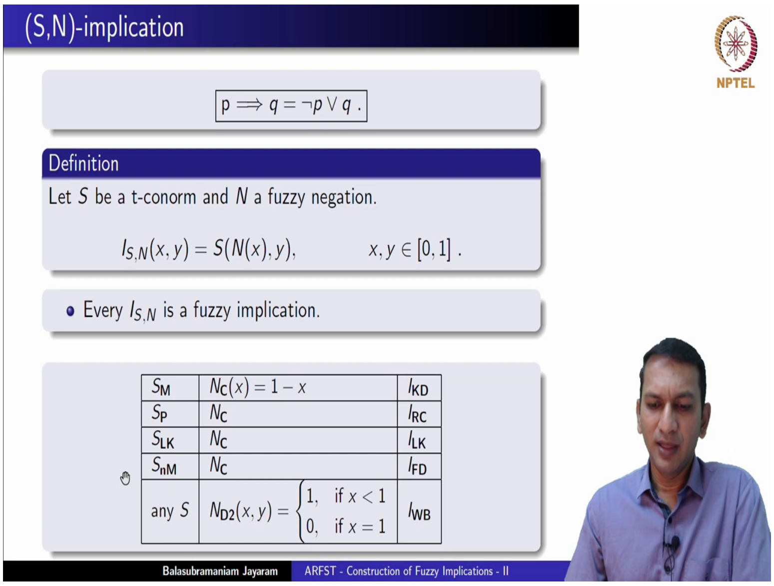
In this lecture, we will have a general introduction to another family, perhaps a very important family of fuzzy implications known as the R-implications.

(Refer Slide Time: 01:05)

(Refer Slide Time: 01:07)



(Refer Slide Time: 01:11)

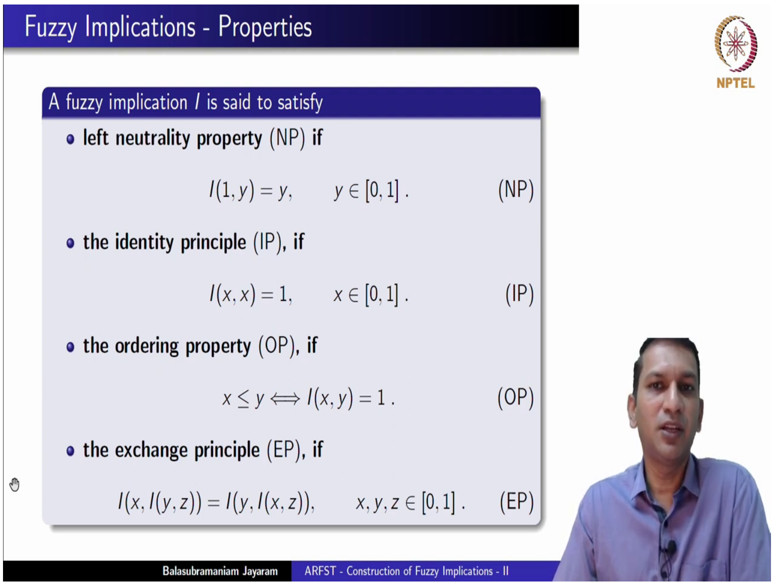


A quick recap of the family of (S, N)-implications; we have seen that the truth table for the classical implication can also be captured by this formula which is also called the material implication p implies q is given as negation p or q; that means, p implies q is true either if p is false or q is true.

Taking q from this, we define the family of S, N-implications. Given a t-conorm S and a fuzzy negation N, we defined a function I as S(N (x), y). We denoted it by I S, N, the S, N, N standing for the t-conorm S and negation N to indicate that this implication is coming from the family of S, N-implications.

We have seen that every such function I_(S, N) is a fuzzy implication and we have also seen that many of the basic fuzzy implications do belong to this family.

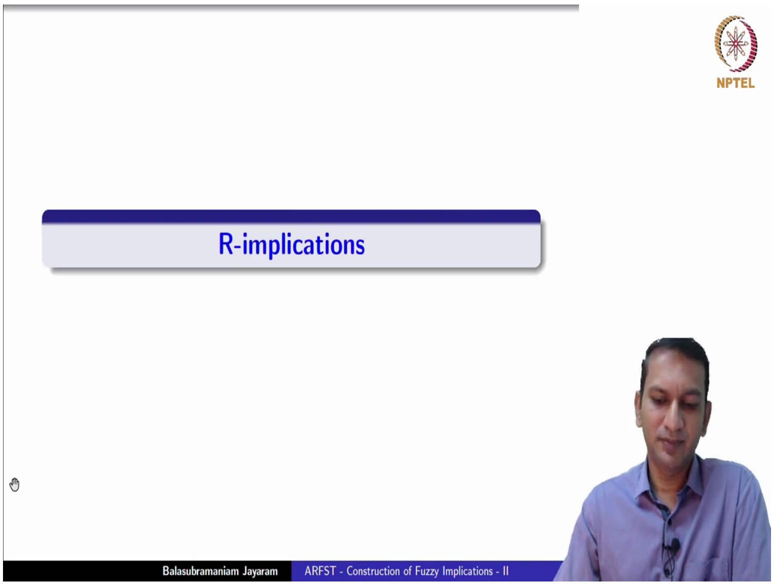
(Refer Slide Time: 02:20)



Then we went on to discuss what desirable properties S, N-implications satisfy. The first of them is the left neutrality property or just simply the neutrality property, which says that I of 1, y should be equal to y. We have seen that this property is held by all S, N-implications. The next property is that of the identity principle. We have seen that not all S, N-implications satisfy this property.

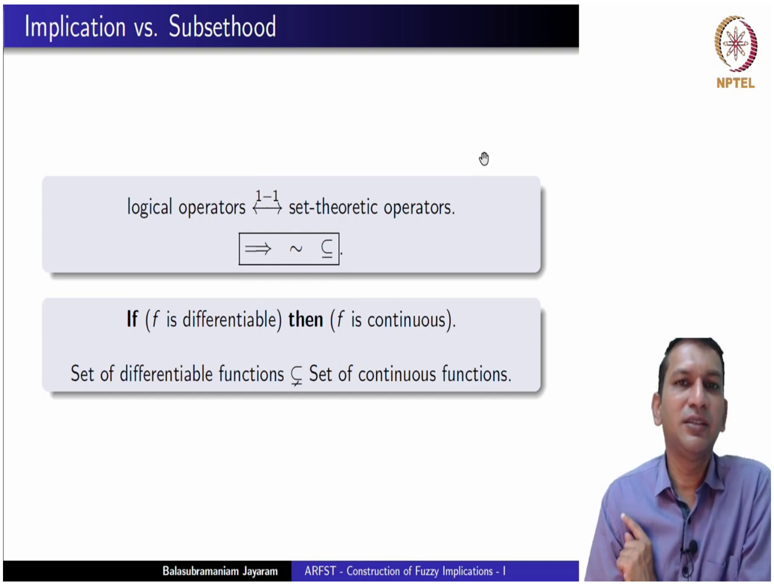
Similarly is the case with ordering principle, not all S, N implication satisfy this. However, the exchange principle is indeed satisfied by all S, N-implications. We have seen this is largely due to the fact that a t-conorm is both commutative and associative. It is from there we are inheriting based on the definition of an S, N implication that all of them do satisfy the exchange principle.

(Refer Slide Time: 03:25)



Now, let us move on to the second family of fuzzy implications that of R-implications. Well, the second family of implications and perhaps arguably the most important family of fuzzy implication is called the R-implication. These have had varied origins, but perhaps a good way to introduce them is to look at some correspondence between logic and set theory.

(Refer Slide Time: 03:55)



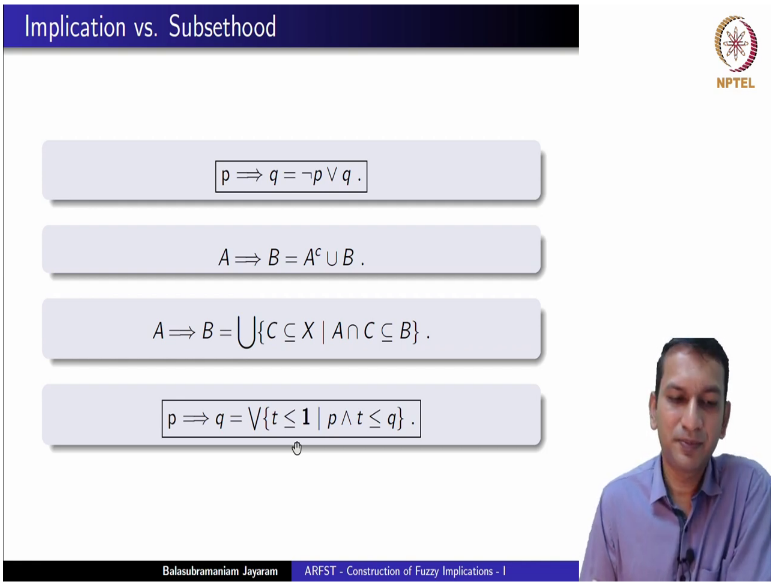
We know that in the classical setting, the classical logical operations are related to corresponding set theoretic operations.

For example, the meet in the Boolean algebra is in fact, related to the intersection in terms of the corresponding subsets of the power set of corresponding subsets of the universal set x. So, we can always relate a classical logic operator or to another equivalent fuzzy set theoretic operator. If you ask about the implication what could it be related to, perhaps it could be related to the subset hood operation. How so?

Now, let us look at this; once again, a very familiar conditional by now. If f is differentiable, then f is continuous. We know for sure that this conditional is valid. Now, what it seems to say is this that the set of differentiable functions is actually contained in the set of continuous functions, that is what we are (Refer Time: 05:01) because if you take an f, if it is differentiable it is also continuous.

So, if you consider set of all differentiable functions, then it is contained the set of all continuous functions. But implication is not just subsethood.

(Refer Slide Time: 05:15)



For instance, look at this formula p implies q; we have related it as or equivalently written it as negation p or q. So, now, when you are looking at when is this conditional true it is not only required that whenever p is true q is true, but even when p is not true it becomes true; that means, if p is false then the conditional does not come into picture.

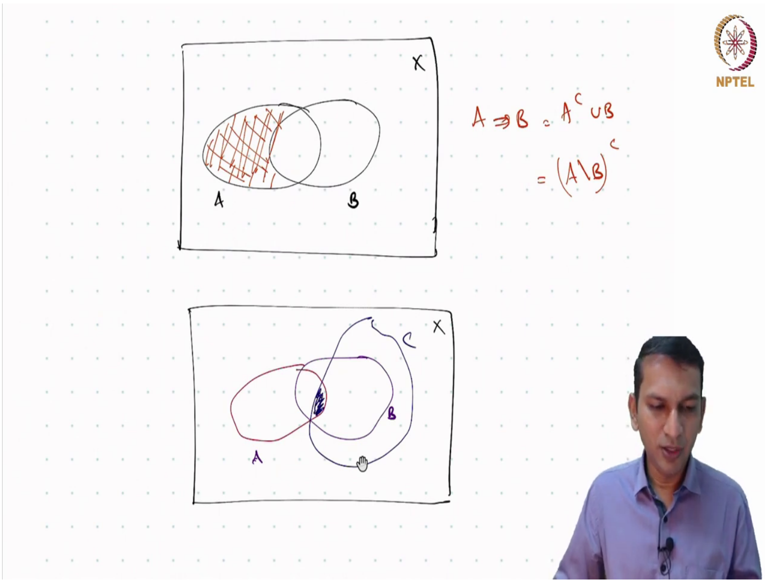
The antecedent is no more, so the conditional does not come into picture. So, you see that negation p or q captures this in fact and says that whenever p does not happen, it is still the conditional is valid. Whereas in the case of subsethood, we want that A is subset of B if and only if every X in A also belongs to B. It does not talk about when X does not belong to A.

So, if X A does not belong to A, then well belong to B or not and that is what this condition is capturing extra.

Well, now in terms of the set theoretic operations if you have to look at it, we could also know we could also write this from a previous knowledge that when you look at it from Boolean algebra point of view p of X with the usual intersection union and complementation they form a Boolean algebra.

So, from that point of view you could write also this implication in some sense as A complement to min B and this is what we have captured, in coming up with the family of S, N-implications. Now, we could also write an equivalent formula like this. Now, this needs a little bit more explanation. Let us look at it. What does it say it says? A implies B can be expressed as union of all those subsets C of X, where X is a universal set such that A intersection C is contained in B.

(Refer Slide Time: 07:12)



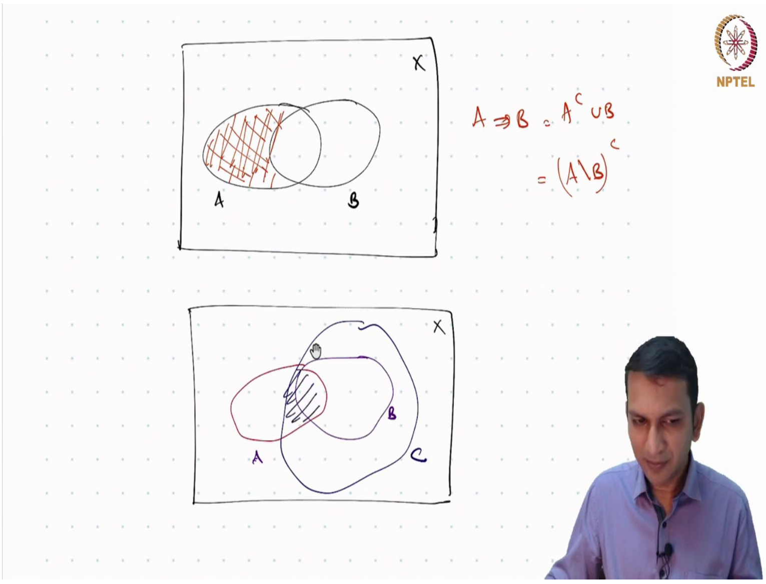
Let us look at this, a Venn diagram point of view. So, this is my universal set X. This is my A and let this be my B. So, this is my A, this is my B. If you go by this first formula A implies B is equivalent to A complement B in B, essentially A complement contains everybody outside of A, union being that is this.

So, only part that is not considered in this is this part; I am just shape this way with some other colour. So, essentially this is; so, this is the part that is not considered. So, you could also write A implies B is equal to A compliment union B as first remove B from A, that is what this part is and take the compliment. So, the parts marked in this orange or red colour, they are the ones that are not part of A implies B.

Now, what does this formula tell us? Let us look at it. Let us say this is X (Refer Time: 08:43), this is A, this is B. It tells us take the C which is a which is a subset of X and look at A intersection C, is it contained in B?

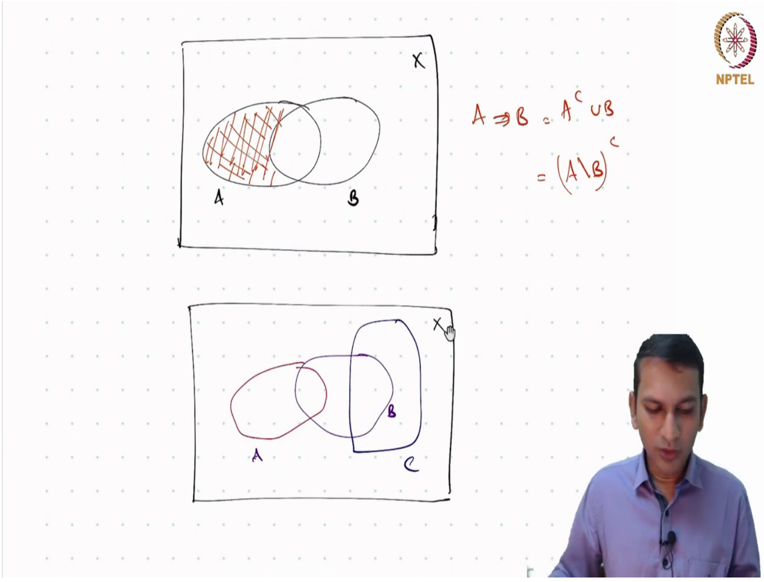
So, let us look at this as sub C. Now, what is A intersection C? What is A intersection C? Clearly, this part. Now, is this contained in B? Yes, it is contained in B. So, this C actually could be considered in this family here. Now, we want not just some C, but we want the union of all of them. So, essentially, we are asking for the largest such C. Do we have any other C like this? Yes. Let us look at this.

(Refer Slide Time: 09:41)



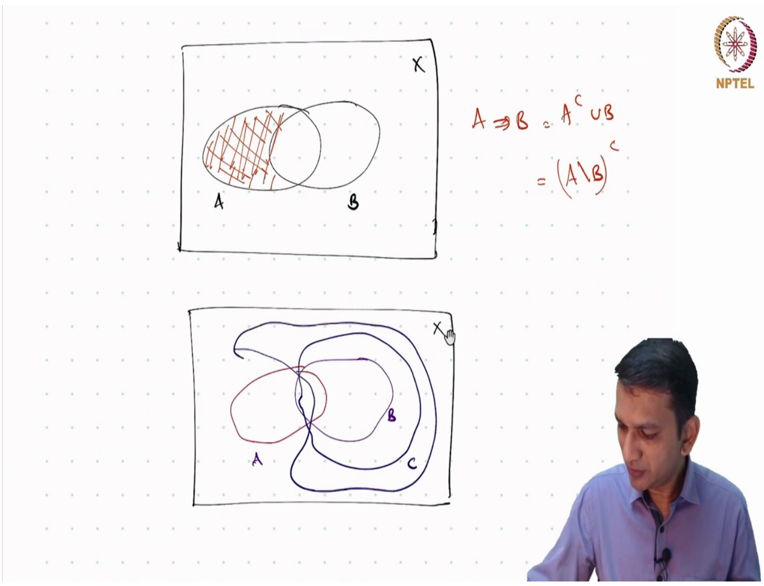
Another C, so this is another C. Now, A intersection C will be this. But unfortunately, if you look at A intersection C, it is not completely contained in B. So, this C is practically useless for us. It will not fall into this set here.

(Refer Slide Time: 10:02)



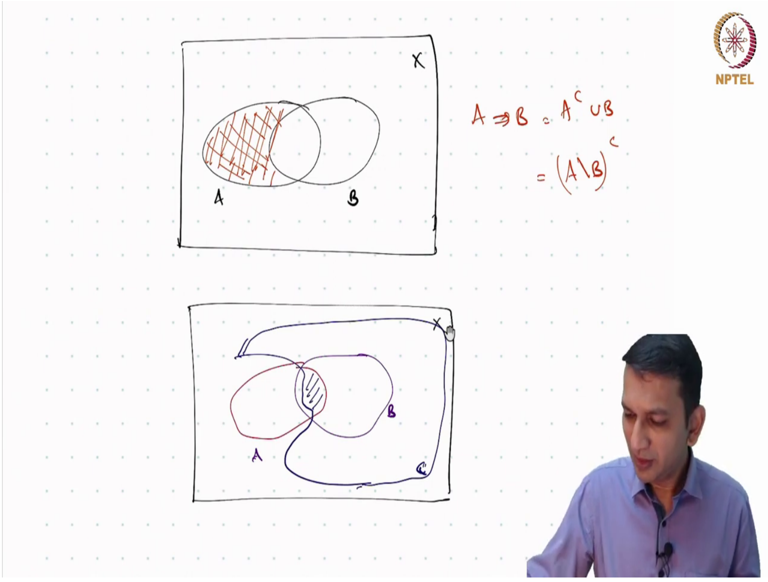
What about this C? A intersection C is empty, but empty set belongs to any other set non-empty set. So, that means, this C also can be considered. Now, the question is how do we find the largest such set C? Well, consider this fellow. You said that this C will belong here.

(Refer Slide Time: 10:26)



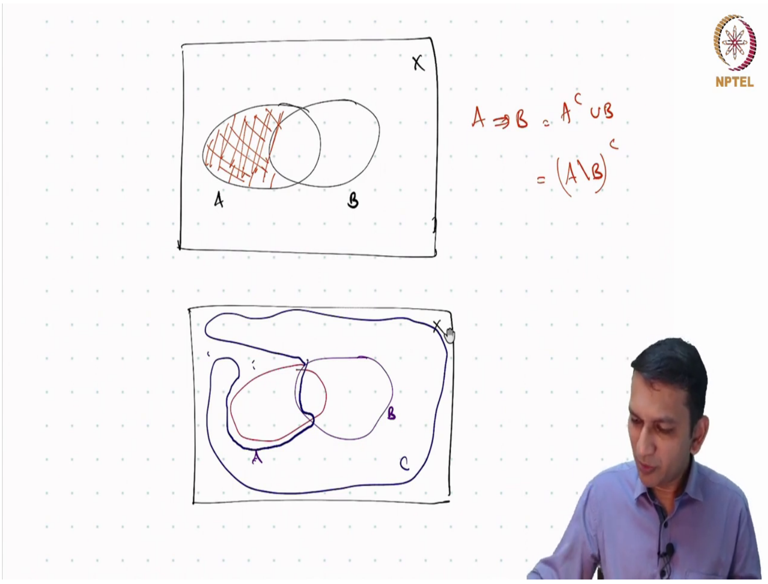
What about this C? Look at this intersection, the intersection with A is this which is contained in B.

(Refer Slide Time: 10:56)



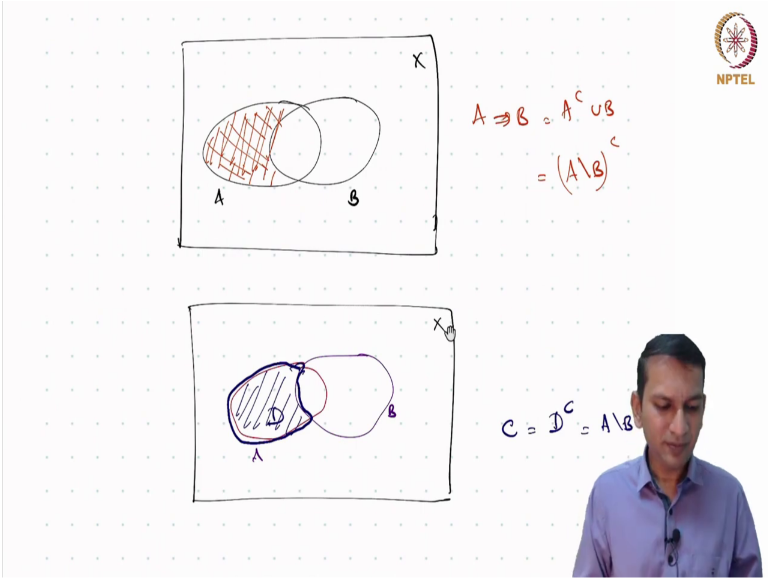
So, now we can try to draw this C little further. Why not consider C in it. So, take this all the way here and here. So, now, if you consider this C the intersection still remains the same.

(Refer Slide Time: 11:20)



So, from here you get an idea what could be the biggest C that we can draw. Clearly, it would look something like this. Take this fellow all of them, sorry. So, this is one C that could look very nice, but still we are missing all these things; why not take all of them.

(Refer Slide Time: 11:40)



So, now when you consider from that point of view you actually see that the biggest set C is essentially that which covers everything outside of this. So, if you consider this to be the set D, then the bigger set C is essentially D complement. But what is D? That is essentially A minus B.

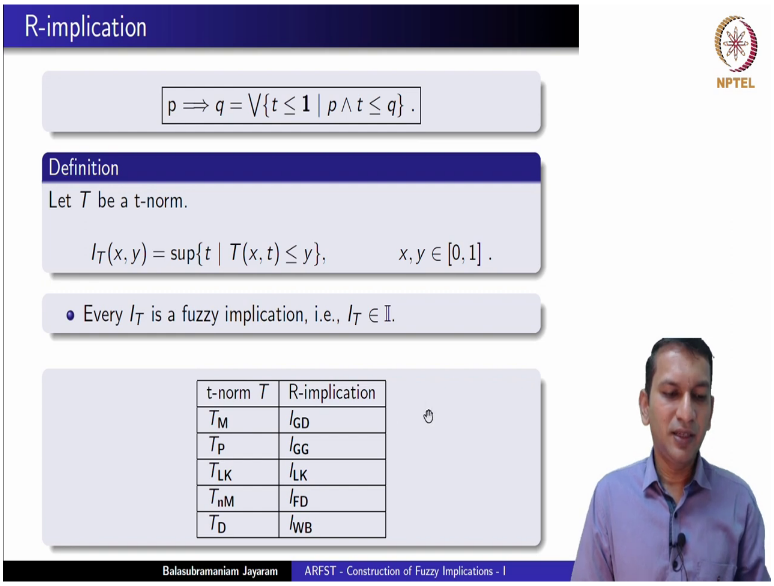
So, you see here from set theoretic point of view we have actually got another way of looking at implications of the conditions. Now, because we do have an idea that the set of all subsets of an X; if you consider it from an algebraic point of view you get a Boolean algebra, a Boolean lattice. So, now, writing this in terms of the lattice operations would mean like this.

So, p implies q is the supremum of all those t less than or equal to 1. Remember this is a bounded lattice, where X is the top most element, supremum over all those t less than or equal to 1, such that p min t is smaller than q is essentially exact transliteration of this formula from the setting of set theory to that of the lattice theory.

So, it is only a change in language. So, now, you have one definition here of p implies q as negation p of q. Now, we have another definition here. In the case of classical logic, when you had only two truth value 0, 1, both these definitions are in fact, equivalent as you can see from the corresponding truth table.

However, in the case of fuzzy logic connectives, in the setting of fuzzy logic, they are actually quite different. And what do we mean by that? Well, let us look ahead.

(Refer Slide Time: 13:23)

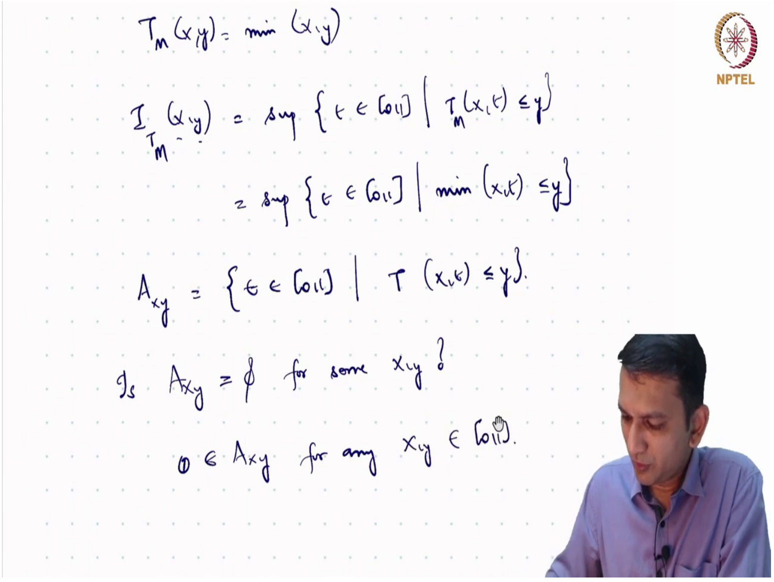


So, now this is the formula that we want and the only operation that we seem to be using is that of a meet, which we know can be captured quite nicely in terms of t-norms. So, let us begin with the t-norm T and translate this formula as follows. So, supremum over all those t element of 0, 1 such that T(x), t is less than or equal to y.

This t is actually coming from the 0, 1 interval which is a set under consideration. So, if you define a function I_T from T, like this, what can be shown? It can be shown that I_T is always a fuzzy implication. That is it is decreasing in the first variable, increasing in the second variable and satisfies all the boundary conditions. This can be seen without much difficulty.

If we consider the sets we will see how to look at this when we discuss some properties, perhaps from there you may be able to easily prove for prove it to yourself that indeed this formula does give a fuzzy implication, ok. If you consider the usual t-norms, what is the kind of R-implications do we get? Let us consider the minimum t-norm.

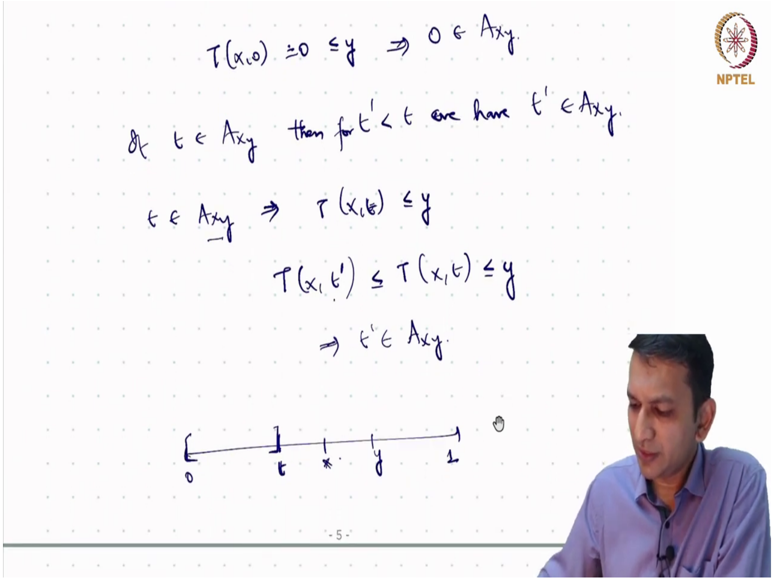
(Refer Slide Time: 15:00)



Now, let us look at what is the corresponding R-implication we get from this. Remember, it is defined as supremum of t element of 0, 1 such that T(x), t is less than or equal to y. Now, if you put M here, then you put T_M here. So, writing this formula like this supremum of t element of 0, 1 such that minimum of x, t is less than equal to y.

So, to help us in this for a fixed x, y let us call this set A of x, y as set of all those t element of 0, 1 such that min of x, t is less than or equal to y. And essentially I_{T_M} of x, y will be supremum of this set. First of all we should see in general how would this set look like for any t, that is if I remove min here and put just T, the first question is A_(x, y) equal to empty for some x, y. Now, this never happens for the simple reason look at this, 0 will always be element of A_(x, y) for any x, y element of 0, 1.

(Refer Slide Time: 16:52)



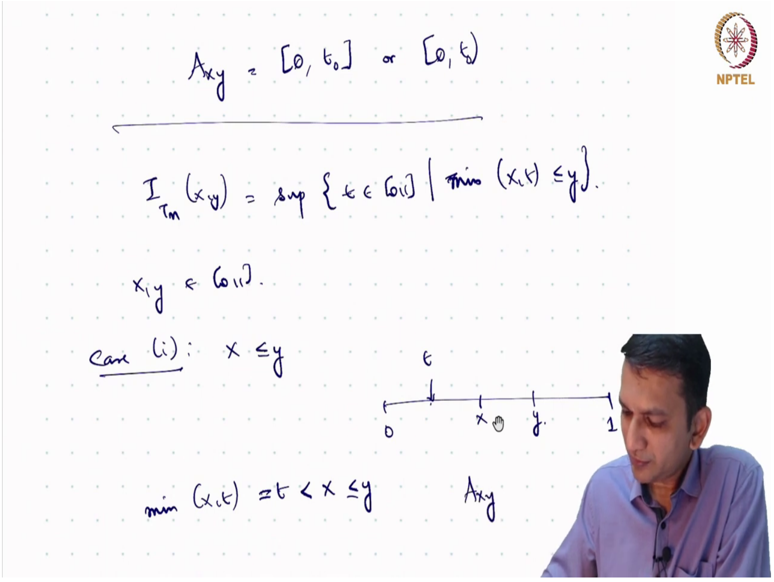
Why so? Because look at this T(x), 0 is equal to 0 and less than or equal to y for any y, which means 0 always belongs to A_(x,y). And not only this, there are many other interesting facts about the set A_(x,y). Now, what can this A_(x,y) be?

If t belongs to A_(x,y) then we can say that any t dash smaller than t also, so t dash t we have t dash belonging to A_(x,y). Well, how do we prove this? Note that if t belongs to A_(x,y) implies T(x), t is less than or equal to y. However, we know that T of t dash is smaller than t; that means, by monotonicity of t the t naught t we know that T(x), t dash is smaller than or equal to T(x), t which is less than or equal to y; that means, this implies t dash also belongs to A_(x, y).

So, essentially what we have is if you look at it like this, you have an x, you have a y. Now, you are looking at A_(x,y) for this x and y, x and y can be which way, x can be smaller than y or greater than y, but we know 0, 1 is a chain. So, both given any two elements they are related under the ordering.

So, if we know that this t here is such that T(x), t is less than equal to y, then essentially the entire interval also evolves; that means, the entire interval is also contained in A_(x,y).

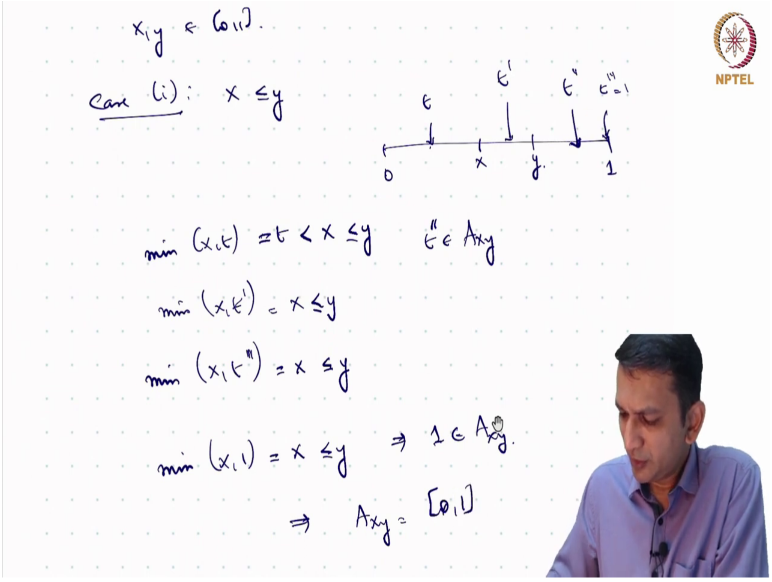
(Refer Slide Time: 18:56)



So, it can be shown, we will see with some examples soon enough that A_(x,y); typically looks like this for some t naught either it is 0, t naught or for some t naught it is close to 0 open t naught. So, essentially A_(x,y) is never empty, ok. So, now, let us come back to I_(T_M) of x, y which is nothing but supremum of t element of 0, 1 such that minimum of x, t is less than or equal to y. Yeah. How would this look like? To understand this let us split the case.

As was mentioned just now let us take x y element of 0, 1. Now, we have two cases, case 1: x could be less than or equal to y or x can be greater than or equal to y. Let us look at this x is less than or equal to y. So, that means, let us assume that x is here and y is here. Now, if I take a t here what happens to min of x, t? Min of x, t, t is smaller than x. So, this is equal is equal to t, but it is less than x and so it is less than y too. So, definitely this t will belong to A_(x,y). So, this t belongs to A_(x,y).

(Refer Slide Time: 20:40)

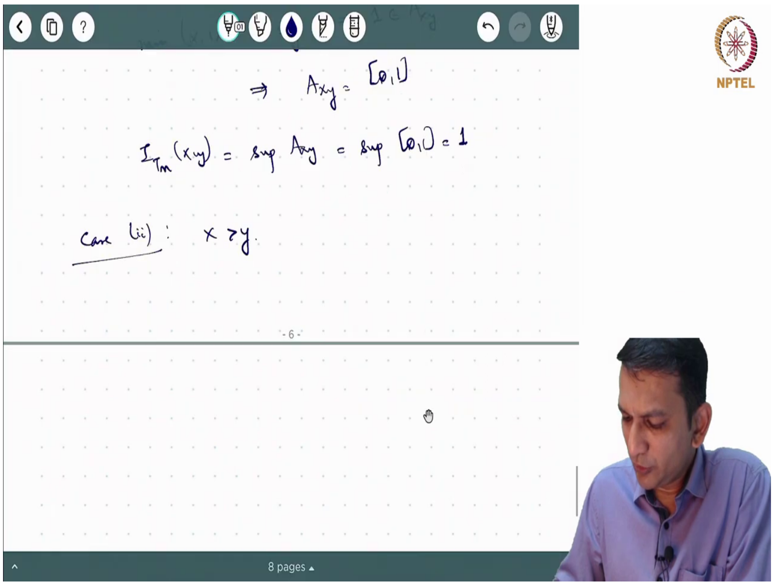


Now, what if I take a t here or a t dash here? Once again minimum of x, t dash is actually equal to x because t dash is greater than, but x is smaller than or equal to y, so that means, minimum x, t dash is less than or equal to y, which means once again t dash also belongs to A_(x,y).

What if I take a t double dash which is bigger than y also? Now, if I look at it minimum of x, t double dash, once again t double dash is bigger than x which means min of x, t double dash is actually equal to x which is still less than or equal to y because that is the case that we are discussing which means t double dash belongs to A_(x,y). Now, how far can we go here is the question.

Now, t dash we can actually push it till 1. Why? Because look at this minimum of x, 1 is actually equal to x. In this case, it is always under the assumption that we have x is less than or equal to y which means 1 also belongs to A_(x,y). But we know that if a t belongs to A_(x,y) then everybody below t also belongs to A_(x,y). This implies A_(x,y) in this case is the entire interval [0, 1].

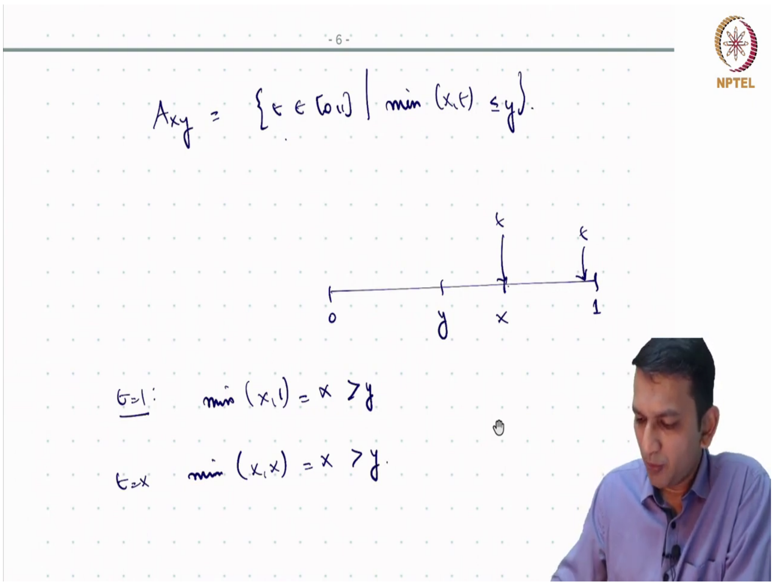
(Refer Slide Time: 22:04)



Now, what is I_(T_M) of x, y? Is nothing but supremum of A_(x,y), which is nothing but supremum of 0, 1, which is from our understanding matrices with respect to the usual ordering on 0, 1, this is actually equal to 1.

Now, let us consider the second case, x greater than y. What happens in this case?

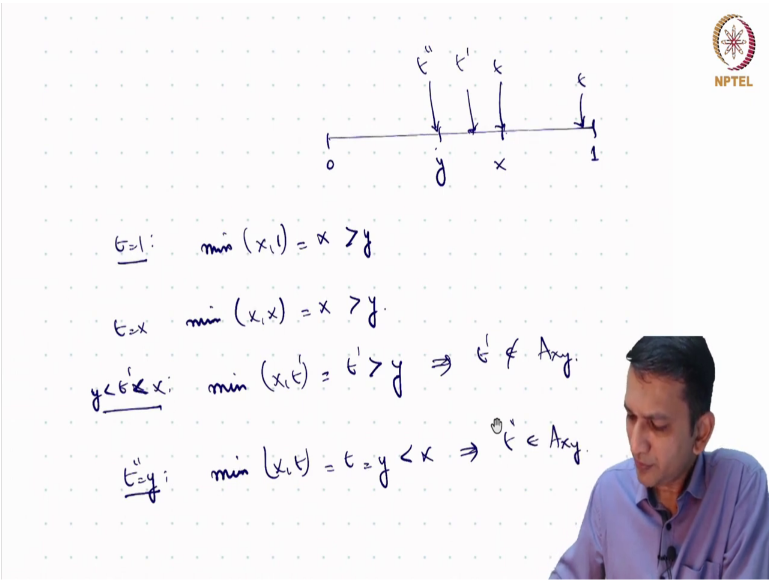
(Refer Slide Time: 22:40)



Once again we need to discuss A_(x,y); A_(x,y) set of all t element of [0, 1], such that min of x, t is less than or equal to y. And note that x is strictly greater than y here. We can ask the same question now, 0, 1; now y is here and x is here, can I have t to be somewhere close to 1. What if t is actually equal to 1? Then we see that min of x, 1 is equal to x, but it is greater than y under the assumption that we have.

So, t is equal to 1 is not going to work. Then what would work for us? What if t comes close to x or what if t is actually equal to x? If t is equal to x, we see that min of x, t is min of x, x which is x, which is greater than y; so, clearly t any t greater than x is not going to work. But what if t is between y and x?

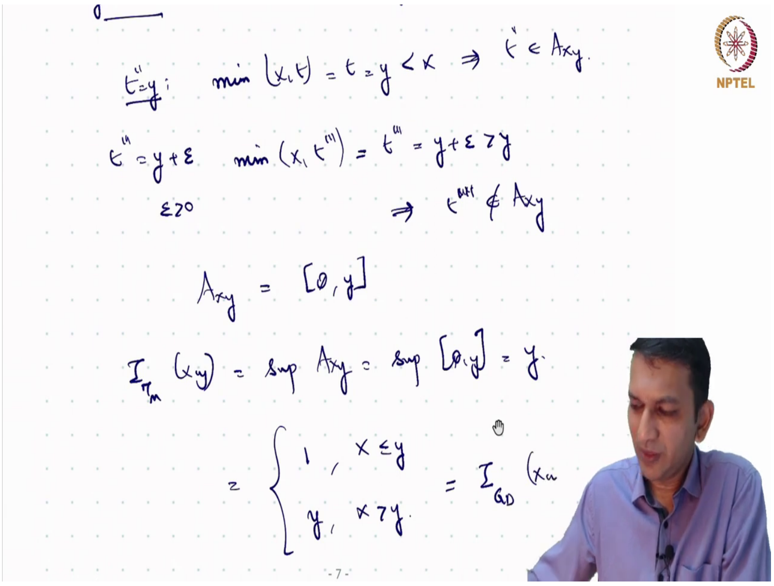
(Refer Slide Time: 23:40)



Once again, we see that min of x, t may be t, but this t is greater than y. This implies for this t, put it as t dash does not belong to A_(x,y). So, if my t dash is here, it does not going to work. How much closer or lower should I need to come towards 0, so that t dash or t can actually belong to x y? It is clear that I can come only till y.

What if my t is equal to y? Well, minimum of x, t is now t because t is equal to y and y is actually smaller than x smaller than x. This implies, I probably put this as t double dash t double dash belongs to A_(x,y).

(Refer Slide Time: 24:50)

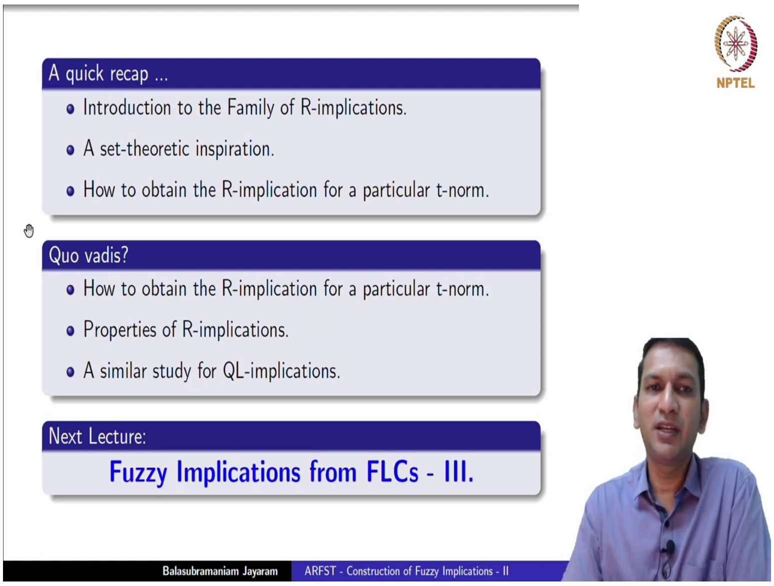


Now, we know that the moment a particular value of t belongs to A_(x,y), every value of t or every value below that, so the entire interval 0 t double dash actually will belong to A_(x,y). What if my t, I take t triple dash as some y plus epsilon; that means, it is just marginally above y?

If you look at this, this is the mini of x, t triple dash is equal to it is still smaller than x. So, it is t triple dash which is y plus epsilon. Now, since epsilon is definitely greater than 0, however small it may be is greater than y this implies that t triple dash does not belong to A_(x,y).

Well, so now, that means, if you look at A_(x,y) we see that this is nothing but the interval 0, y and y included. So, in this case, I_(T_M) of x, y is supremum of A_(x,y), this turns out to be supremum of 0, y is equal to y. So, from this we get the formula that it is 1, if x is less than or equal to y, and it is y, if x is greater than y. Now, we know that is essentially Godel implication that we have.

(Refer Slide Time: 26:08)



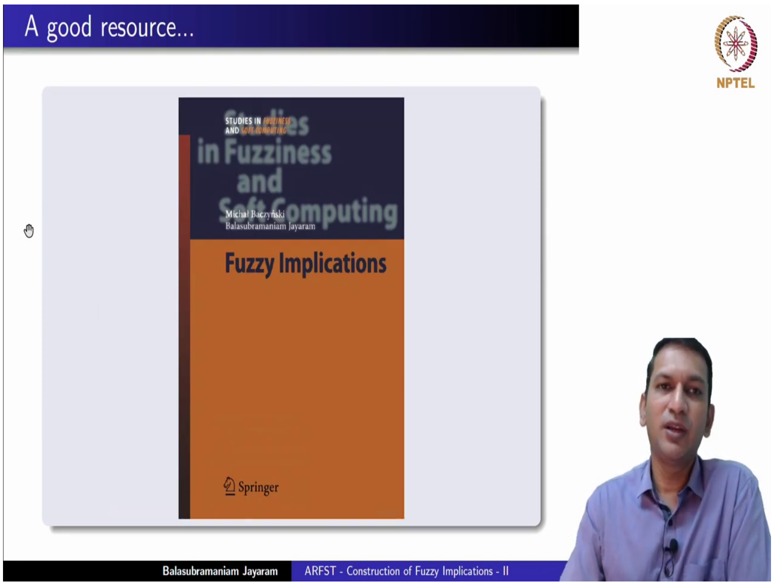
So far we have given a gentle introduction to the family of R-implications. We have seen the formula itself is inspired from set theoretic equivalence between the implication, the condition that we have represented as negation p or q. We introduced it as A complement union B. And then from there we saw, how you could look at it as finding the largest subset C whose intersection at A is contained in B.

We have seen how to obtain the R-implication for a particular t-norm that of the minimum t-norm. In the next lecture, we will look at obtaining the formula for yet another R-implication from another t-norm perhaps the product t-norm, and then we will discuss the properties the family of R-implications thus possess vis-a-vis, the identity principle, ordering property, the exchange principle, and the neutrality property.

And finally, we will also do a similar study of yet another family of fuzzy implications called the QL-implications which are is quite interesting in that. They are actually obtained from all the 3 other fuzzy logic connectives that we have seen that of t-conorm, t-norm, and a negation.

In the next lecture, we will deal with this. And with that we will come to the end of constructing fuzzy implications from other fuzzy logic connectives. As was mentioned already, there are many other ways of obtaining fuzzy implications, but we will restrict ourselves only to these 3 families in this lecture series.

(Refer Slide Time: 27:54)



Once again, a good resource for looking into for the topics that we have covered in this lecture is the book on Fuzzy Implications. Glad that you could join us today for this lecture. I am looking forward to meeting you again in the next lecture.

Thank you once again.