Approximate Reasoning using Fuzzy Set Theory
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Lecture – 18
Construction of Fuzzy Implications - I

Hello and welcome to the next of the lectures in this week 4. In the course title Approximate Reasoning Using Fuzzy Set Theory. A course offered over the NPTEL platform.

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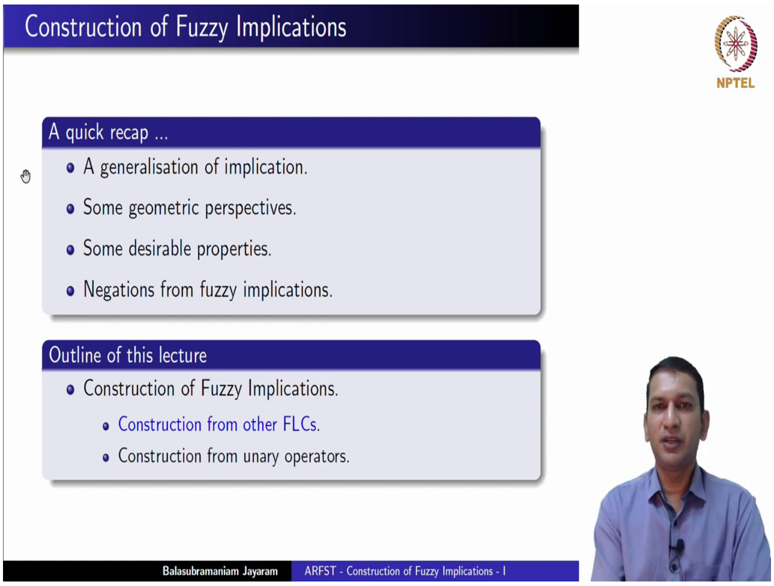

So, far we have seen a generalisation of the classical implication to the fuzzy setting. We have seen how to decode the axioms that we have come up for in terms of the definition of a fuzzy implication. From a geometric perspective that is we have looked at how the three d plots of fuzzy implications would look like. What are the essentials there, we have also seen some desirable properties that we expect a fuzzy implication to have.

Specifically we have seen how to obtain fuzzy negations from fuzzy implications. In this lecture we will deal with how to construct fuzzy implications. There are myriad ways of constructing fuzzy implications. It is still a very hot research topic to come up with families of fuzzy implications satisfying some specific properties. In this lecture series we will only single out two particular ways of constructing fuzzy implications and discuss them in detail.

The first of them is to obtain fuzzy implications from existing or non fuzzy logic connectives. In our case we have seen t-norms, t-conorms and fuzzy negations as the fuzzy logic connectives so far. So, we will try to construct fuzzy implications from these three fuzzy logic connectives. We could also construct fuzzy implications from unary operators much like we constructed t-norms using additive generators which are themselves just unary functions.

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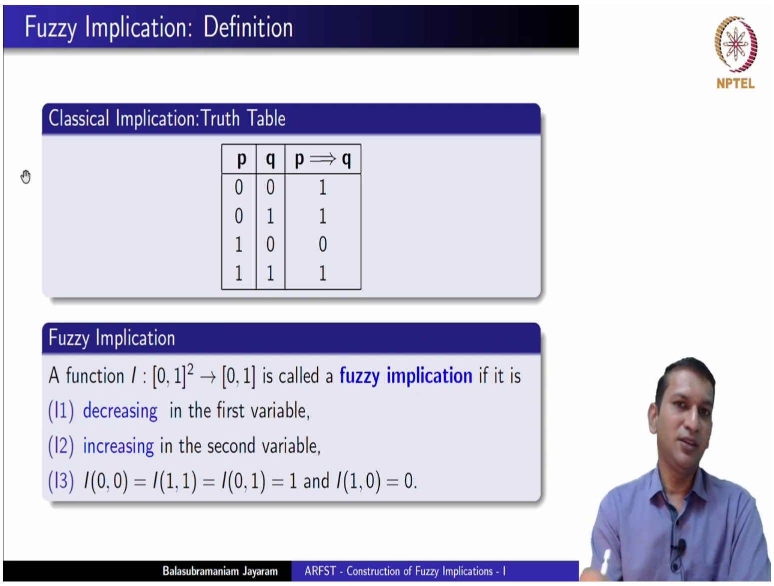

Once again, we know that this is the truth table of the classical implication based on this we have come up with the definition of fuzzy implication as follows. A binary function on this [0

1] interval, which is mixed monotonic; that means, it is decreasing in the first variable and increasing in the second variable. Once again, not in the strict sense and we ask for the boundary conditions. These four boundary conditions which are essentially what appears in the classical implication truth table.

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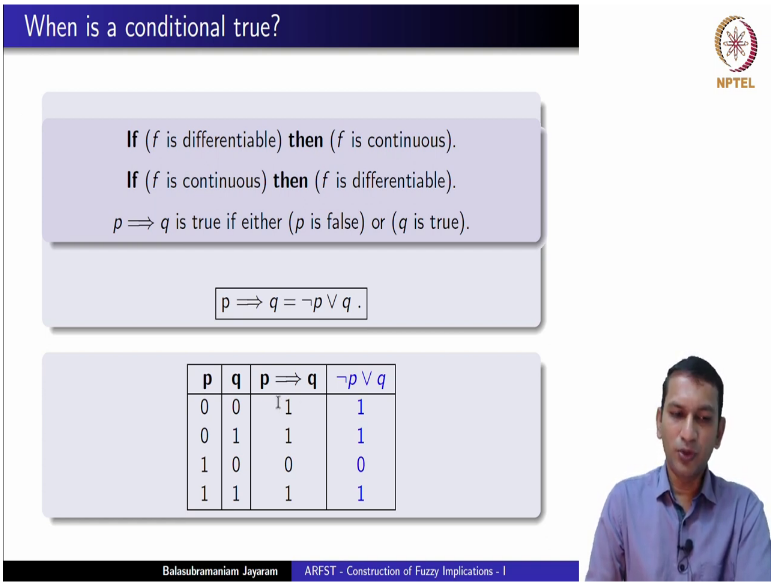

Now, let us look at these two conditionals by now these are very familiar to you. And we ask the questions ask this question when is a conditional, when is this or these conditionals true? We have seen that these are true or any conditional can be considered as true if either the antecedent p is false or q is true.

Now, this could also be equivalently formulated in terms of classical logic operations as follows: p implies q is actually equal to negation p or q. Note that whenever p is false negation p will make it true, which means true or something else would be true. Now, are we justified in capturing this essence through this formula? Well, let us look at the truth table for the classical implication. This is what we had as the output or the truth value of the implication p implies q conditional p implies q.

And now, if you actually apply this formula negation p or q for the corresponding values of p and q. We see that both these values coincide and for all possible combinations of p and q and since we are talking about binary logic classical logic. We have only four possibilities combined for p and q. And we see that all of them are actually coinciding with what we have as the truth table for classical implication. So, it appears that this is actually a valid formula to capture when a given conditional will have a particular truth value.

Once again recall that we are discussing truth values of compound statements in the sense of being able to construct them from the corresponding truth values of the components.

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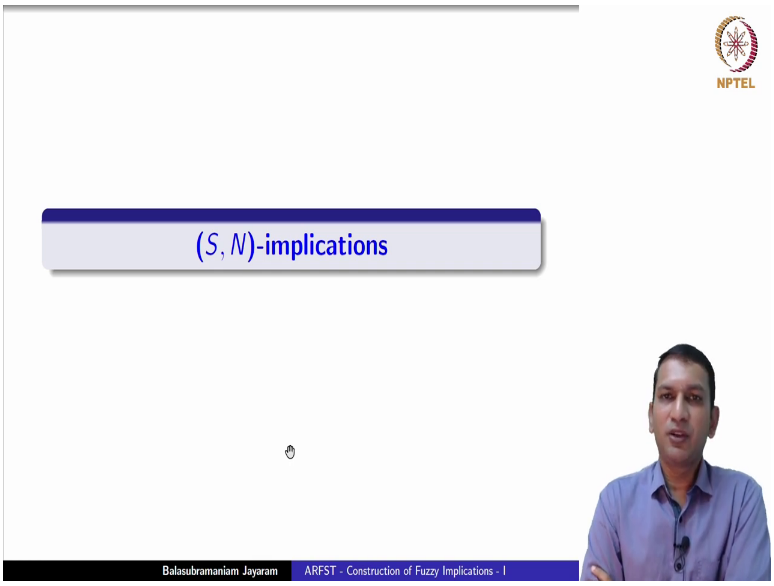

This particular formula which in the classical logic setting is called the material implication led to introducing the first perhaps the first family of fuzzy implication themselves which are called s n implications. Let us look at that definition.

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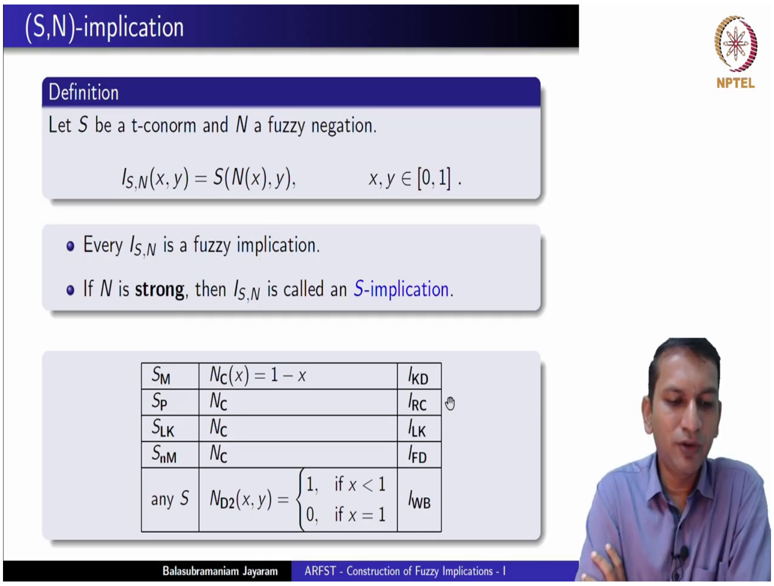

So, now we had negation p or q. So, we need a negation and we need a disjunction which in the case of fuzzy logic connectives or fuzzy set theory will immediately translate or at least used to translate into t-conorms and negations in the early days of research. So, we start with the t-conorm S and a fuzzy negation and define a function I as follows: S of N x, y. We will denote this as I with subscript S N. So, I_(S, N) is what we would denote this as and consider this function. It is easy to see that I_(S, N) is a fuzzy implication for every t-conorm S and any negation N.

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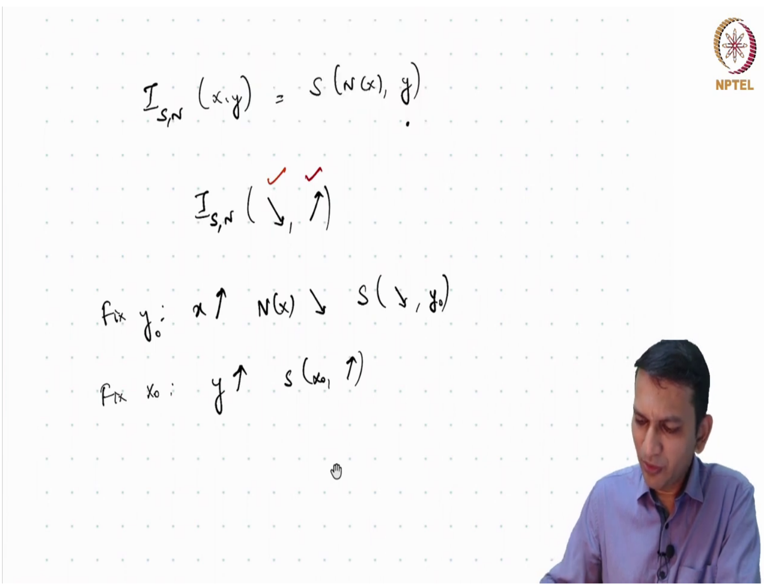

Let us look at this. This is the formula we have for this to be an implication what we want is I_(S, N) should be decreasing in the first variable and increasing in the second variable and should satisfy the boundary conditions. Now, let us look at this if we fix y let us fix y and increase x. As x increases we know that N of x decreases and S is an increasing function.

So, in the first variable as N of x decreases for a fixed y naught S also will decrease; that means, this property is valid. Similarly, if you fix an x naught and increase y then we see that S of as y increases S will increase, because S is an increasing function its a monotonic function in both variables it is commutative anyway. So, if it is increasing in one it is also increasing in the other. So, we see that both these properties are immediately satisfied. Now, what about the boundary conditions?

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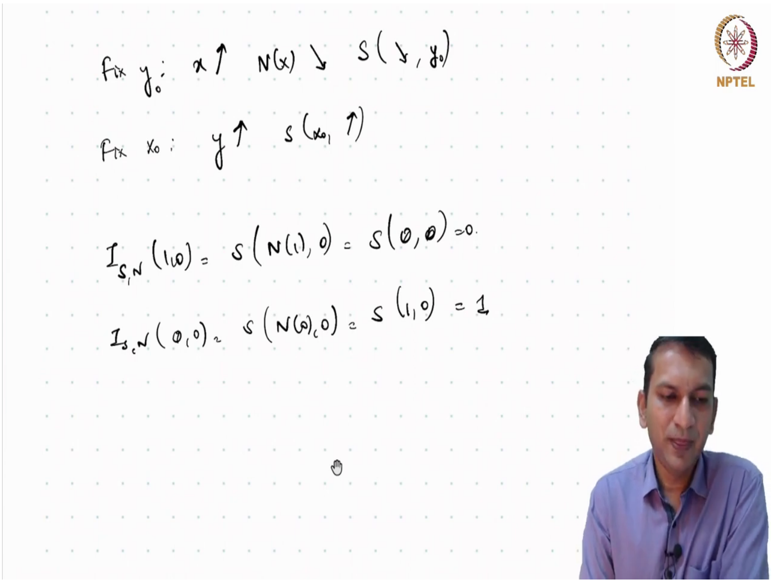

Just look at this I_(S,N) of 1, 0 is nothing but S of N of 1, 0, this is S of N of 1 for any negation N of 1 is 0. So, S of 0, 0 is 0. Similarly, we can show that S of 0, 0 is nothing but S of N of 0, 0, N of 0 is 1. So, it is S of 1, 0 which is actually 1. So, similarly we can check it out for the other boundary conditions and you will see that we actually get that all the four boundary conditions are indeed satisfied and this I_(S,N) is a fuzzy implication.

A word about the nomenclature in the beginning early days of research into fuzzy implications people typically construct strong or involutive negation and in such a case I_(S,N) was called an S implication instead of a S N implication where the N was strong. Let us look at some examples of S N implications. And you will immediately see on the rightmost column that you have the familiar basic fuzzy implications appearing immediately.

Let us take a couple of them here S_M stands for the maximum t -conorm and N_C is the usual classical negation 1-x. Let us use this formula and see what is it that we get.

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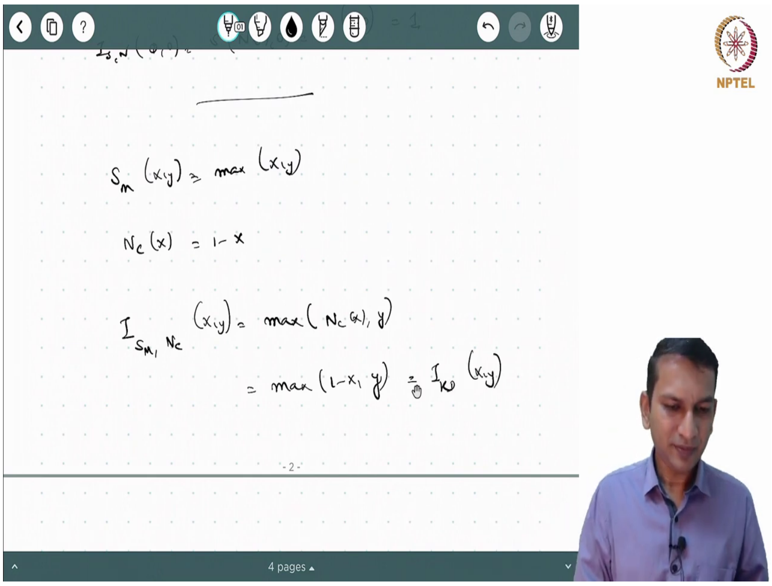

So, S_M of x y is max of x y, N C of x is 1 minus x. So, let us look at what is I_{S_M}, N C of x y this is max of N C of x, y, which is nothing but max of 1 minus x which as we know is essentially the clean beams implication. So, you see here that 5 of the 9 basic implication that we consider are. In fact, S N implications perhaps we will take one more.

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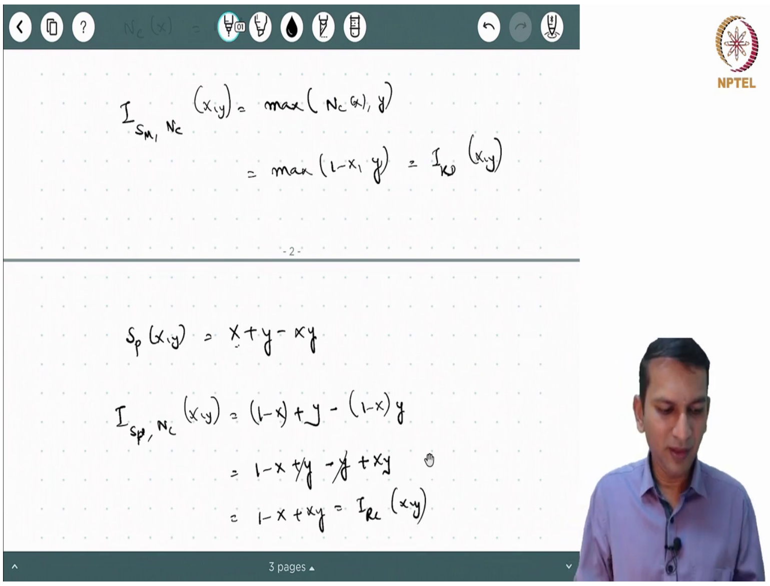

We have the S P t-conorm which is given like this is the dual of the product t norm. And once again we consider classical negation 1 minus x. So, now, I S P, N C of x y will be instead of x. We have to put 1 minus x plus y minus instead of x you have to put 1 minus x into y. This turns out to be 1 minus x plus y minus y plus x y and we cancel what you get is 1 minus x plus x y which is essentially Reichenbach implication. That is what we have integrated.

And similarly if you work out for the rest of them you will get these S N implications. Note that interestingly if we consider N D2, which is the largest fuzzy negation that you could have which is one almost everywhere except at the point x is equal to 1 and any t-conorm S, we will get the Weber implication.

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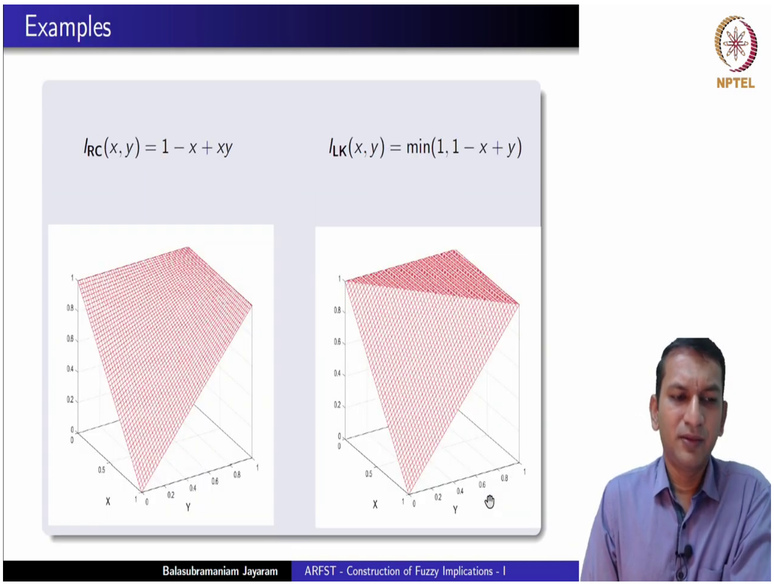

Just to recall the geometry of these fuzzy implications how the 3D plots would look like look at this, this is the Reichenbach implication. And this is the Lukasiewicz implication. These are once again S N implications. In fact, you could call them S implications also, because they the negations that they have been obtained from a strong negations 1 minus x.

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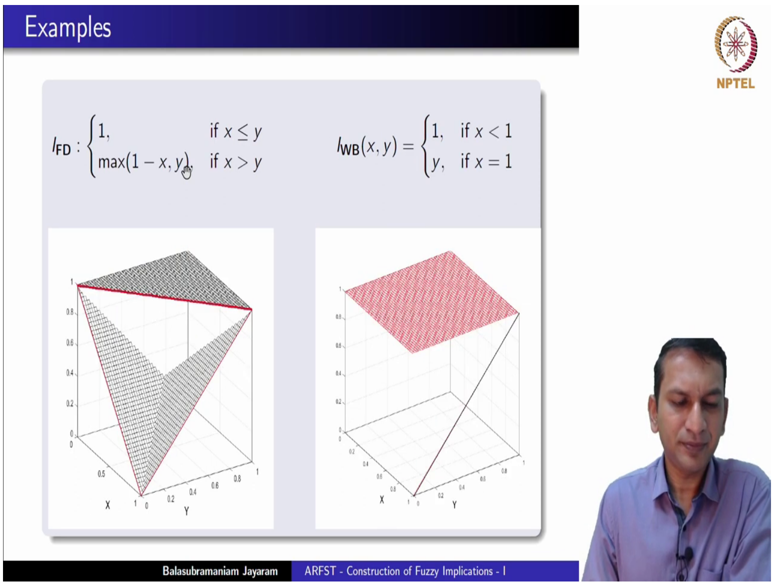

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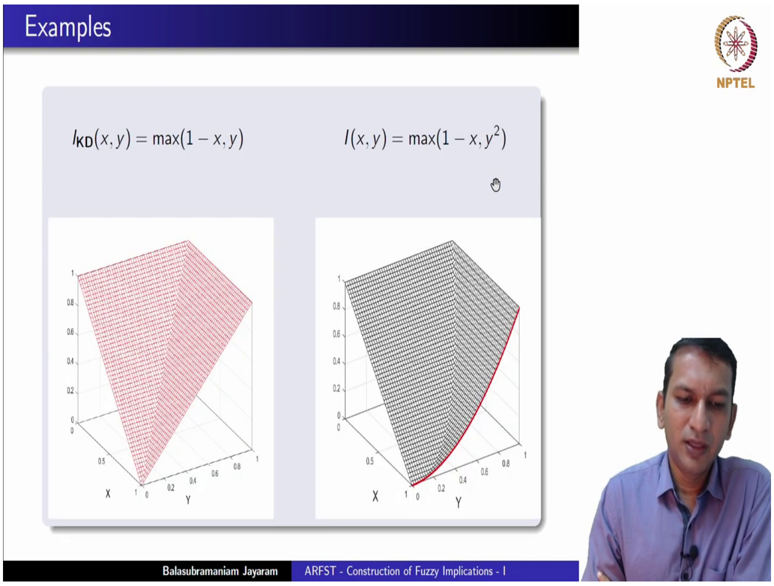

There is the Fodor implication, this is the Weber implication there is the Kleene-Dienes implication. On the right you find another implication; however, it is not an S N implication the reason for that would become very clear. If you want one immediately ok.

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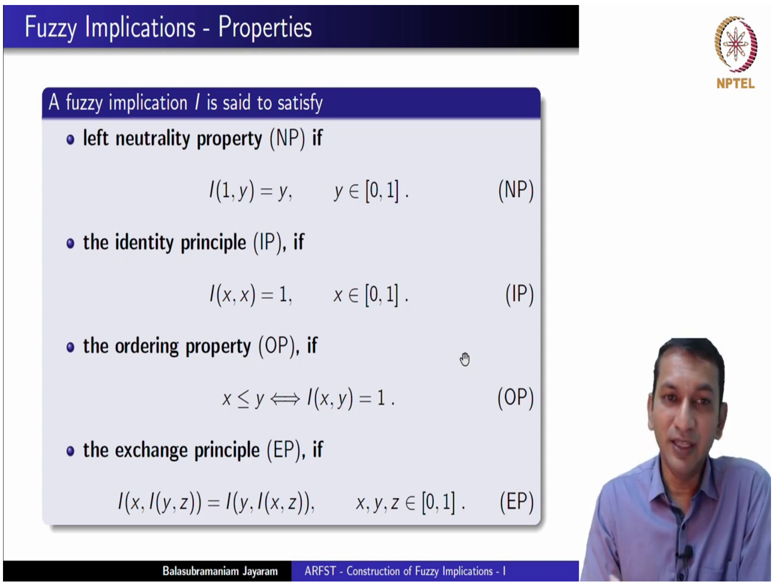

So, let us look at the S N family of implications from the perspective of what desirable properties they satisfy. And we have listed out four such desirable properties so far in previous lecture, the first of them is the neutrality property; that means, when x is equal to 1 then I of 1, y should be equal to y. Essentially we saw this was discussing about the right boundary of the implication from the perspective.

Perspective that we have; that means, we are looking at the right boundary of this particular region from the perspective that we have. The neutrality property is I of 1, y is y. Next, we discuss the identity principle which talks about what should be the value that the function the implication should assume when you walk along the diagonal; that means, y is equal to epsilon if it is 1.

We call it a fuzzy implication that satisfies the identity principle. The ordering property extends the sense is that it will not only be 1 on the diagonal clearly because of mixed monotonicity it can be proven that above the diagonal also it is 1, but ordering principle says the ordering property says that it should only be 1 up to the diagonal and not beyond it. That is what is insisted by this if and only if condition. Finally, we saw that exchange principle is some kind of it might remind you of associativity.

It is related to associativity when you generate fuzzy implications in particular way for example, S N implications, they derive the that they have this property exchange principle through the associativity of the underlying t-conorm we will see that presently. The exchange principle says that if you have I of x, I of y, I of y z. You could essentially interchange y and x and this would be equal to I of y, I of x z.

So, this is where the nomenclature of calling it an exchange principle or exchange property comes into picture. Now, the next few slides we will discuss what are these what properties from these and s n implication is likely to have.

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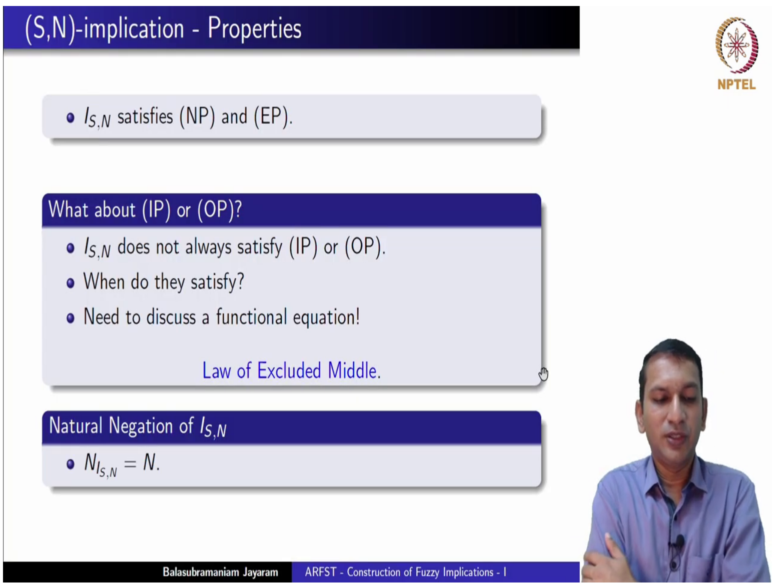

First thing that you can see is that an S N implication always satisfies neutrality property and the exchange principle. Remember exchange principle is not easy to see in terms of geometry. But we will try to prove it very simply.

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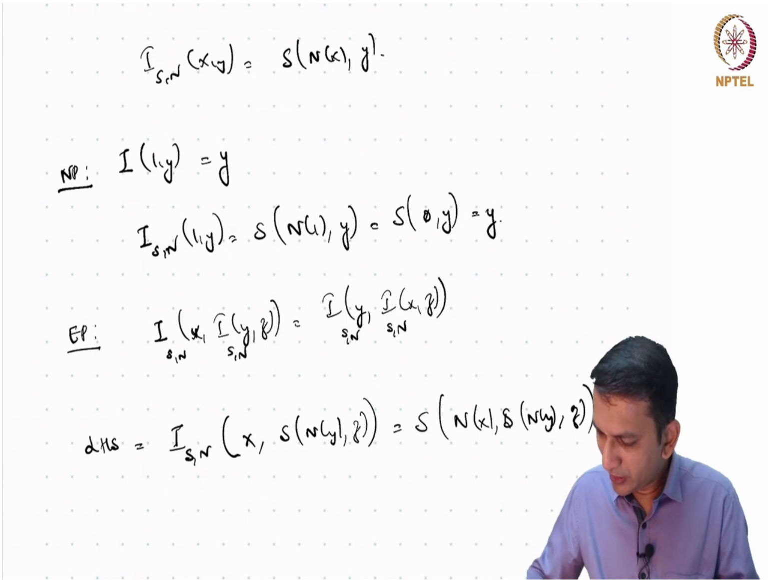

So, I_(S,N) of x y is given like this S of N of x, y. We note that neutrality is this property and clearly if you put I_(S,N) of 1, y. It is nothing but S of N of 1, y which is S of N of 1 is 0 from a y. We know that 0 is actually the identity for any t-conorm (Refer Time: 14:50). Let us look at the exchange principle what does it say? I of x, y x, y I of y, z is I of y, I of x, z.

This is what we need to prove for the S N implication. Let us consider the LHS. Now, if you substitute slowly and what we have is. Let us unfurl from the inside. So, this is S of S of y, z this will be S of N of x, S of N of y, z.

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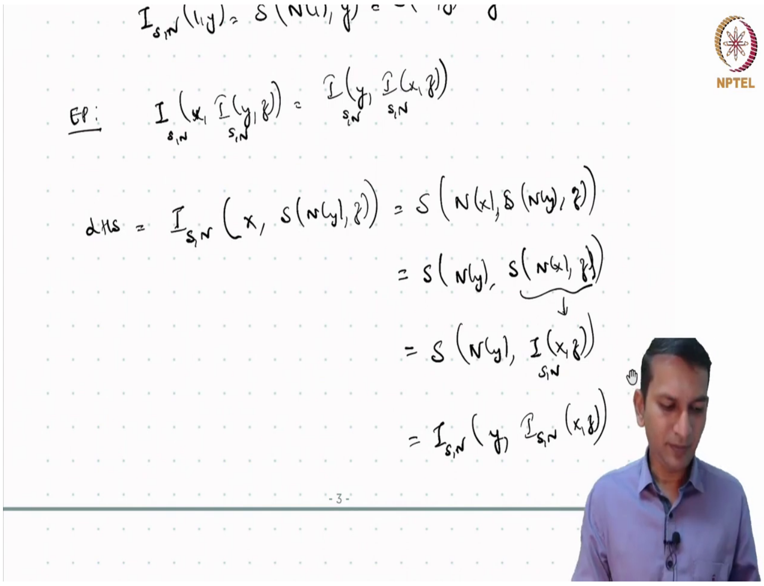

Now, we know that S is a t-conorm which means it is both commutative and associative. So, now, using this both commutativity and associativity I can we can write this as S of N of y N of x, z. Now, what is this? Once again you see that this is nothing but S of N, y. Now, this part I can write it as I of x, z for I_(S,N) of x, z and now this is nothing but ISN of y, I_{S_M} of x, z. And this essentially what we wanted to prove that is the exchange principle.

So, as you can see the associativity and commutativity of the t-conorm underlying t-conorm really helps us in ensuring that the fuzzy implications have S N implications have the exchange principle. What about the other properties of IP, the identity principle or the ordinary property. It can be shown that they do not always the fuzzy implications coming from this family.

Do not always satisfy IP or OP. If you ask the question when do they satisfy either of them or both of them well we need to discuss a particular functional equation which we have seen earlier if you recall as the law of excluded middle when we discussed lattices. We will have a separate lecture on discussing functional equations involving the fuzzy logic connectives.

So, for the moment we will skip this part about discussing when do they have IP or OP finally, if you look at the natural negation of an I_(S,N). What would it be?

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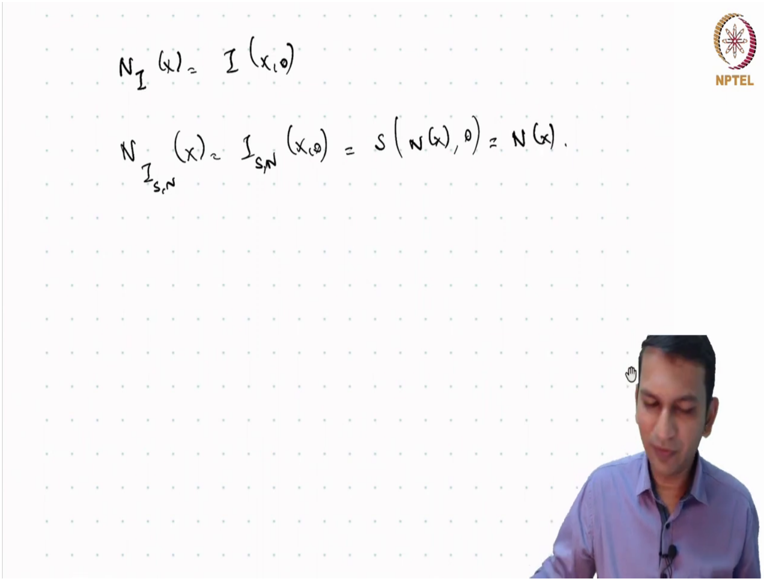

What is natural negation of an implication, given an N of I of x is nothing but I(x, 0). So, in the case of I_(S,N) this would be I_(S,N) of x, 0. This is nothing but S of N of x from 0. Note that 0 is actually the identity of the t-conorm of any t-conorm. So, what we get is it is N of x. So, the natural negation of an (S, N) implication is nothing but the negation that you originally employed to generate this fuzzy implication.

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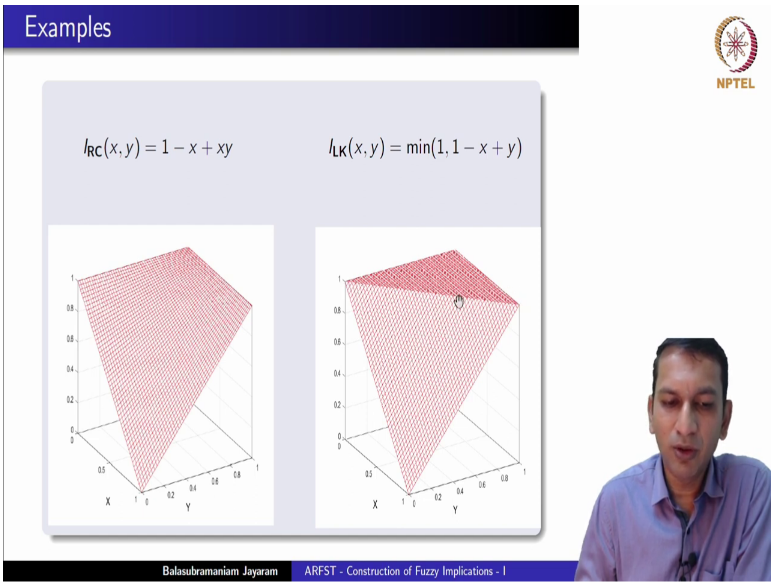

Let us look at some of these S N implications and see what properties they have from a geometric perspective. So, now, this is the Reichenbach implication. You see that it has a neutrality property; that means, that x is equal to 1 you have this y. If you look at what is the its natural negation you see clearly.

It is 1 minus x. That can also be seen by putting y is equal to 0 here you get 1 minus x. If you put x is equal to 1 here, you get it is y and. So, is the case with Lukasiewicz implication. How in the case of Lukasiewicz implication you see that it has the identity property; that means, along the diagonal it is 1. In fact, it has ordering property also because it is one only above the diagonal and now below it from our perspective. So, to speak whereas, the Reichenbach implication neither has IP nor OP.

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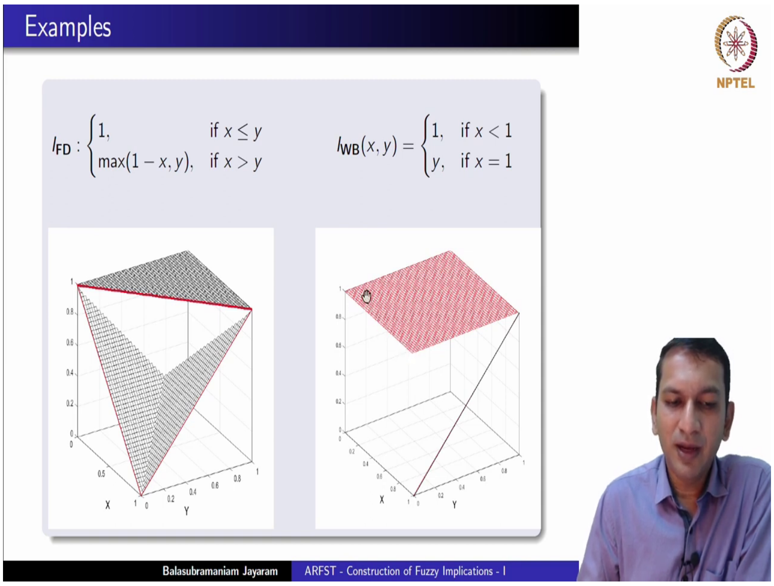

There is a Fodor implication, once again it is an (S, N)- implication you can see that it has a neutrality property as any (S, N)- implication would have. Its natural negation was one minus x and it has both OP if it has OP definitely it also has IP. and this is the Weber implication once again you see that it is it has this neutrality property the natural negation is N_D2 the largest one and you see here along the diagonal.

If you travel it takes the value 1, but it does not have the ordering property simply because even when x is greater than y, I Weber of x, y is actually equal to 1.

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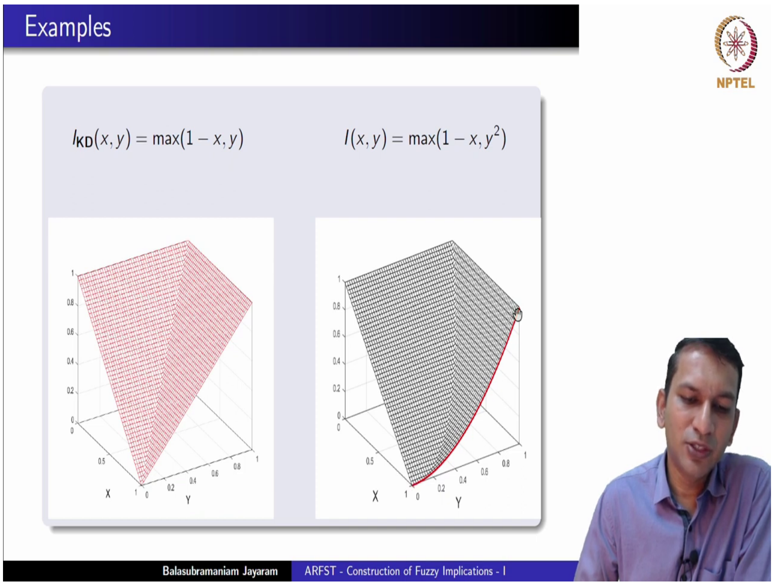

There is the Kleene-Dienes implication, once again you see that it satisfies neither IP nor OP. This is the other implication that we have seen a little while earlier you see that it cannot be an (S, N)-implication, because it does not satisfy the neutrality property; however, it has such a formula. So, it essentially max(1-x, y^2). So, you can look at it as S of N x, y square, where N is the classical negation and the t-conorm S is the maximum t-conorm; however, this is not classified as an S N implication even though its a fuzzy implication.

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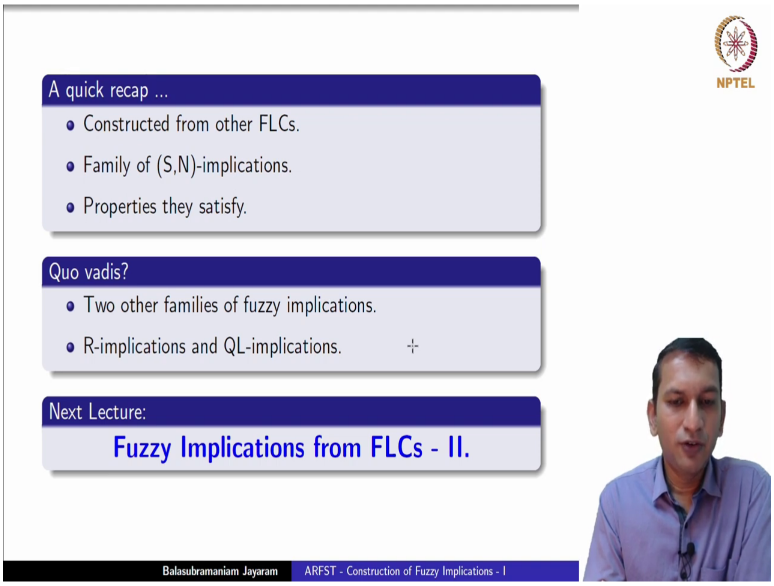

Well, so far we have discussed constructing fuzzy implications from other fuzzy logic connectives. Specifically we have looked at the family of (S, N)- implications clearly the notation. S, N here stands for t-conorm S and the negation N from which such implications have been obtained. We have also discussed that the properties that they satisfy specifically the neutrality, identity, exchange principle on the ordering property. And interestingly we found that the natural negation of an (S, N)-implication turns out to be the original fuzzy negation from which the fuzzy implication itself was constructed.

What next? There are many such families that we could construct. We will look at two other families of fuzzy implications. Constructed from fuzzy logic connectives. Namely, the families of R-implications and QL-implications. R-implications are perhaps one of the most important families of fuzzy implications. Their importance can hardly be exaggerated and QL-implications they actually find their inspiration perhaps from the quantum logic setting. We will look into both these families as we have done for (S, N)- implications.

That is in the next lecture again we will look at constructing fuzzy implications from fuzzy logic connectives. Constructing fuzzy logic fuzzy implications from unary operations will be taken up in the next lecture following that.

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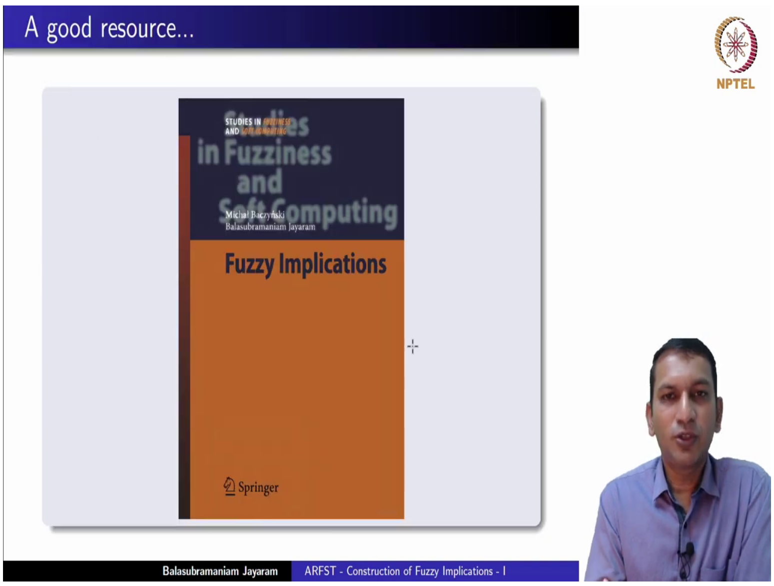

And for the topics that we have covered in this lecture a good resource could be the book of fuzzy implications. Glad that you could join us today for this lecture and looking forward to meeting you in the next lecture.

Thank you once again.