Approximate Reasoning using Fuzzy Set Theory

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Lecture - 15
T-Norms: Complementation and Duality

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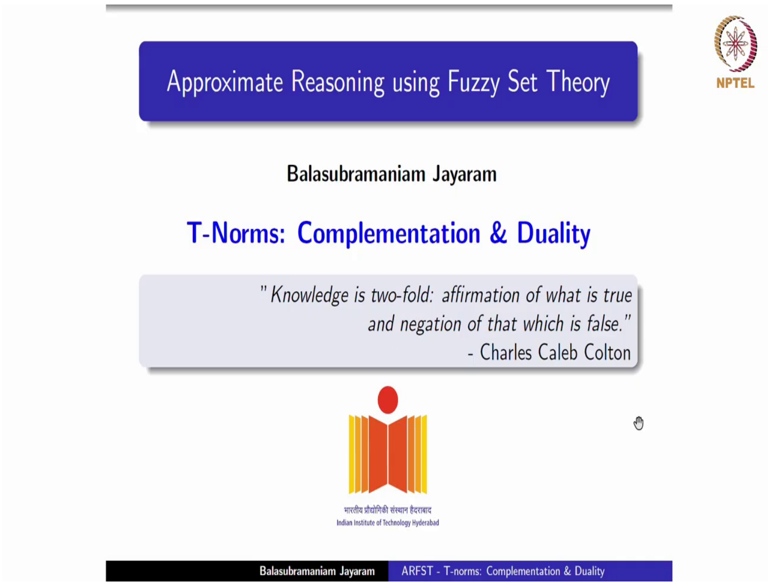

Hello and welcome to the last of the lectures in this week under the course titled Approximate Reasoning using Fuzzy Set Theory. A course offered over the NPTEL platform during the course of this week we have seen one particular generalization of conjunctions.

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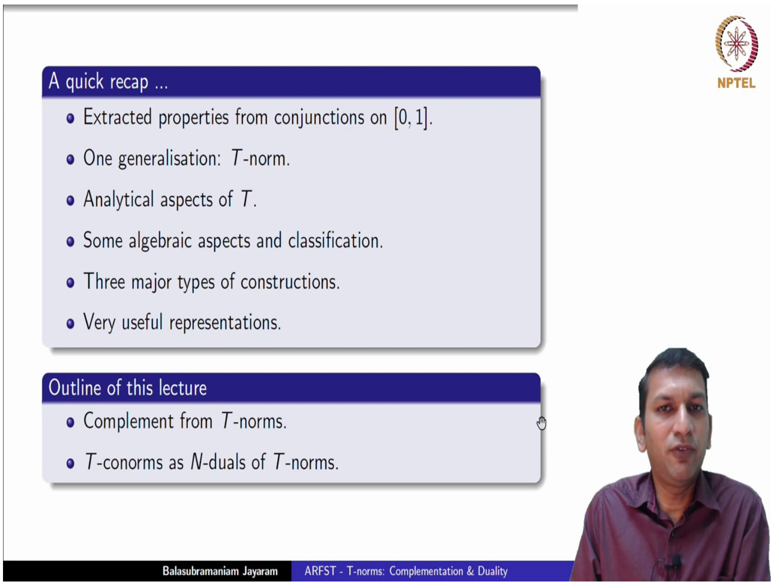

Namely that of triangular norm or T-norm for short. We have seen the different kinds of continuity that we could relate to with respect to a T-norm. We also saw some algebraic aspects of T-norms, both at the elemental level and also at the level of a function and offered a few classifications for the on the set of T-norms.

We also saw 3 major ways of constructing T-norms and how they led to useful and insightful representations. In this lecture we will look at obtaining a complement from a T-norm and also discuss fuzzy disjunctions; again, one particular generalization of a fuzzy disjunction as a T-conorm which can also be seen as an N dual of a T-norm.

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We have seen what a complementary lattices.

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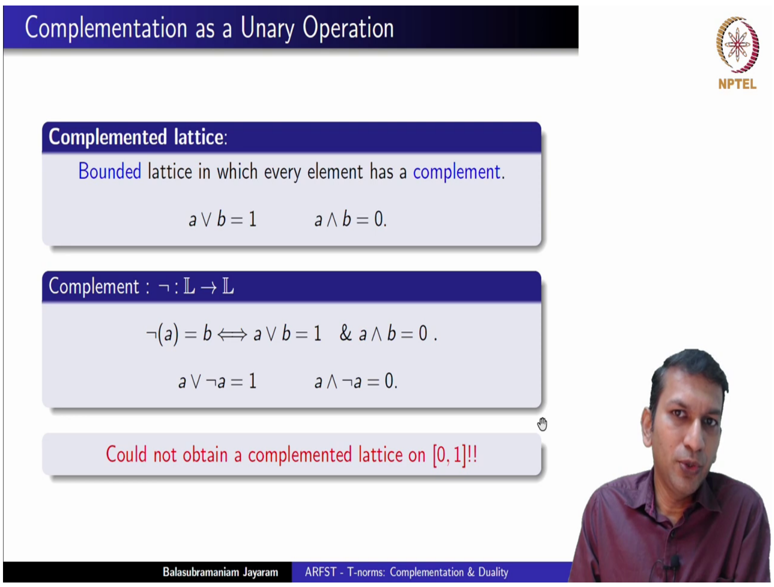

It is a bounded lattice in which every element has a complement. Complement of an element means given an a, b is said to be its complement if a joint b is 1 and a meet b is 0. Then we said we could look at this operation itself as a function from L to L.

That means negation of a is b if and only if a joint b is 1 and a meet b is 0. Unfortunately, on the set of on the unit interval we could not obtain a complemented lattice with the operation that we have seen.

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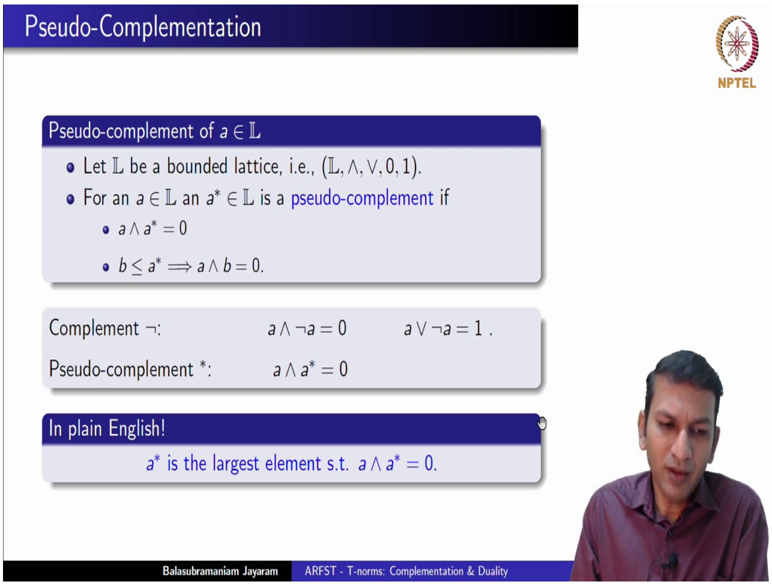

We have also seen that we could introduce another operation called the pseudo complement. Once again we need a bounded lattice L. We say an element a star is a pseudo complement of a if these 2 properties are valid.

And when you compare the complement to the pseudo complement, we see it satisfies the first of the two conditions, this pseudo complement satisfies first of the two conditions that you expect of a complement. And if you recall the second condition is what in some sense we have introduced as the stones identity. What does pseudo complement of an a stand for?

In plain English it says a star is the largest element such that a meet a star is 0. Now let us try to generalize this definition for pseudo complement to our setting.

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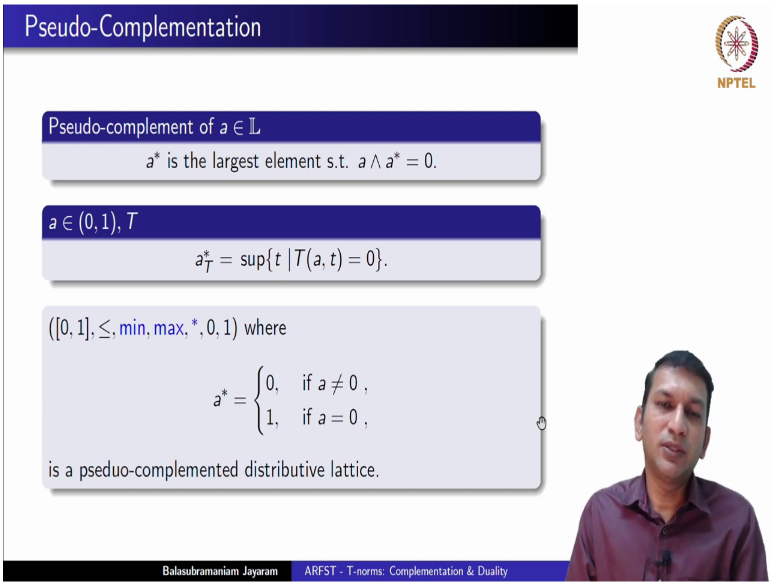

Where, we look at L as [0, 1] and meet as a T-norm. So, given an a from open (0, 1) because anywhere we know 0 maps to 1 and 1 maps to 0 under complementation. Take any a from the open interval (0, 1) for a fixed T we could perhaps define a T star the pseudo complement of a with respect to T as follows.

So, take a T such that T of a t is 0 and we are looking at the largest such T. So, that is why the supremum comes into picture. So, we could look at supremum of this set the set which consists of all those t such that T of a t is 0. Now if you look at this particular pseudo complement that we have obtained to make this structure a pseudo complemented distributive lattice.

It is clear that this a star actually has been obtained by the formula above when t was it was the corresponding minimum T-norm.

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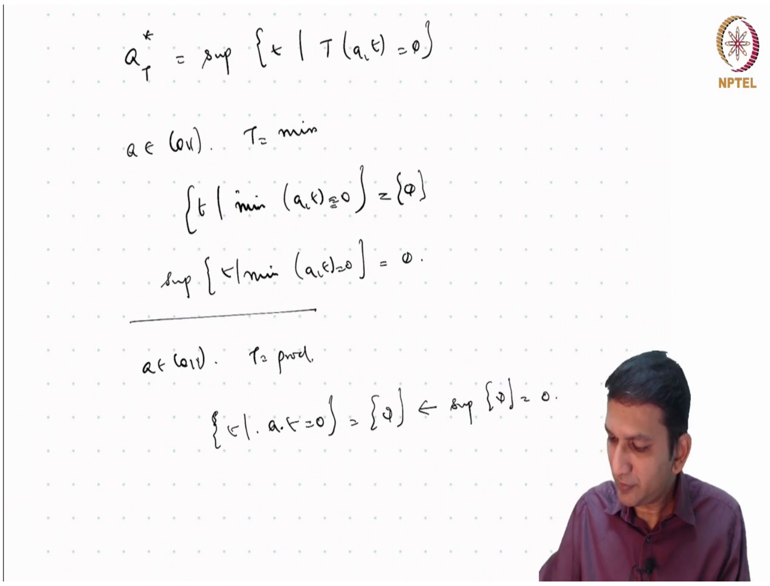

Let us look at this the definition for a T star was given this way supremum of t with the T of a, t is equal to 0. Now look at a coming from open 0 and T is the minimum T. Now if you wonder what is the set of minimum a, t is equal to 0. Now, this set consists only of 0, because if you want to take minimum of a, something to be 0 if a is not 0 it is coming from open interval (0, 1).

So, then t has to be 0. So, now, if you apply supremum on this set the supremum is also 0, but this is a set 0 this is just an element 0. So, you see that this is exactly what we have got in this definition. So, now taking Q from this let us define a natural negation of a T-norm.

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We will denote it by N T the N coming from the T-norm t.

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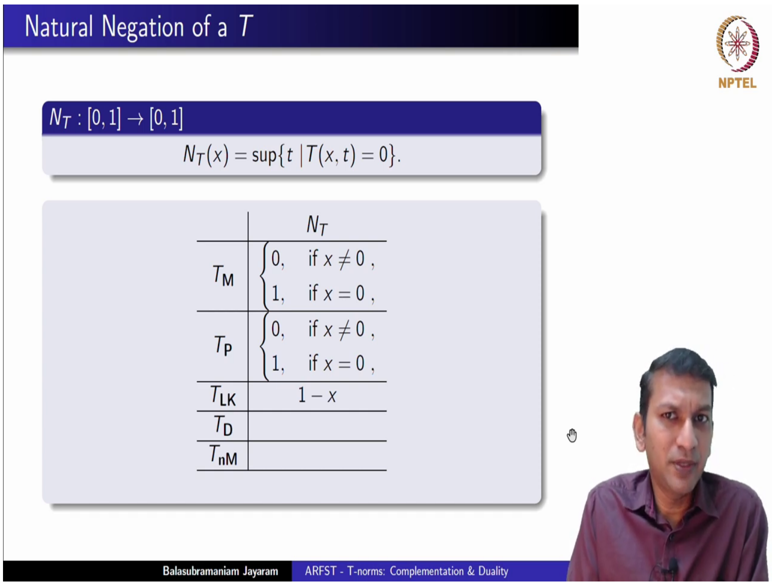

As a function from 0 1 to 0 1; as we saw complementation as a function on a. Let us define it like this supremum over all those t such that T of x t is equal to 0. Let us look at what could be the natural negations for the typical 5 examples of T-norms that we have been considered.

We have just constructed that for the T M the minimum T-norm this is the natural negation for it. How would it look like for T P. Once again a comes from 0 1 and T is the product T

LK. And you will see set of all T such that a dot t is equal to 0. Now a is not 0 then t has to be 0. So, then this set is 0, which means its supremum is also 0.

So, which means we get essentially the same natural negation also for the product T-norm. Is it different for the Lukasiewicz T-norm?

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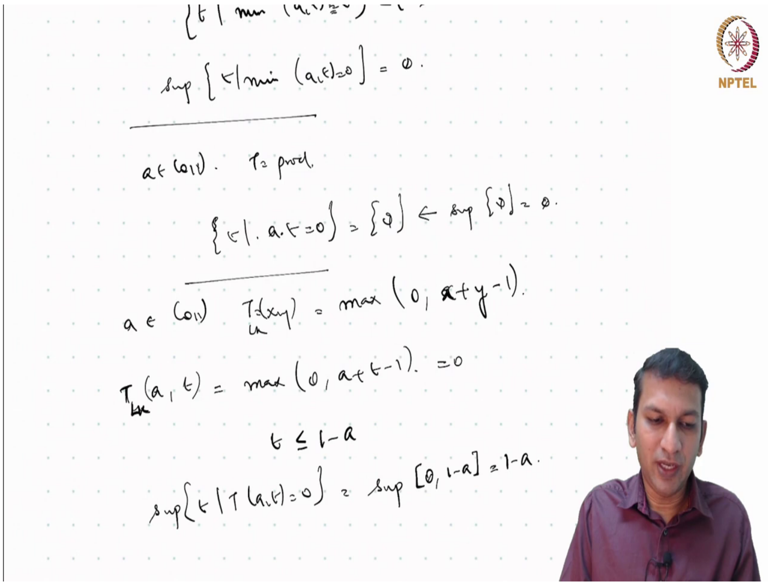

Let us look at it. Once again a comes from 0 1 and T xy is a Lukasiewicz T-norm which is maximum of 0, a x plus y minus 1. So, now, if you take a t such that T LK of a t. If you want this to be 0 look at this maximum of 0, a plus T minus 1.

I want this to be 0; that means, a plus T minus 1 is 0, alright. When can this happen? It can happen only if t is less than or equal to 1 minus a, if t is equal to 1 minus a t is 0, if t is any number less than 1 minus a, it is again 0 because of monotonicity of the T-norm. So, the supremum of this will actually be. So, if you look at this t such that, T of a, T is equal to 0. The set then consist of 0 is possible 1 minus a inclusive 1.

So, if you are looking at supremum of this this 1 minus a. So, now, this happens for every element a from [0 1]. Thus, we get that the natural negation obtained from Lukasiewicz T-norm is in fact 1 minus x. We could do the same for the other T-norms also, but let us have a geometric perspective of this.

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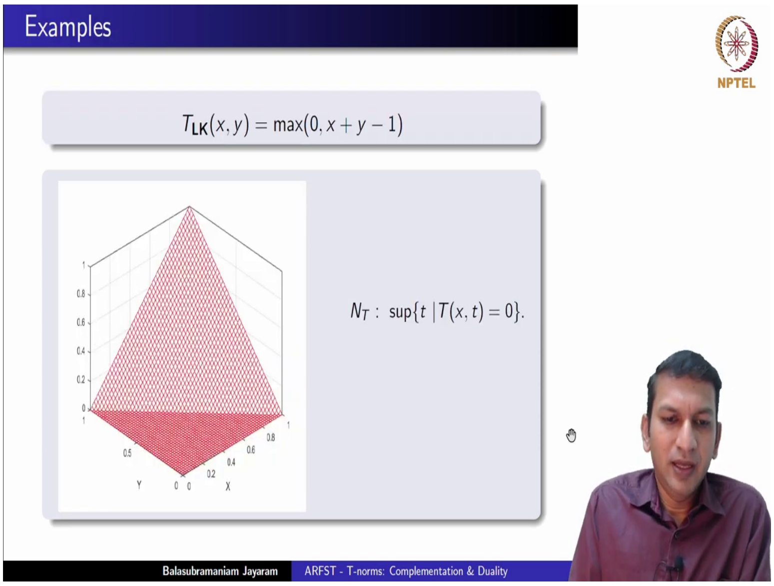

Now, this is the Lukasiewicz T-norm the graph of it. And what is the natural negation it is the supremum of all those t such that T of x T is 0. Now, what does it say it says for a given x you find all those t such that T of x t is 0. So, somehow it seems to be talking about the 0 region of the T-norm. So, essentially what lies on the floor on the x y plane. Now if you look at this Lukasiewicz T-norm.

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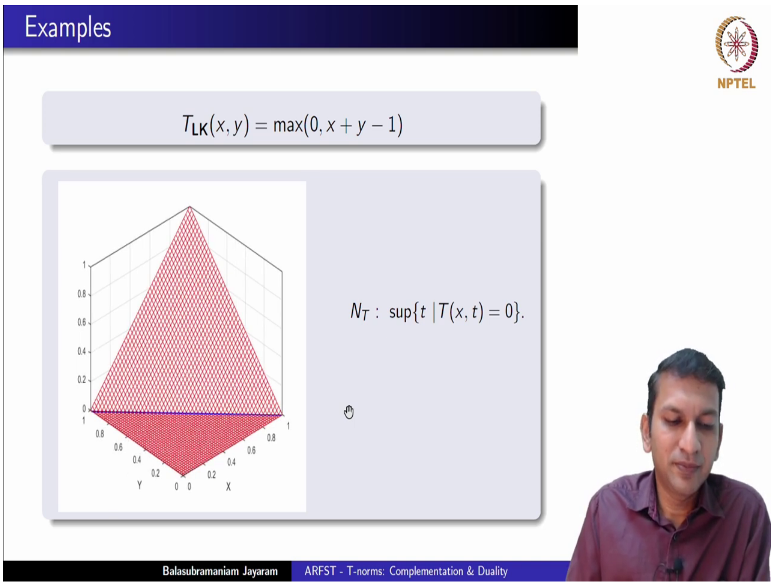

Now, let us look at the boundary of the 0 region which is now marked in blue. Remember it is a continuous T-norm. So, the boundary belongs to the floor as well to the rising graph rising surface.

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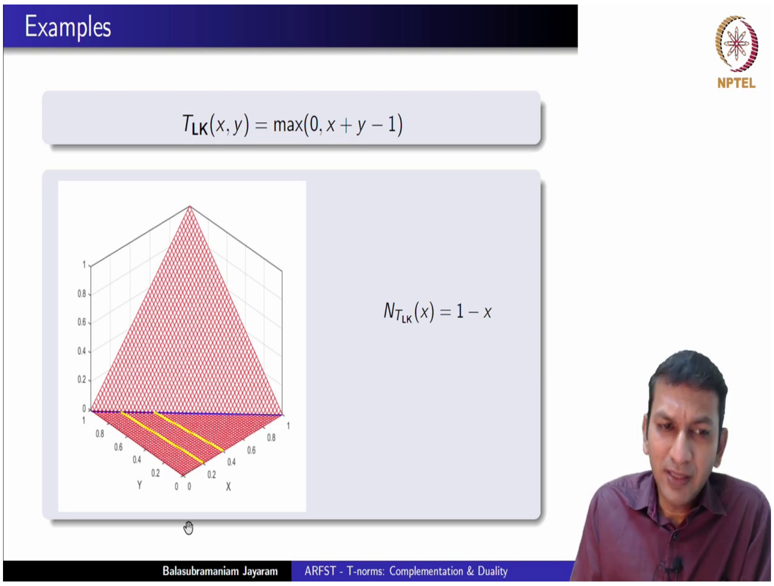

But now let us pick one element, let us say 0.2. Now all we are seeing is what if a is 0.2 we are looking at what is that t, what are all those t such that T of TLK of 0.2, t is 0. Now that means, you are looking at all those points on this yellow line, because for every point on the yellow line every t on this yellow line T of T_LK of 0.2, t is 0.

So, the supremum if you are looking for it essentially it says how far can you walk? That means, we can walk only till the edge of the 0 region. So, essentially what you are characterizing is the boundary of the T-norm. Same way if you take 0.4 how far can you walk?

We know that we can walk only till 0.6. Just as in this case when a was 0.2. You could walk only till point a the moment it is greater than point a, then if T of 0.2, t is no more is no more 0 its not 0. So, that is how you actually get the natural negation of the Lukasiewicz T-norm 1 minus x.

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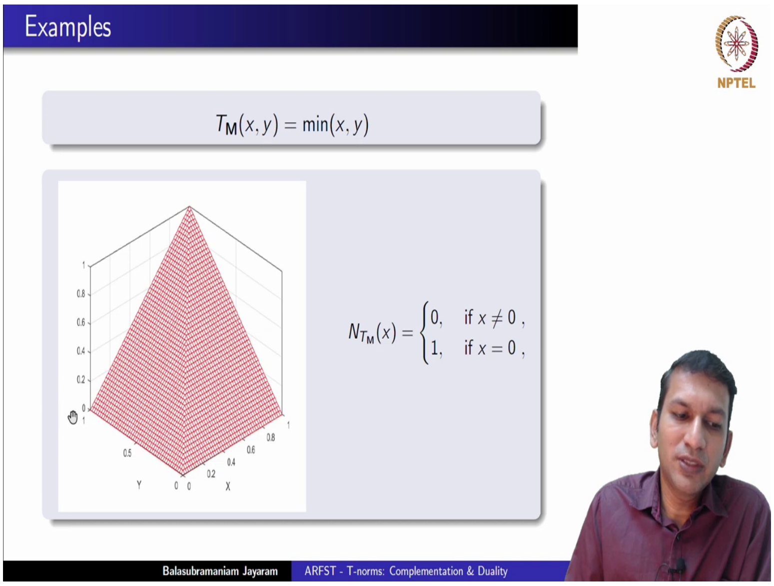

Let us explore this for a few more things. In hindsight or in the light of what we have discussed, if you look at the minimum T-norm where is it 0. Only on this boundary; that means, when x is not 0 it has to be 0 and it is 0 here. And when x is 0 of course, it can go till 1.

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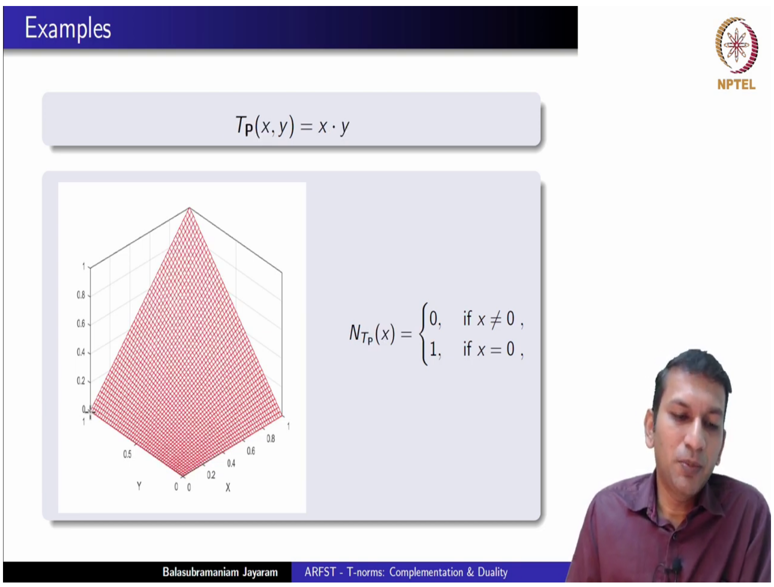

So, is the case with the product T-norm. Because there 0 regions are only on the boundary where x is 0 or y is 1.

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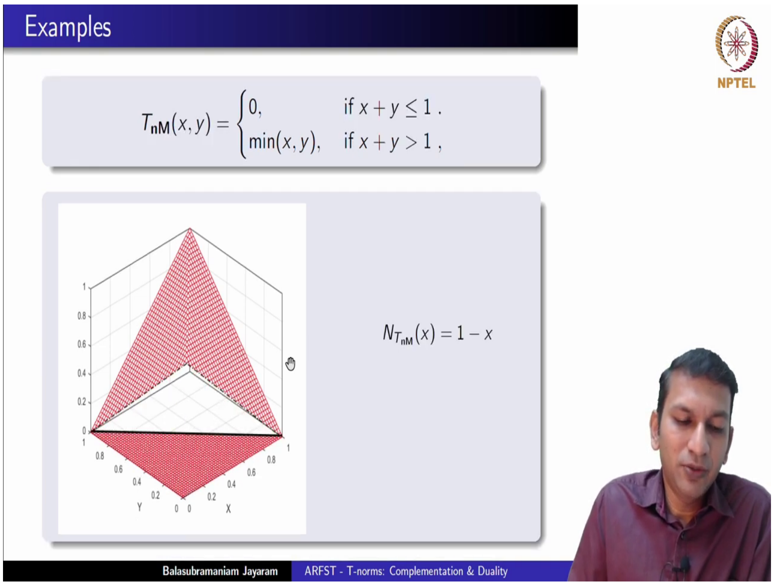

Now consider the nilpotent minimum. Now clearly you have this 0 region, part that is on the floor and rest of it actually above. Now you see that this black line tells you the boundary belongs to the floor and this dotted black line tells you the boundary does not belong to the surface that is above the floor.

So, once again it is much like the Lukasiewicz T-norm setting. If you start with any x you can only go up to 1 minus x that is the boundary of the 0 region.

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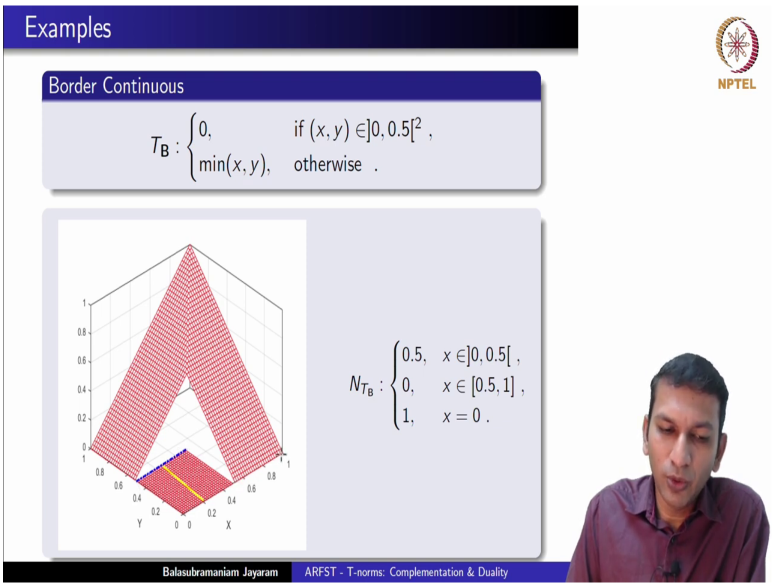

So, far we have considered left continuous T-norms; that means, T-norms are continuous T-norms for whom the boundary belong to the 0 region. Now let us consider some T-norms for whom the boundary of the 0 region actually does not belong.

You look at this on the open 0, 0.5 square is where it is 0. So; that means, the blue line is a boundary of the 0 region, but does not belong to the 0 region. That is why it is shown as a dotted; however, if you take a point and walk. So, start from 0.2 how far can you go, you can go almost till 0.5. The moment you touch 0.5 the minimum operation applies on, but you can walk almost till 0.5.

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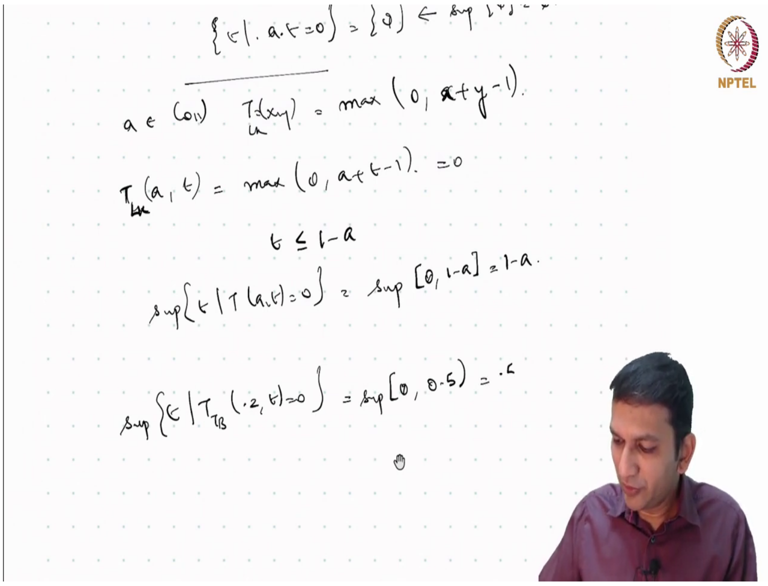

So, essentially if you look at set of all t such that T TB of 0.2, t is equal to 0. You see that this set is of course, 0 belongs 0, 0.5 open. The moment you put 0.5 you are no more in this region you fall in this domain, which means it becomes minimum of 0.5, 0.2 which is 0.2 and that is what you see here.

Now, this is the set and you want to apply supremum on this. From what we have seen earlier we know the supremum of this set is in fact 0.5. And you see that from 0 to open 0.5 you can walk for each one of those points you can walk actually till 0.5. So, when you see it like this it is easy to construct the natural negation from the graph of the function itself if you know it.

So, essentially it is the boundary of the 0 region for 0 it is 1 between 0 and 0.5 open it is 0.5 and from 0.5 and to 1 closed it is actually 0.

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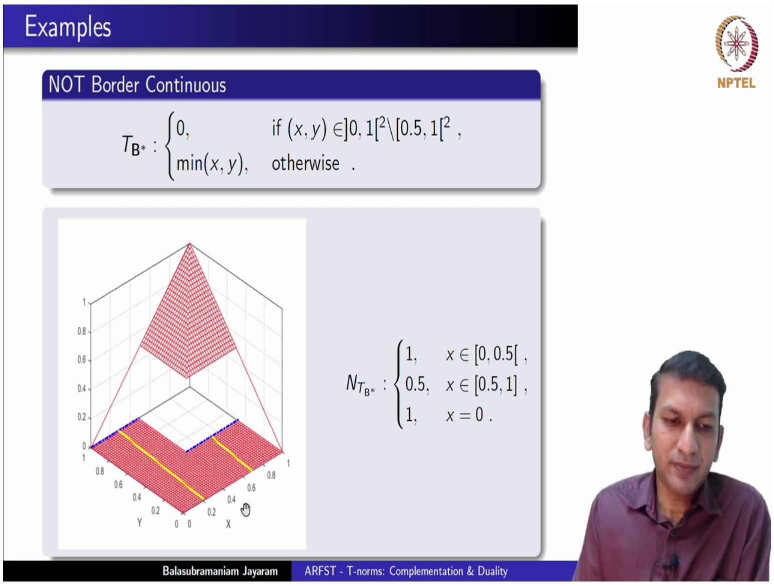

Now let us look at this particular T-norm which we have seen is neither left continuous nor border continuous because at 1 it is not continuous. So, now, if you take any point here 0.7, it is clear you can only walk till this point which is in our case 0.5.

You see that once again these blue lines are dashed or dotted, which means the boundary does not belong to the floor of this graph does not belong to the 0 region of it. Similarly, if you start here at 0.2 you can almost walk till 1, which means this is the natural negation of this particular T-norm. So, between 0 and open 0.5 you can almost walk till 1, but 1 is not there.

Remember because at 1 it has to be the identity. So, it can be here, but till 1 you could walk. Similarly you can for say 0.7 you could walk till 0.5 almost 3.5, but not touch 0.5. So, you see that the natural negation somehow is capturing the boundary of the 0 region.

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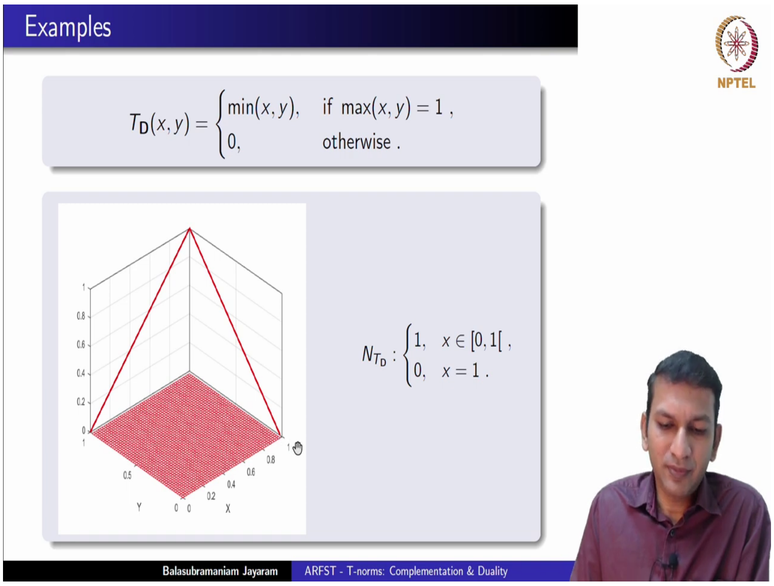

And for the T D it is very clear now that everywhere it is 1 except at one which is 0.

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Well from this you could also talk about negations in general not really being obtained from T-norm what is a fuzzy negation.

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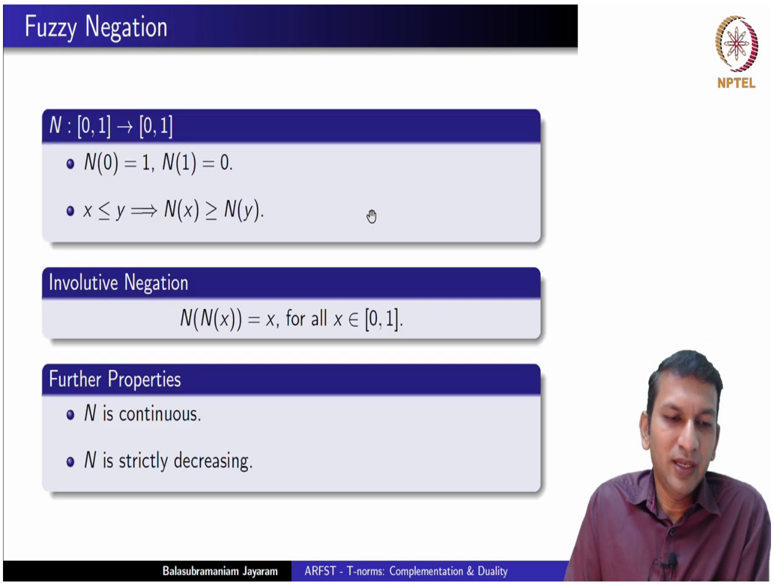

So, this is just a complementation for us so; that means, a negation from as a function from 0 1 to 0 1. What do we expect of it the boundary conditions N of 0 is 1, N of 1 is 0 and order reversing function, means when x is less than or equal to y it which should imply N of x is greater than equal to N of y.

Of course, we can ask for other some interesting or useful desirable properties, that of involution. We say negation is involutive that in case of classical complementation N of N of x is x and then N composite itself gives you identity. We could also ask for few other interesting properties that of continuity and of strict decreasing; that means, when x is not equal to y, N of x is not equal to N of y.

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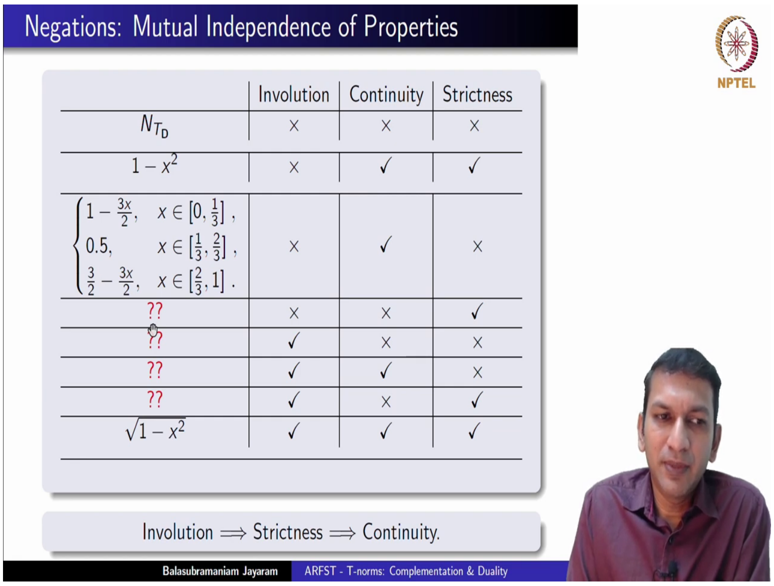

It is interesting to see whether these three properties that you have added. Note that this is the standard definition of a negation all we want is boundary property and this monotonicity. Monotonicity in the sense of decreasingness, everything else is what we would like it to have desirable property. So, let us look at the desirable properties, how are they related?

For example, if you look at the natural negation that you get from product or minimum or the drastic T-norm. You will see that they are neither involutive nor continuous nor are they strict, because they are constant in most of the places either 0 or 1. If you look at this particular function 1 minus x square it is not an involution.

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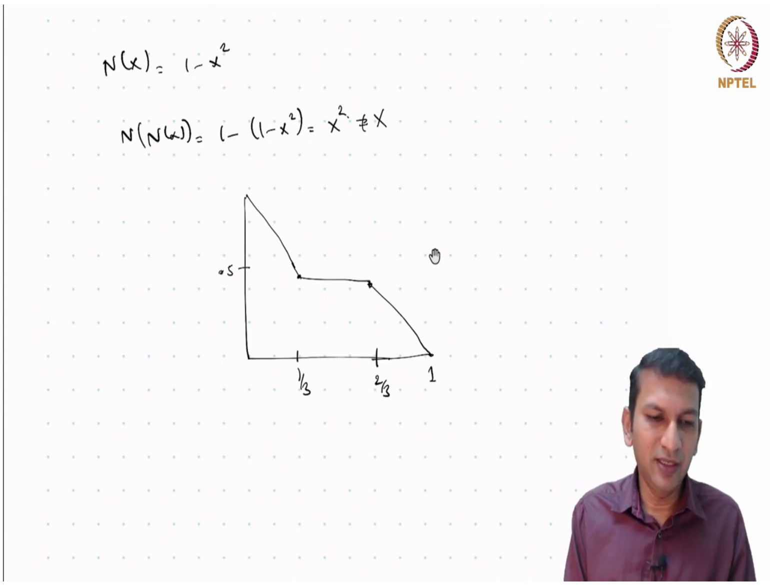

Clearly N of N of x is not x this can be seen. N of x is 1 minus x square N of N of x is 1 minus 1 minus x square you will see that it is actually x square and not equal to x. So, this is not involutive. Of course, it is continuous and we see that it is strictly decreasing. If you look at this function it shows, so in fact the graph of this function would look something like this.

So, up to 1 by 3 it falls from here to here, then between 1 by 3 and 2 by 3 it is 0.5, 1 here then it comes to 0. So, you can see here it is continuous, but not strictly decreasing and it is not also involuted. Now what about other options? If you see you cannot find any example such that it is only strict and not continuous or involutive, it is involutive but neither continuous nor strict and other such options.

Why is it so? We will see that presently. One last example if you look at root 1 minus x square or the natural usual classical negation of 1 minus x, it is involutive continuous and strict. Now let us try to answer why we are not able to find examples in these 4 rows. If you start with an involution it is N of N of x is x. We can also show that such a negation will also be strict; that means, it will also be strictly decreasing.

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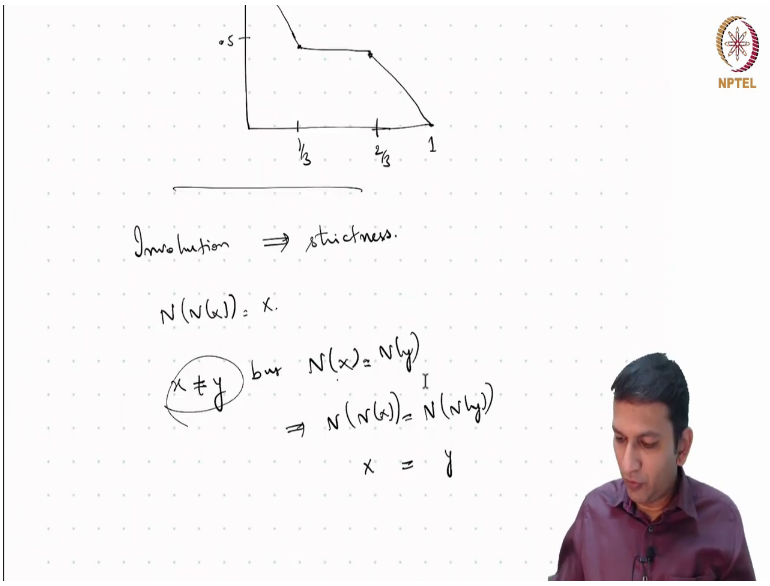

It is actually quite simple to see, what do we have? We need to show involution implies strictness. So, involution means N of N of x is equal to x. So, let us for the moment assume that x is not equal to y, but it is not 1 to 1; that means, N of x is equal to N of y, but this means if I apply N once more and there is a function where N of N of x should be equal to 1.

But because it is involuted N of N of x is x this is y and it says x is equal to y contrary to our assumption. So that means, essentially, we are saying that N of x is N of everything x has to be equal to y. So, clearly if it is involutive you know that it is also strict, but it can also be proven because of this monotonicity strictly decreasing and because of the boundary condition that such a negation has to be continuous.

The proof is not very difficult, but it can be seen we can work this out if the context warrants, we will look into similar such groups little later on. So, involution implies strictness and strictness implies continuity of the negation well.

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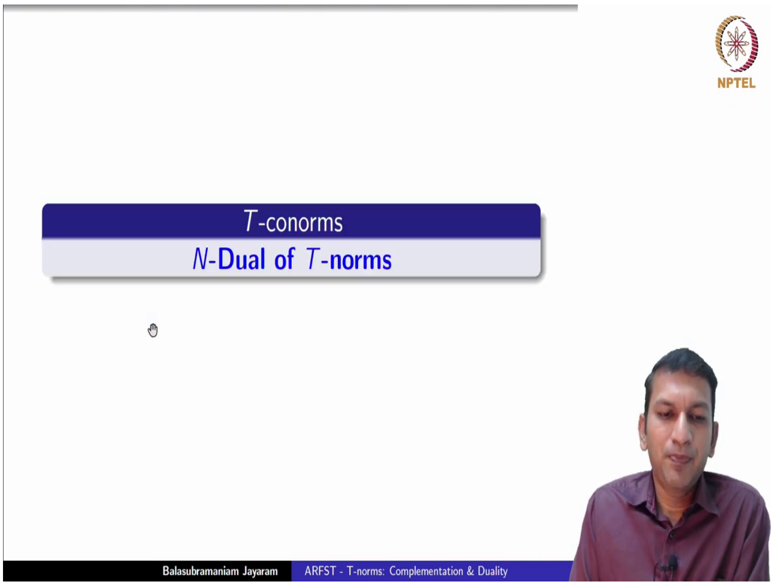

So, much for the complementation operation there is a lot more to it we could talk about fixed points, equilibrium points and different kinds of negations. But as such is a if the context warrants later on, we will look at such special types of negations or special properties of negations, for the moment.

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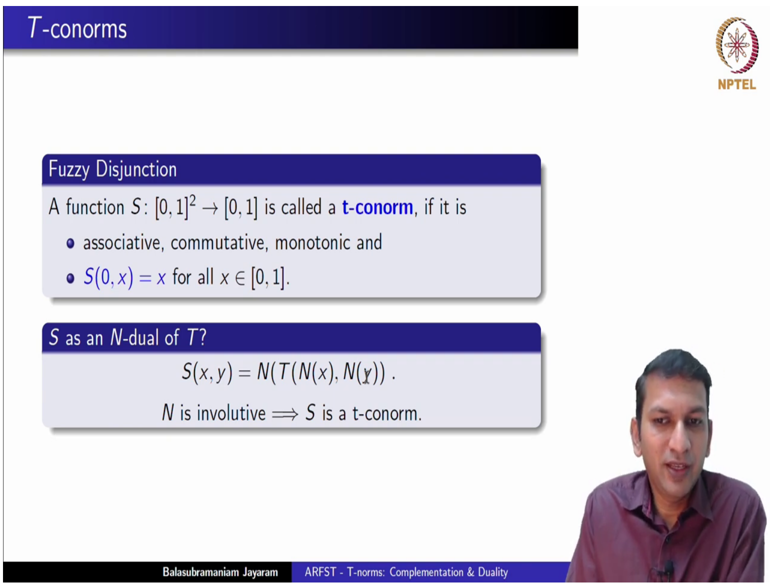

Let us look at fuzzy disjunction. One particular generalization of a fuzzy disjunction is called a T-conorm triangular conorm. It is a function binary function on 0 1, which is associative commutative monotonic increasing and 0 is the identity for it, unlike the T-norm where 1 was the identity for a T-conorm 0 is the identity.

So, you see that almost everything is same, looks like only the identities got switched swapped 0 to 1, 1 to 0. There 0 was an annihilator here 0 is the identity. Of course, it can be shown that 1 is annihilator here. Now the question is can we relate S and T through this negation? What if we define take a T-norm T and define S of x plus N of T of N x, N y, will this become a T-conorm?

Now, let us look at it, what do we need for it? We need this to be commutative, S is commutative as is clear because T is commutative. It is it monotonic? Note that as x increases N decreases as N decreases T in the first variable decreases, but N is a negation. So, again it increases. So, which means it is monotonic if you put 0 here then this is essentially T of N of 0, which is 1 and what we get is for 0 to be the identity element.

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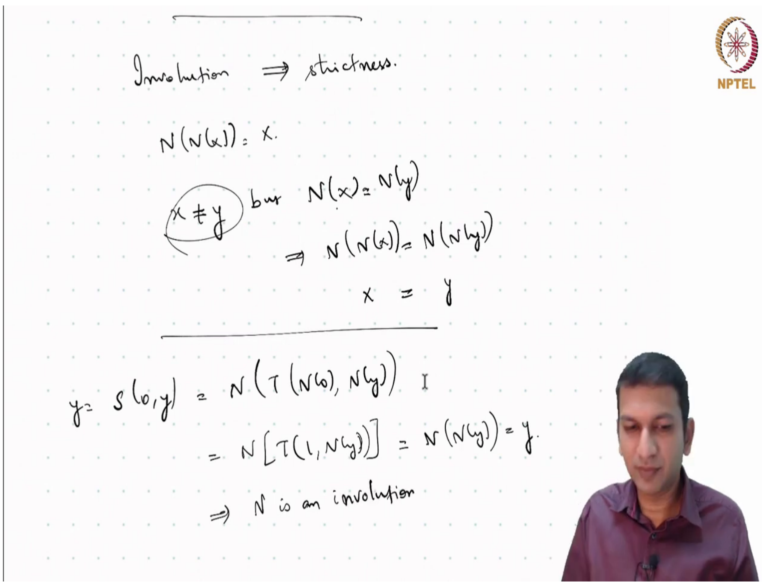

We want that N of T of N of 0, N of y this is equal to N of T of 1, N y this is N of N of y. So, now if you want this to be y it is clear that this implies N has to be in involution. Now what about associativity yes, if N is an involution. Then you can easily see that it will also be associated much like in the case of taking the additive generator f circle f inverse because N circle N will be identity.

So, if you use an involutive negation, then every T-norm gives rise to a T-conorm, which is called the dual T-conorm of the T-norm T.

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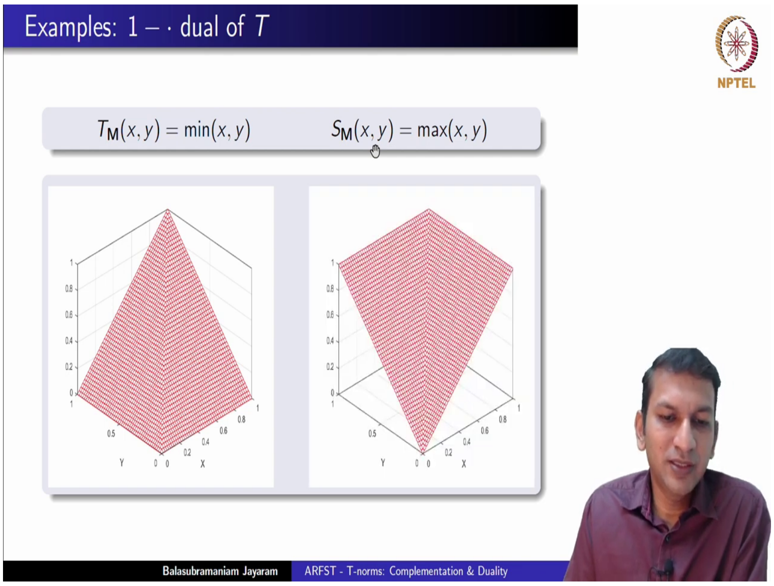

Let us look at some examples with respect to 1 minus x as an equation it is easier to plot and easier to see the relationships. So, if you have minimum the 1 minus x dual of minimum is the maximum. So, graph of minimum looks like this the maximum almost inverted one.

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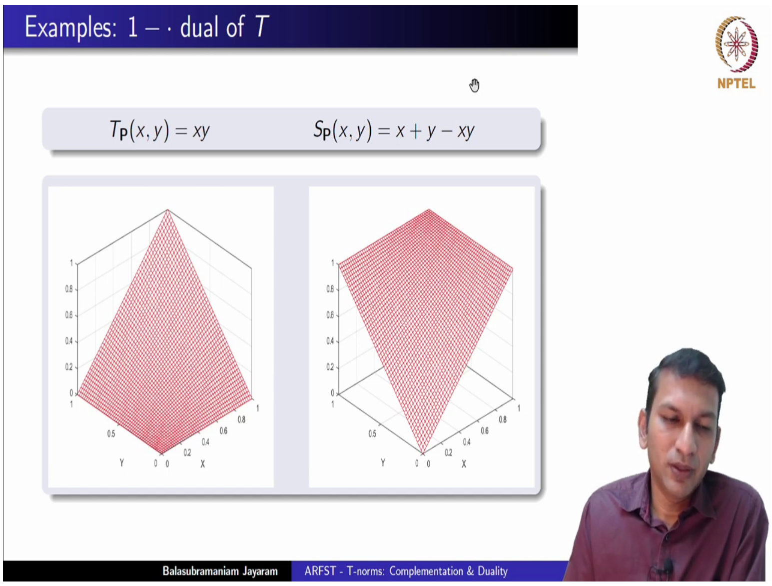

In the case of product this is the corresponding duality conorm x plus y minus x y.

The graph of product is on the left and almost an inverted dual of that is the T-conorm which is dual of the product T-norm.

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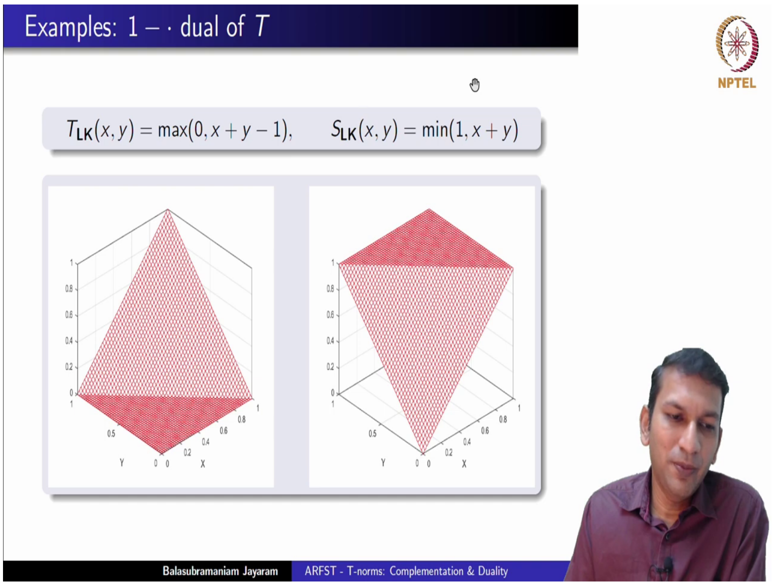

For the Lukasiewicz T-norm this is the corresponding Lukasiewicz T-norm.

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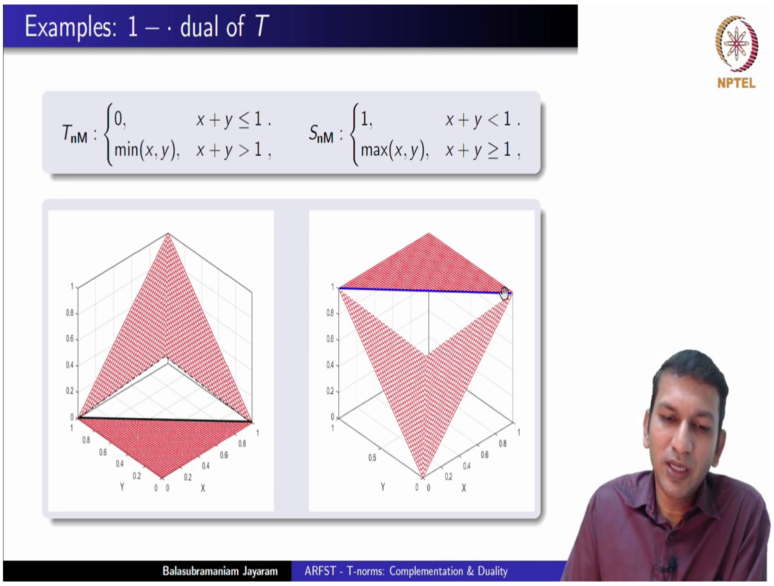

What is interesting is for the nilpotent minimum, you see here with an nilpotent minimum the boundary of the 0 region belongs to itself. So, this is in that sense for every increasing sequence the limit will belong to it. So, it is a left continuous T-norm. If you look at the dual

1 minus x dual of it this is what you have this blue is the boundary of the region where it takes the value 1 and it is clear these ragged edges do not belong to this region; that means, the boundary does not belong here, which means for a decreasing sequences it is true and hence it is a right continuous T-conorm, the dual of left continuity is we will apply here.

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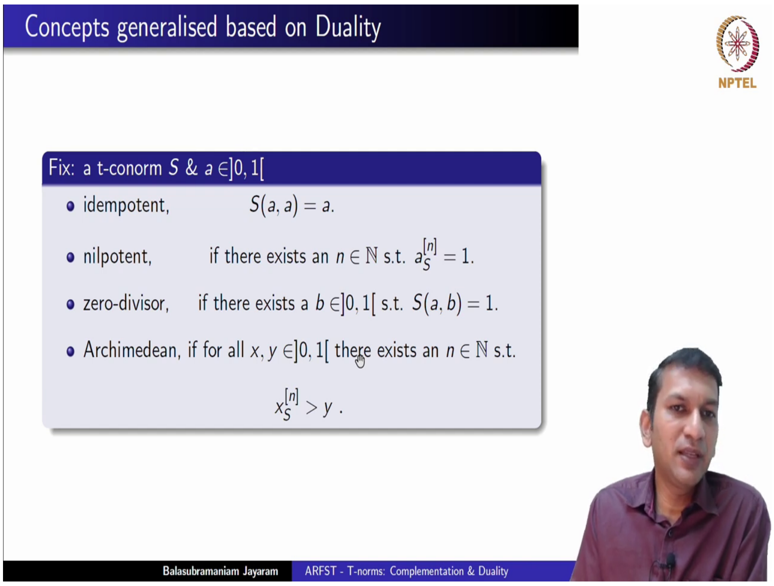

Finally, all the concepts that we have discussed can be generalized based on duality. For example, you take an element a and affix a T-conorm as idempotence remains as the same nilpotence means if you take an a and operate it on itself. So, finately many times with respect to S it should reach 1, zero divisor means here an a should have a b such that S of a b is 1. This might sound like a misnomer 0 divisible it is going to 1.

Remember 1 is the annihilator for the T-conorm. So, in that sense this is the 0 element and so this is a zero divisor. Finally, Arhimedean once again if you take 2 elements and apply a T-conorm it is going to increase bigger becomes bigger than the Arhimedean that you consider, which means Arhimedean is also defined accordingly. You have taken x for some n it should be such that if you operate x on itself n times with respect to S then that should be bigger than y.

So, similarly almost every concept can be generalized based on duality. Once again we will look at specific properties when the context works.

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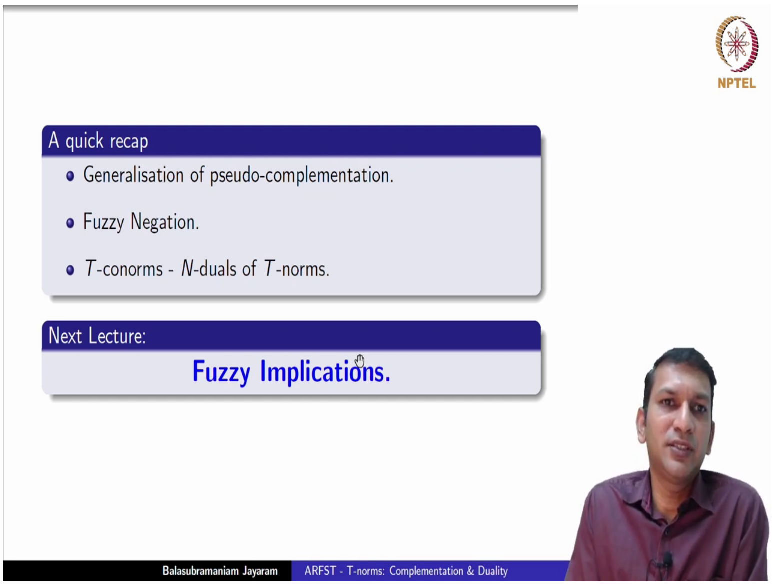

Finally, in this lecture what we have seen is a generalization of a pseudo complementation, which is what we call the natural negation of a T-norm from a T-norm. So, its natural negation of T-norm from there we generalized what could be a fuzzy negation.

Just the boundary and the monotonic conditions were imposed on it, other properties were desirable. Finally, we looked at one particular generalization of fuzzy disjunction which we call the T-conorms and we saw them as N duals of T-norms. What next? So, in this week we have discussed only about conjunctions, largely about conjunctions and the generalization and all the algebraic analytic aspects related to them.

We have also seen about fuzzy negation and T-conorms, which are generalization of complementation and disjunction. Next, we look at fuzzy implications which are implications which are one of the most important operations from the point of view of logic and hence from the point of view of reasoning. In the next week we will look at fuzzy implications almost along the same lines as we have seen T-norms.

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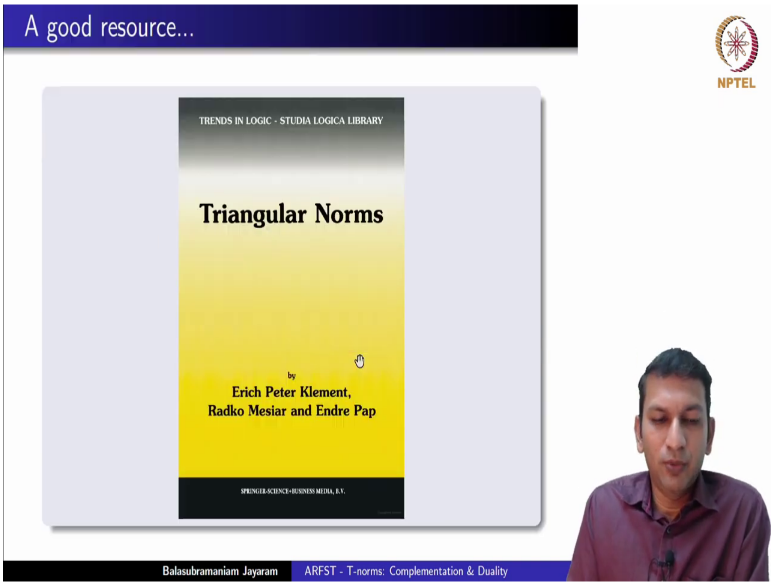

Once again for the topics covered in this lecture a very good resource it is the book by Klement Mesiar Pap titled Triangular Norms.

Thank you once again for joining me in this lecture and hope to see you soon in the next lecture.

Thank you.