Approximate Reasoning using Fuzzy Set Theory
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Lecture - 12
Triangular Norms: Analytical Aspects

Hello and welcome to the second of the lectures in this week under the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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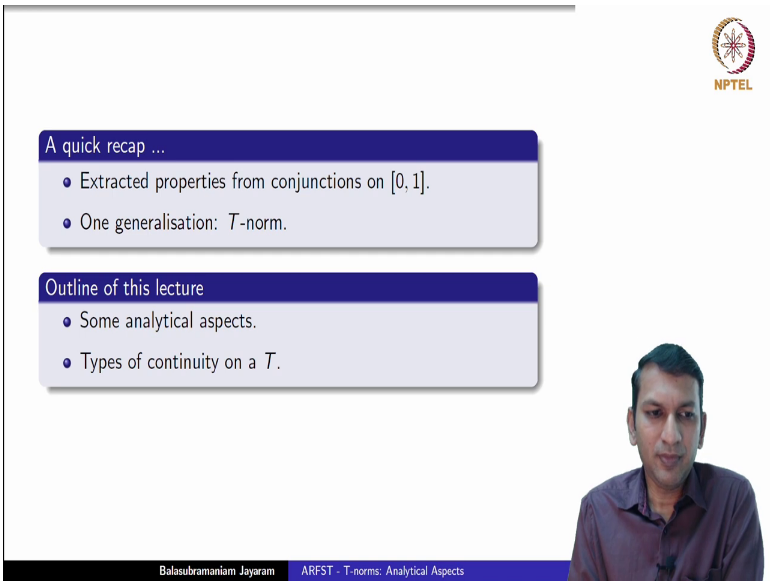

In the last lecture we looked at extracting the properties of conjunctions that we would like to have on the unit interval. And gave an axiomatic definition of one particular generalization of classical conjunction to fuzzy set theory vis a vis the triangular norm. In this lecture we will look at some analytical aspects of these triangular norms or T-norms for short. In fact, we will discuss different types of continuity that you can define with respect to a T-norm.

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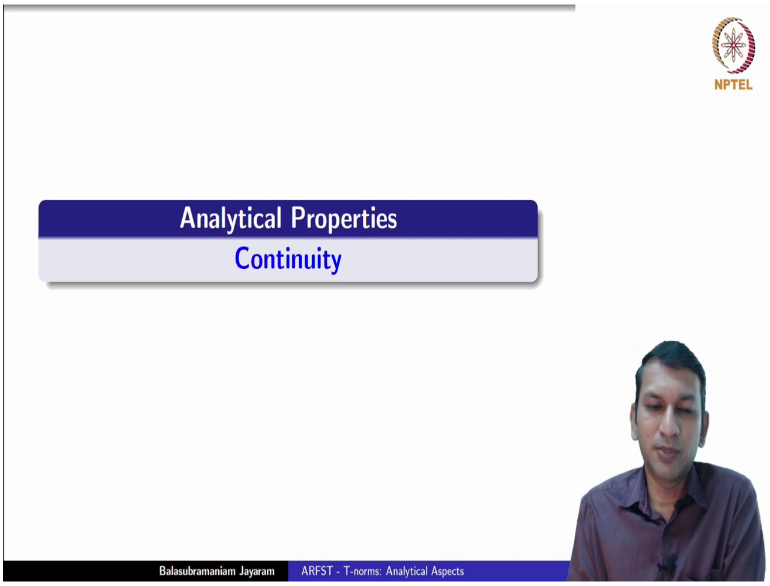

Let us begin with the usual definition of continuity the usual concept that we are all familiar with. So, if I have a T-norm T, let us take two convergent sequences where the elements come from the unit interval [0, 1].

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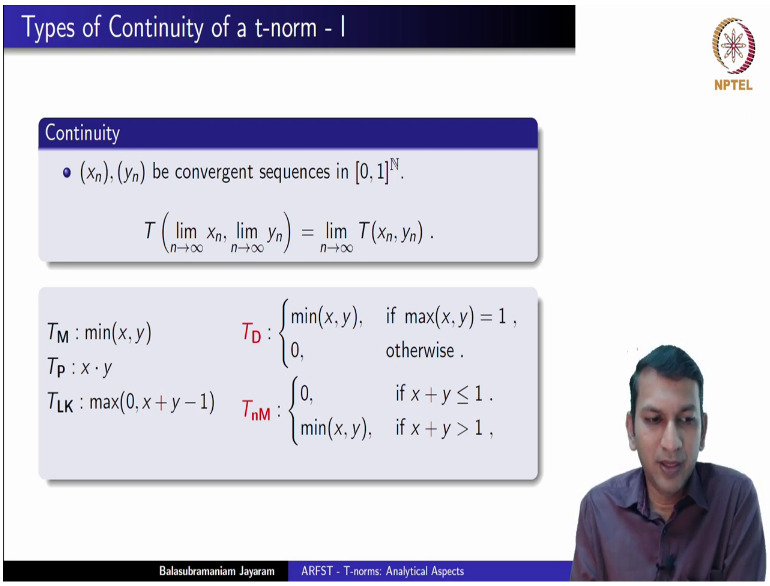

Now, by definition the usual definition of continuity we know that this is what should happen for any pair of convergent sequences; that means, we should be able to pull out the limiting process from inside the function and both these sides should be equal. Now, if I look if you look at the 5 T-norms that we have always considered; the minimum, product, Lukasiewicz, drastic, and nM we do not know what it stood for so far; it is called the Nilpotent minimum.

In the previous lecture we have seen that we can actually get it from the minimum by cutting out some parts of it and pushing it on to the floor on to the zero region. It is from there we obtain this nomenclature of nilpotent minimum, what is this nilpotence? We will see this in the next lecture when we discuss the algebraic aspects of T-norms.

For the moment we will continue to call this T-norm as nilpotent, minima, T-norm that is the abbreviation that we have given or the symbolism we have given as T_{nM}. Now, we have seen the graphs of these T-norms, we know the graph of a T-norm it is actually embedded within the unit cube. Now, this is the graph of the minimum T-norm we see clearly it is continuous.

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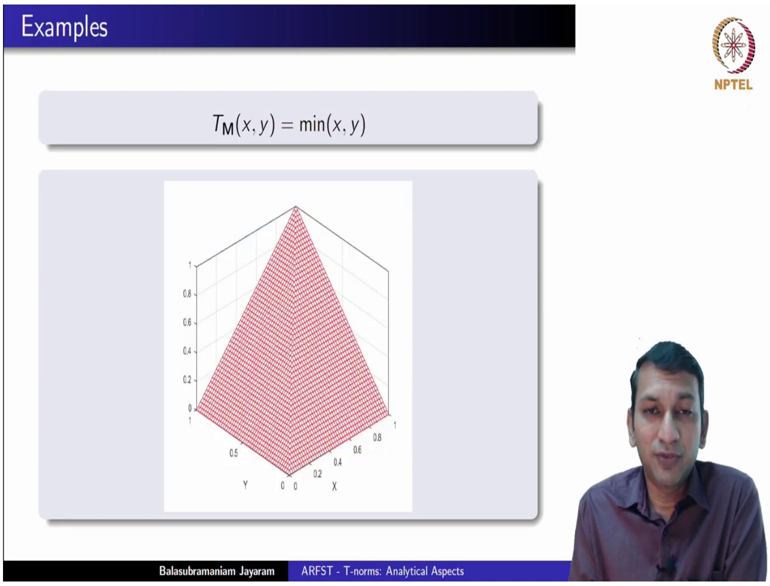

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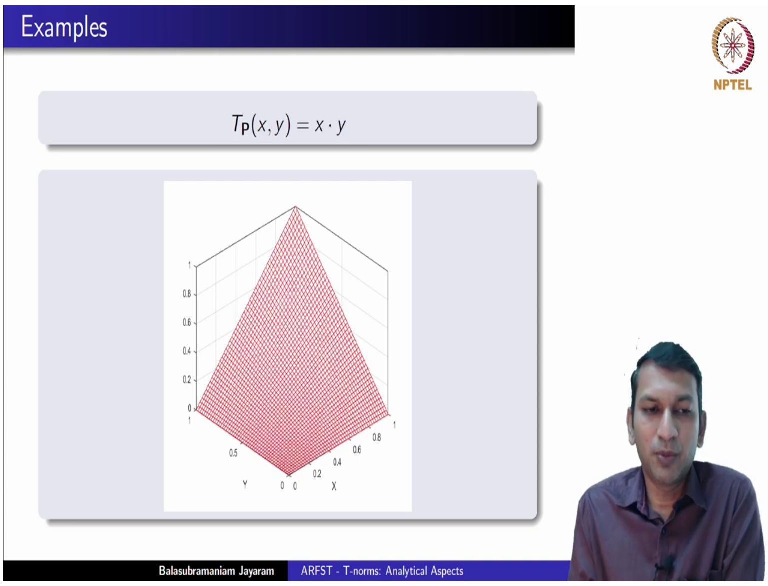

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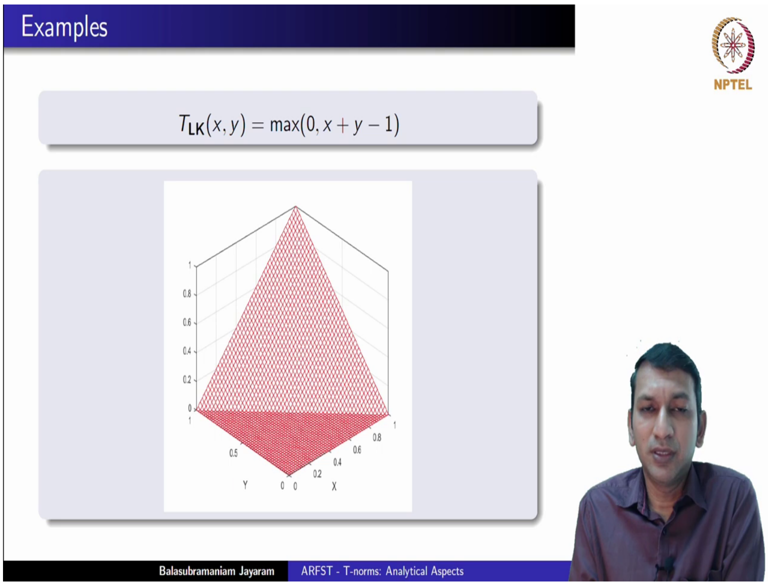

Similarly, the graph of the product T-norm and graph of the Lukasiewicz T-norm, clearly all of these are continuous.

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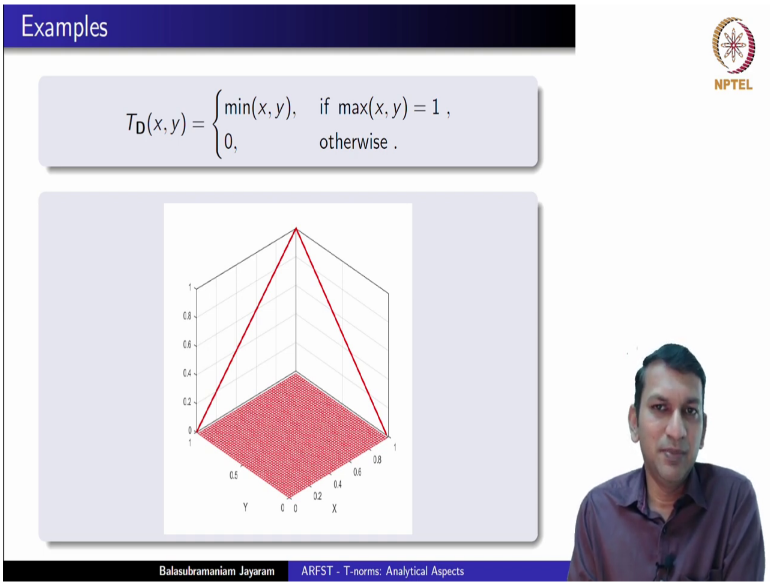

Now, if we consider the drastic T-norm, it is immediately clear it is not continuous.

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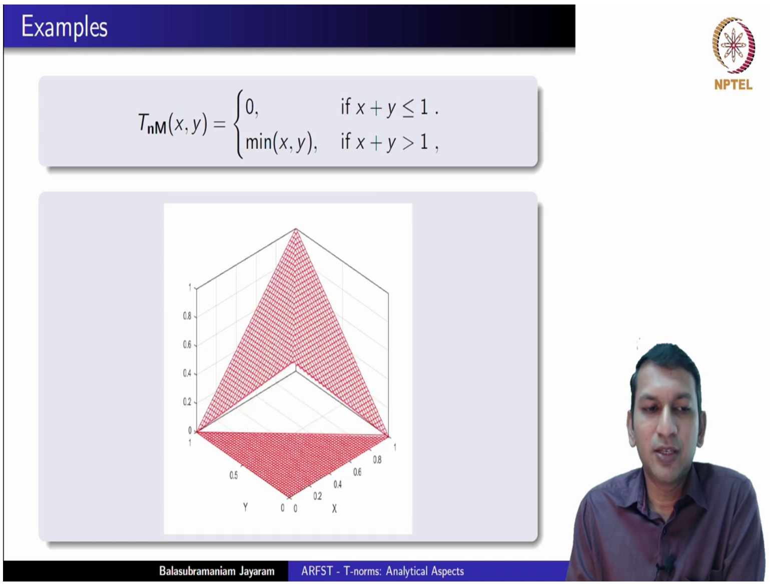

And so, is the case with nilpotent minimum T-norm; however, even when these are not continuous, they may satisfy some weaker forms of continuity.

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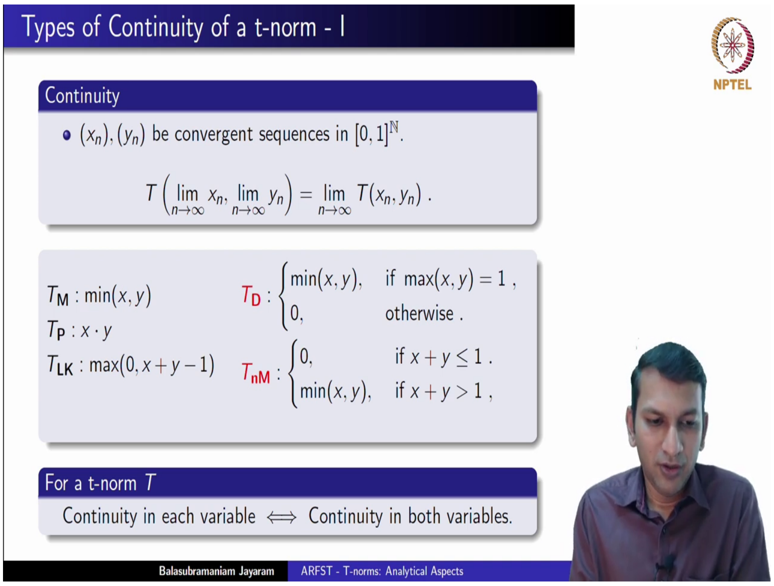

Before going to that we will look at an interesting result that we have on T-norms themselves. So, we saw that this is what we define as a continuity of the T-norm, the usual definition and we have seen these 5 examples among which drastic and nilpotent minimum are actually not continuous. What is interesting is, when it is a T-norm continuity in each of the variables is equivalent to having continuity in both the variables; now, this is not normal in the case of real functions.

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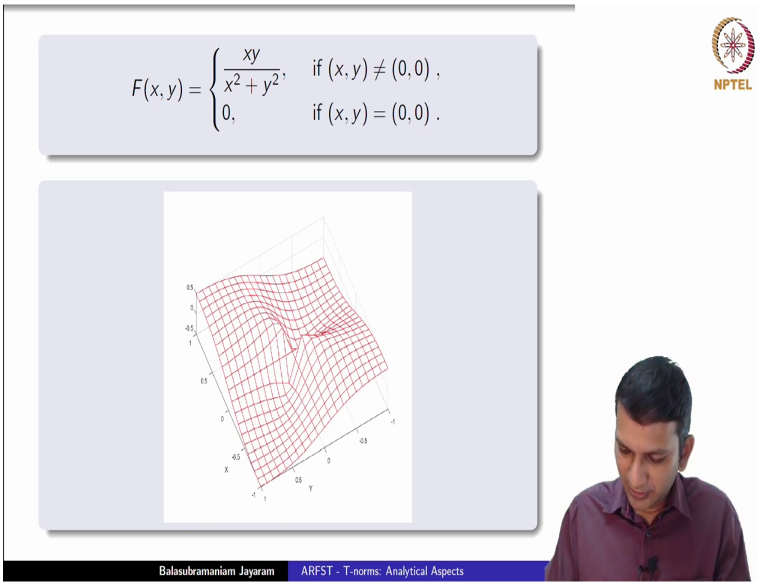

For instance, consider this function F which is from R square to R, let us define this as xy/x^2+y^2.

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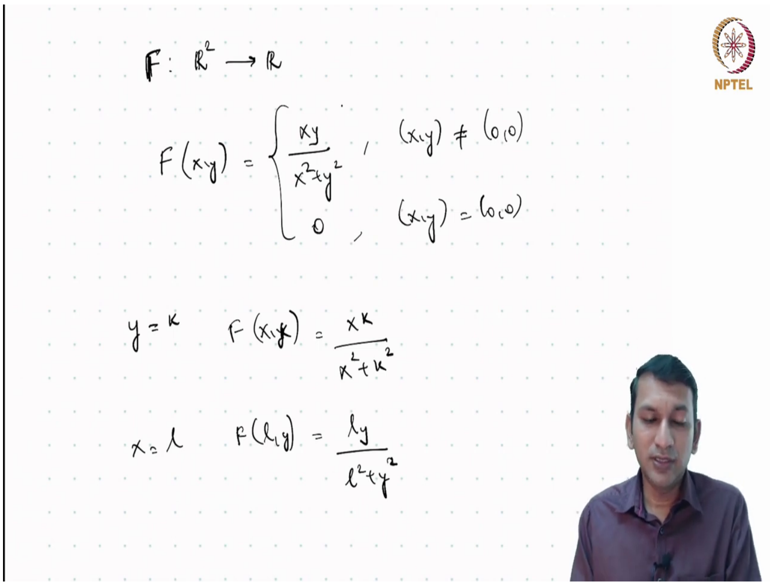

Clearly, we know that at 0 it is not defined; so, let us define this only at points away from the origin it is not the origin and at the origin let us define it as 0. Now, if you plot this the graph of this function this is how it looks like. Now, it can immediately be verified that if you take this function let us rewrite this here just 0. Now, we see that if you want to travel along the line y is equal to k; that means, if you want to travel along the x axis parallel to the x axis.

We see that if you substitute this is essentially $x^k/x^2 + k^2$ and clearly at 0 it is 0 and this function is continuous. Similarly, if you take x is equal to l; that means, if you want to travel parallel to the y axis, we see that F(l,y) the term going to be 1 by $l^2 + y^2$. So; that means, individually if you travel along the along a direction parallel to x axis or y axis these are function this is a function is continuous. Now, why do I say is not continuous in both the variables?

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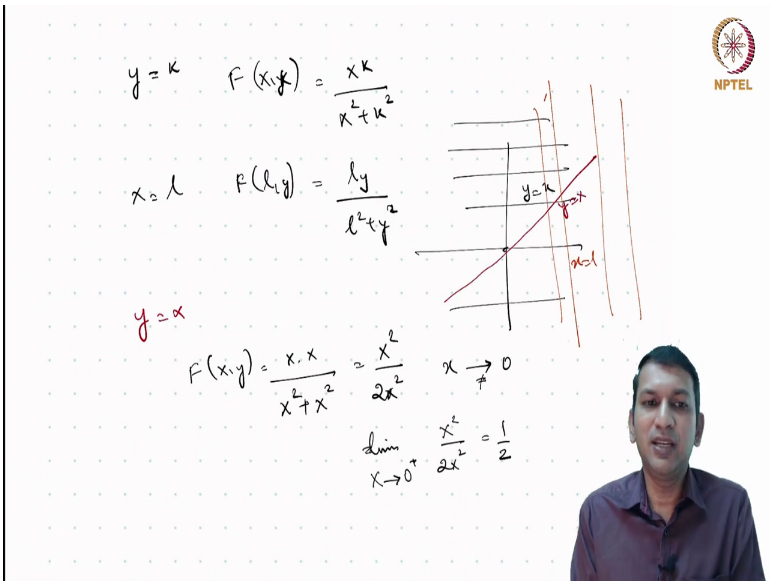

Now, essentially on R^2 and make a origin here, this is the y is equal to l. So, if you travel like this it is continuous it is x is equal to l line if you travel along these directions it is continuous. But, now let us travel along a different direction; that means, along the line y is equal to x.

Now, if you travel along the line y is equal to x what you will see is F(xy) is $x.x/x^2 + x^2$ which is $x^2/ 2.x^2$. Now, if x is only tending to 0 not really 0 and into, but not equal to 0. Then we actually can cancel this and say limit of x tending to be 0 plus or 0 minus x square by 2 x square is actually half.

But, as x becomes 0, then it is F of 0 because y is equal to x then we want the value 0. So; that means, as you go along the y is equal to x line the diagonal from either side 0 plus or 0 minus, you see that the function value remains a constant half, but suddenly at zero it drops to 0.

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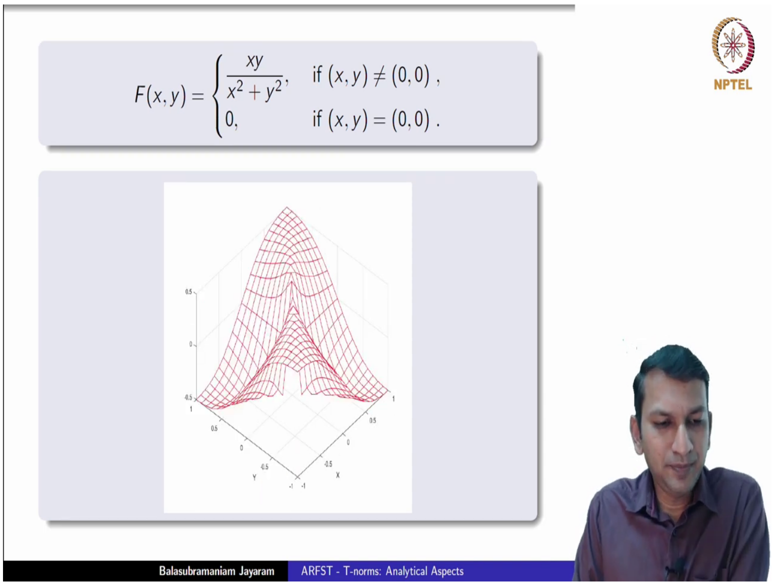

Now, this can be seen here look at this; so, minus 1 minus 1 is here 0 is here. So, as you go along the diagonal you see the graph is almost a constant here and suddenly it dips. So, when you come here it is a constant at half and suddenly it goes to 0; so, clearly this function while being continuous in each of the variables it is not continuous in both the variables.

So, now, what helps us in getting this result about T-norms that if it is continuous in each of the variables then it is continuous in both the variables. It is simply because of the monotonicity of T-norms; it can be proven if you have a monotonic function from $[0, 1]^2$ to $[0, 1]$ binary function. If it is continuous in each of the variables, then it will also be continuous in both the variables.

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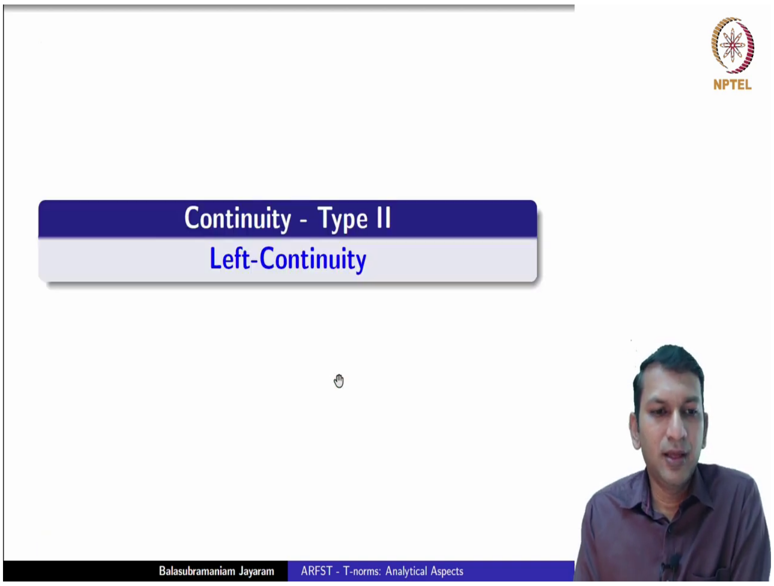

Let us look at the second type of continuity that of left continuity.

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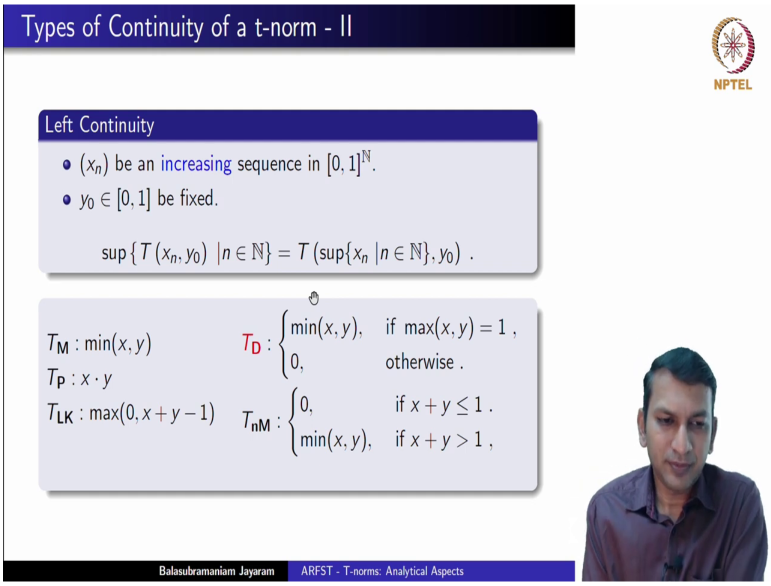

So, for this we take a sequence, but an increasing sequence; let us fix a point y_0, we say T is left continuous at the point y_0 if this equality holds. Once again let us try to decode this inequality.

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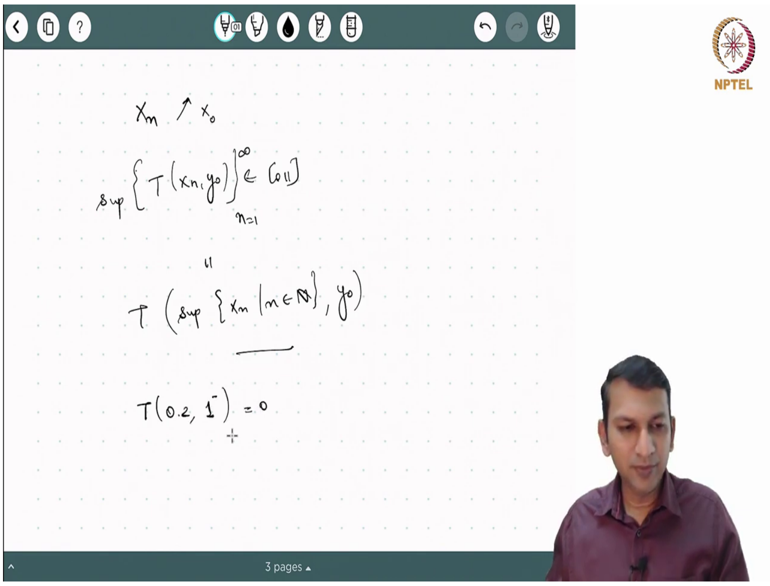

So, what do we have first, we have an increasing sequence let us say it converges to some x_0 and we are looking at T of x n, y_0; so, these are values in 0 1. Now, what we do is then we take all these M is equal to 1 to infinity and take the supremum, remember these are values in the 0 1 interval n is equal to 1 to infinity these are the values you are going to take.

These are values in the unit interval 0 1 which as we know is a complete lattice which means supremum of any subset of 0 1 will exist. So, the left-hand side of this equality exists. Now, look at the right-hand side what is it that we are looking at? Looking at supremum of x n essentially and these are values from the 0 1 interval which means again supremum will exist. Now, we take T of the supremum, y_0, we want that these two should be equal; so, this is left continuity for you.

Now, this is at the level of an equation, but how do we understand it geometrically. So, that let us take the same set of five T-norms, we can see that four of these T-norms are left continuous clearly every continuous T-norm is also left continuous. But, now the nilpotent minimum which was not a continuous T-norm, now has suddenly become left continuous; whereas, drastic still continues to be non-left continuous.

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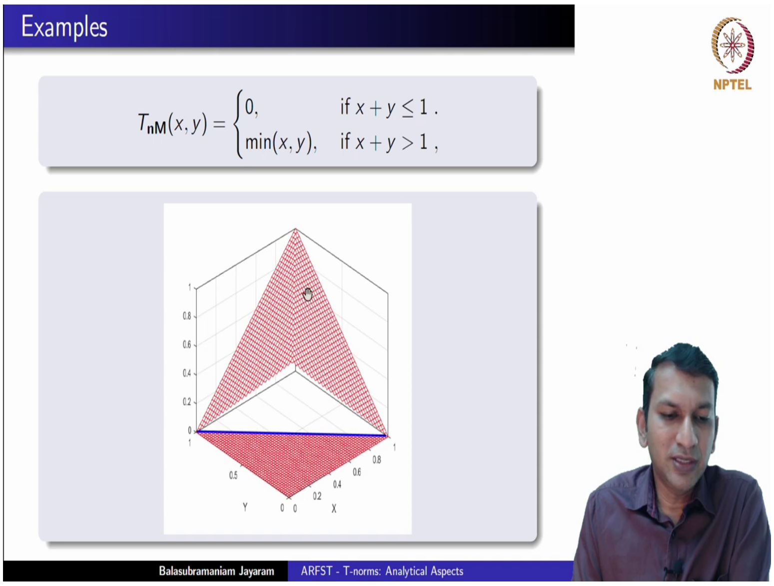

Look at this, this is the graph of the nilpotent minimum. And we see here that for any increasing sequence that we take the supremum the value at T of x, y_0 is also the, is the converge the convergence limit of the sequence that you could consider as T of x 1, y_0. So, essentially it says as you go along the y axis parallel to the y axis or parallel to the x axis, the boundary of any place, where you stop that should actually belong to you.

So, now, you see here for this particular T-norm whose graph we are seeing on the screen, if you notice when you come from the right that is from the with the decreasing sequence from say from one. You see that this is a kind of a jagged or rugged edge which means it does not belong the boundary does not belong boundary belongs to the flow.

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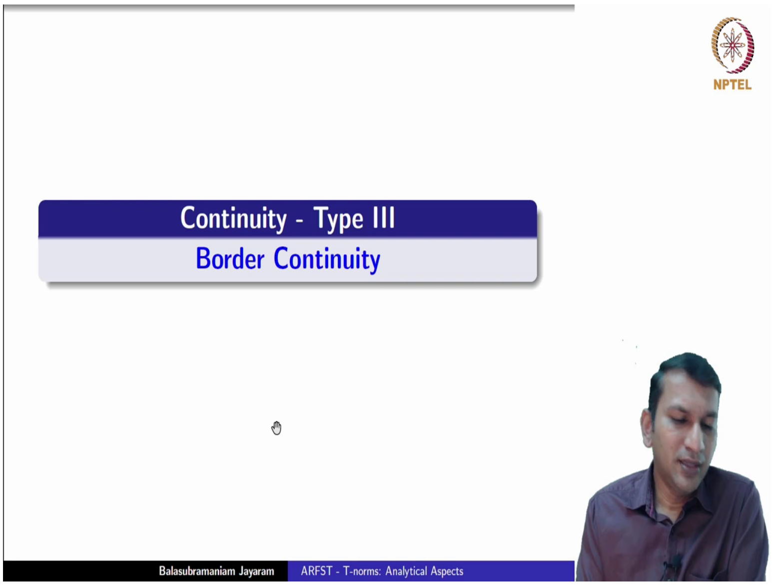

So, this is left continuity for you in terms of geometry. There is a third type of continuity that we can define for a T-norm which is called water continuity, what is this?

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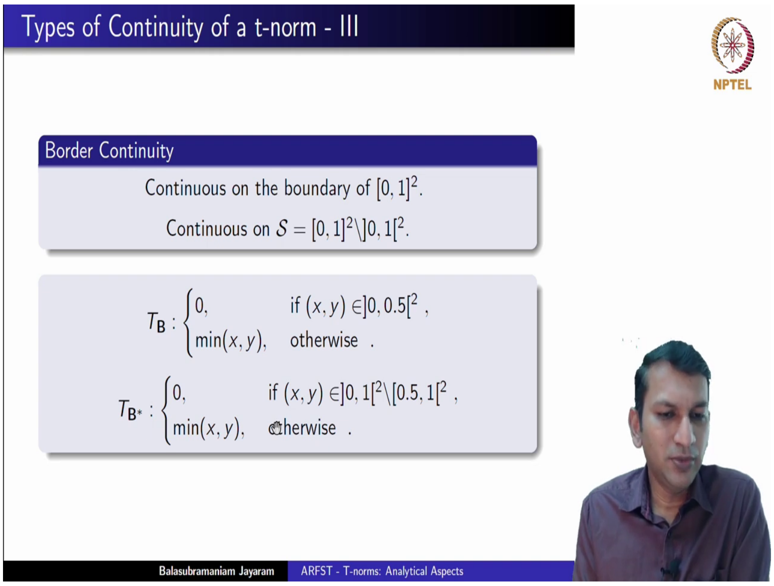

We expect the T-norm to be continuous on the boundary of 0 1 square. What is the boundary of 0 1 square? The set S where essentially only the edges of the unit square; so, we remove the open unit square from the total unit square. Let us look at some examples these are two T-norms that we can think off.

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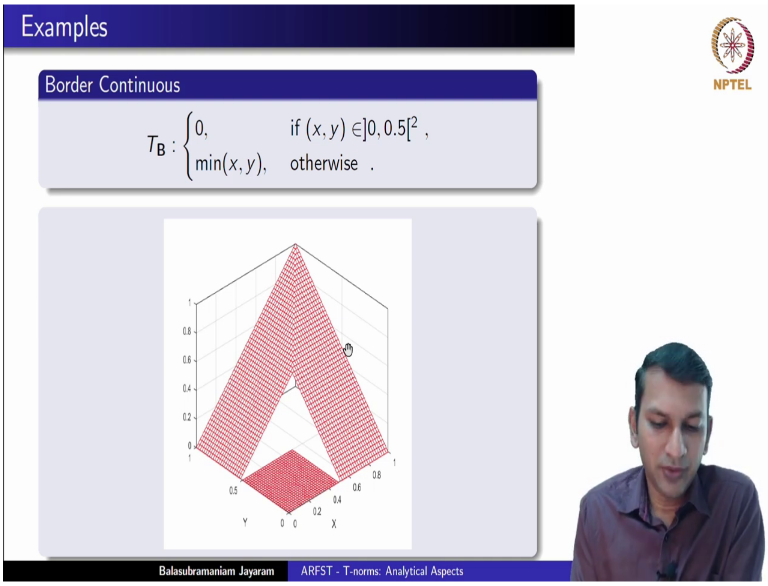

If you look at this T nom this is the graph of this T-norm, you see here it is 0 on the open 0 open 0.5 square. So; that means, as you go from if you take an increasing sequence from here 0.5 does not belong to this; the supremum of the sequence will be 0.5, but 0.5 does not belong to this region. So, at as you move along it is 0 and at 0, 0.5, it becomes at 0, point; so as you move along from here.

So, let us take at 0.2, as you move along till 0.5 it is 0 at 0.2, 0.5 it becomes 0.2; so, this is not a left continuous T-norm.; However, it is a water continuous T-norm what we mean by that, at the boundary when y is equal to 1 or x is equal to 1, it is continuous. So, if you take any sequence that converges to 1. We want that it is in some sense left continuous at the boundary. Because you can only consider increasing sequences lying within the unit square to converge towards one; so, this is a T-norm which is not left continuous, but water continuous.

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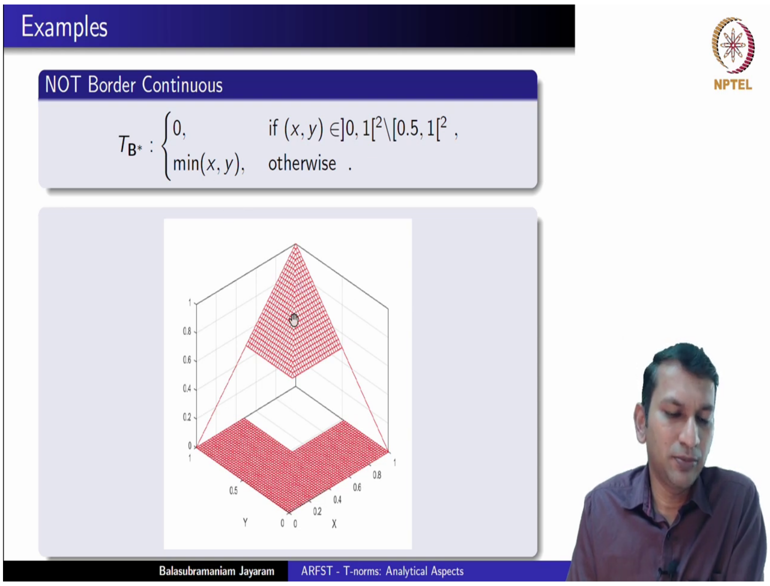

Now, look at this one most of him much like the drastic, T-norm on these parts is 0 and on this part is actually the minimum norm. Now, if you it is once again clear if you take x to be 0.2 and travel along a sequence that goes towards 1, you see that 0.2 p of 0.2, 1 minus if you were to write it like this T of 0.2, 1 minus then we see that it is 0.

Whereas at one; obviously, T of 0.2 1 has to be 0.2, because we know that one is the identity. So, clearly it is not left continuous at one and this is the concept that we talk of as the border continuity; so, this is a T-norm which is neither left continuous nor border continuous.

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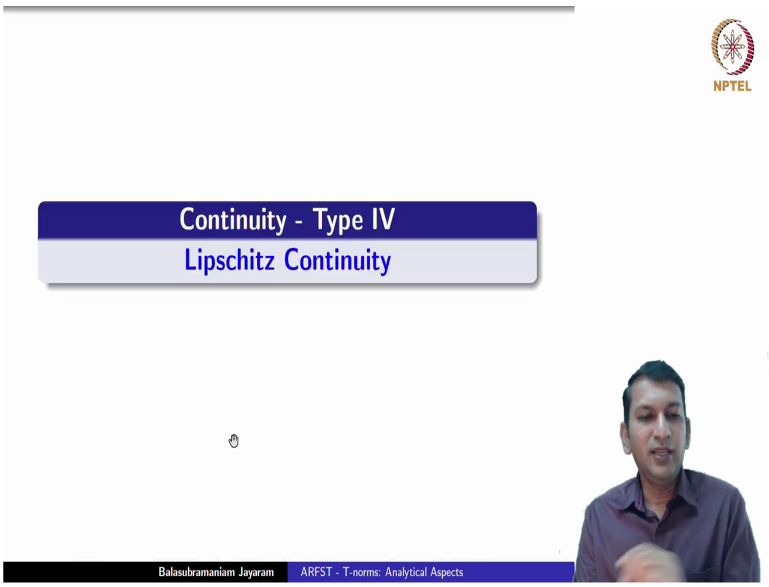

Finally, let us look at one other type of continuity which is of course, discussed normally for real value functions also that are Lipschitz continuity. How do we define Lipschitz continuity in the case of a T-norm? This is how we define it.

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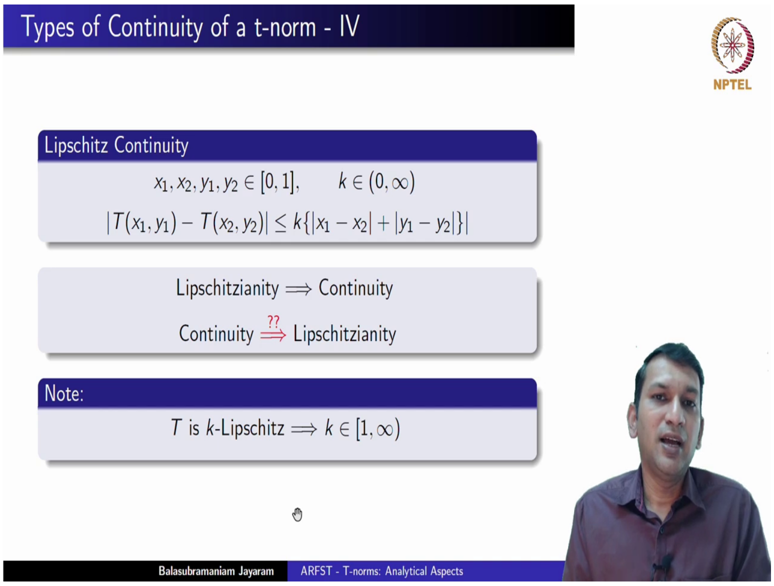

Let us take four points x 1 x 2 and y 1 y 2from unit interval 0 1 and k which will become the Lipschitz constant from anywhere between 0 and infinity. We say T is Lipschitz continuous the T-norm T is Lipschitz continuous, if mod T x 1, y 1 minus T x 2, y 2 is less than or equal to k times mod x 1 minus x 2 plus mod y 1 minus y 2. Now, this is what we define as the Lipschitz continuity equation. So, any T-norm if it satisfies this inequality for a particular k, then we know that we call the T-norm Lipschitz continuous.

Now, as is normal also in real analysis, any Lipschitz continuous function is also continuous the way that we understand between two sequences convergent sequences and interchanging the limits. However, in the case of real functions we know that continuity does not imply Lipschitzianity.

Now, is it going to be different in the case of T-norms well, why this why does this question arise? Because, we have seen that in the case of continuity in real valued functions normal function that you find in real analysis. That continuity in each of the variables need not necessarily mean continuity in both the variables; however, for T-norms because they are monotonic in nature it was the case.

So, now, you suspect if continuity would always imply Lipschitzianity, let us explore this in a little more detail. The first thing that you should notice what could be the constant k? It can be easily seen that if T is k Lipschitz then the constant k has to be greater than or equal to 1. This can be seen quite easily we have to look at it; so, this equation for Lipschitzianity is this k times mod x 1 x 2 (Refer Time: 17:20).

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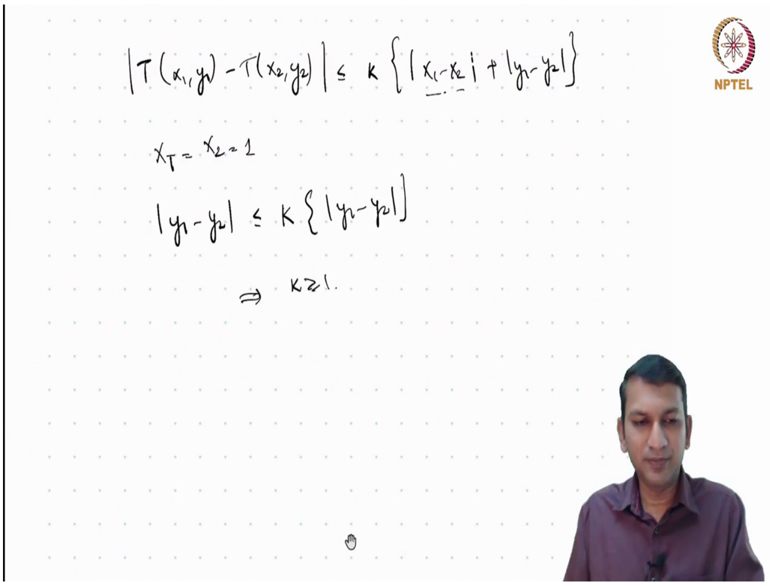

Now, let us take x 1 and x 2 to be 1 then what we know is by since 1 is identity for T-norm, what we have is on the left-hand side? We have mod y 1 minus y 2 this should be less than k times x 1 minus x 2 is 0 mod y 1 minus y 2. So, from here it clearly shows that a has to be greater than or equal to 1; so, that is the first thing that we see. The next question is, is every continuous T-norm also Lipschitz continuous well, let us look at this particular T-norm.

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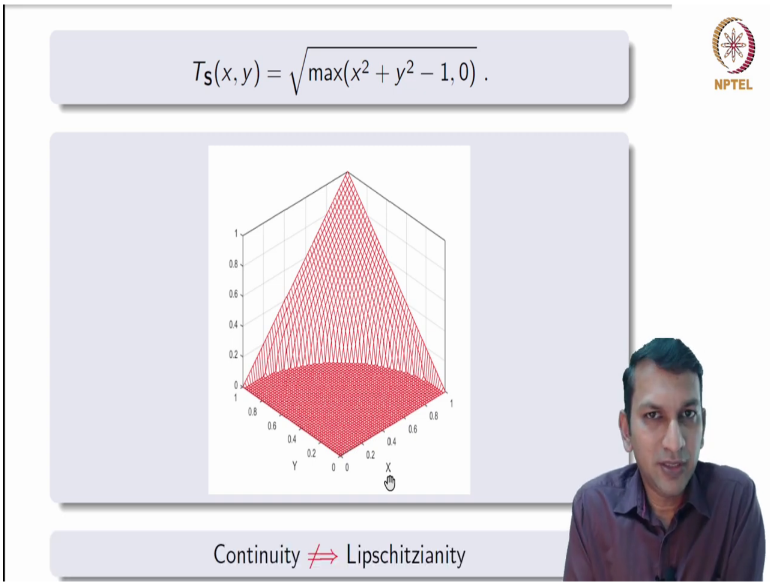

How does it look like? It looks like this; so, definitely it is a continuous T-norm, but let us explore its Lipschitzianity to discuss its Lipschitzianity means to discuss this equation.

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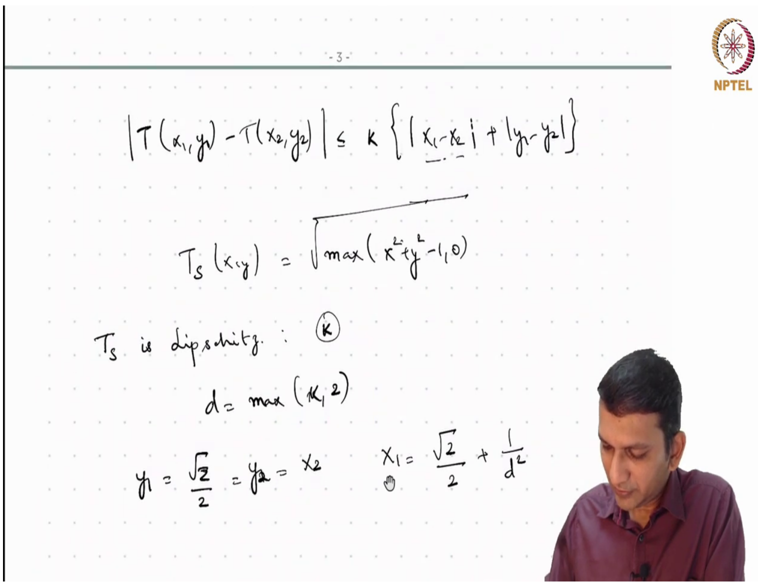

Let us consider the point y 1 is equal to root 2 by 2 and let this be also equal to y 2 and x 2, let us look at x 1. So, to begin with let us assume that we have the T s is Lipschitz what is T s; so, you see here T s this is root of maximum of x square plus y square minus 1, 0. So, this is let us assume that T s is Lipschitz; that means, they should exist a constant k with respect to which it is Lipschitz continues.

So, now, let us define d to be maximum of k, 1, note that k itself is actually greater, than 1 greater than or equal to 1. So, we assume we take d to be either k or if k smaller than 2, then it is 1; so, at least k d is at least 2. Now, consider these values y 1 is root 2 by 2 and it is also equal to y 2 and it is also equal to x 2. For x 2, let us consider this 1 root 2 by 2 plus 1 by Dd square; where, d is given is given; now, we want to substitute this in this equation.

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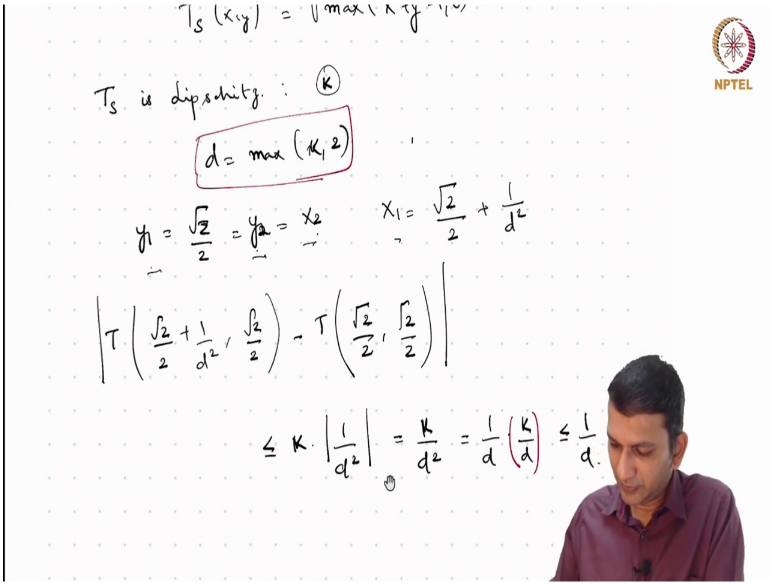

Now, T of x 1, y 1 that is root 2 by 2 plus 1 by d square root 2 by 2 minus T of root 2 by 2, root 2 by 2. Now, we know that if it is Lipschitz continuous it should be less than or equal to k times mod y 1 minus y 2 plus x 1 minus x 2, but y 1 and y 2 are same which means that is 0, x 1 minus x 2 here would be 1 by d square mod, but this is essentially k by d square.

Now, we can write this as 1 by d into k by d, note that by our assumption or by a choice of T d is definitely greater than k which means this quantity is actually less than 1; that means, this is going to be smaller than 1 by d. So, now, what we have seen is the left-hand side we have seen is actually going to be less than or equal to 1 by d.

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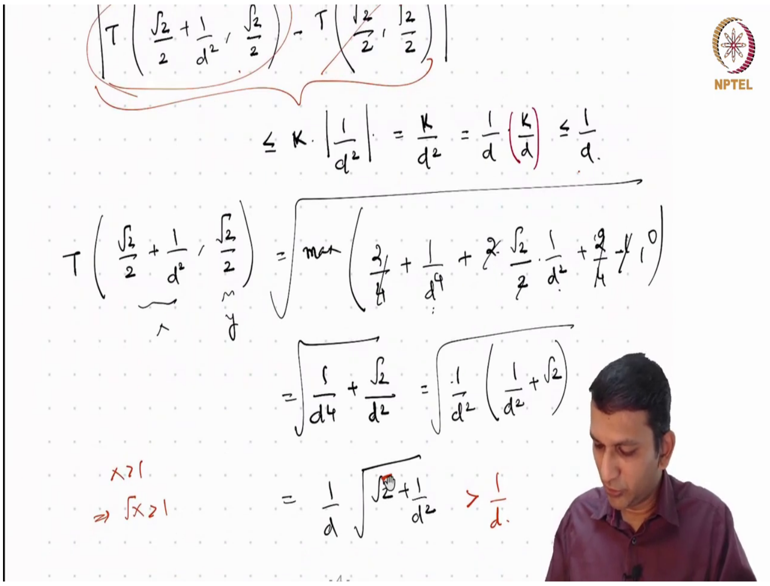

But, let us now consider what each of these terms is going to turn out to be, now T of root 2 by 2 plus 1 by d square, root 2 by 2; now, the formula is root of max of x square plus y square minus 1, 0. So, now first let us look at taking maximum of x square this is our x, this is our y x square; so, that is 2 by 4 plus 1 by d by power 4 plus 2 times d 2 which is 2 into root 2 by 2 into 1 by d square.

So, that is x square plus y square which is 2 by 4 minus 1, 0; now, 2 by 4 is 1 by 2. So, 1 by 2 1 by 2 and minus 1 if you cancel you get 0 and note that d is greater than 0. So, 1 by d power 4 is positive 1 by d square is positive into root 2 it is positive; so, this will turn out to be 1 by d by 4 plus root 2 by d square; now, under root this. So, now, if you write this you could write it as 1 by d square into 1 by d square plus root 2 under root of that and that is written 1 by d into under root of root 2 plus 1 by d square.

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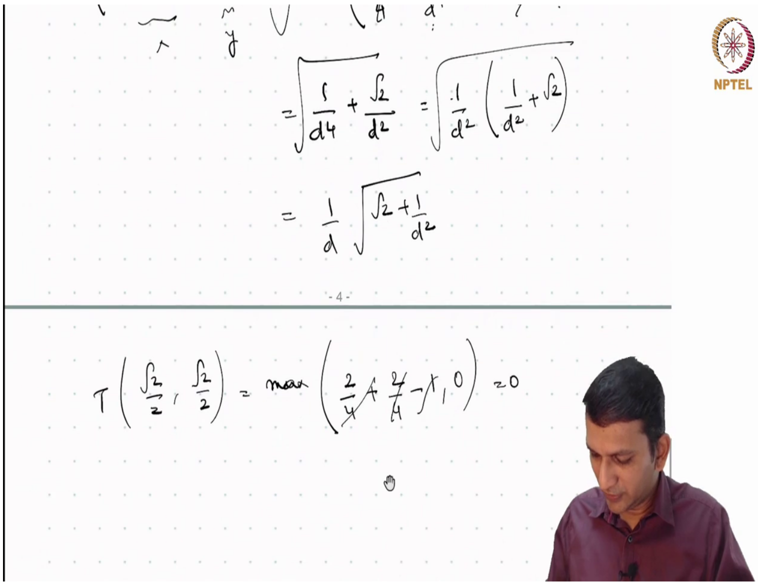

Now, what about T of root 2 by 2, root 2 by 2 and this is max of x square 2 by 4 plus y square minus 1, 0; now, this will be 0. So, now, if you actually look at LHS which is this part, we know that this is 0 and this is the quantity that we are looking at.

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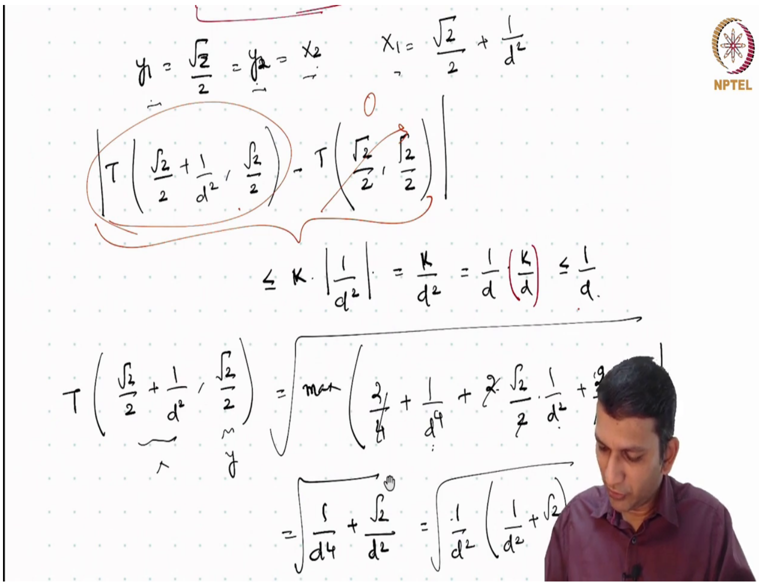

But we assume that this quantity if it were a Lipschitz is less than or equal to 1 by 2, but now what is it that we have found out that this quantity is actually equal to this quantity. Now, 1 by d square is positive due to this 1.4 and 4 if you take an approximation; so, this is greater than 1. We know that if x is greater than 1 then root x is also greater than 1 so; that means, this is strictly greater than 1 by root 2 ok. So, from the definition of T we get that this quantity is going to be greater than or equal to 1 by d.

However, if it was Lipschitz with some k, we find that has to be less than or equal to 1 by D which clearly is a contradiction. So; that means, for this T-norm which we as we clearly see is continuous it is not Lipschitz 0. So, we see that continuity does not imply Lipschitzianity even in the case of T-norms much like usual real value functions.

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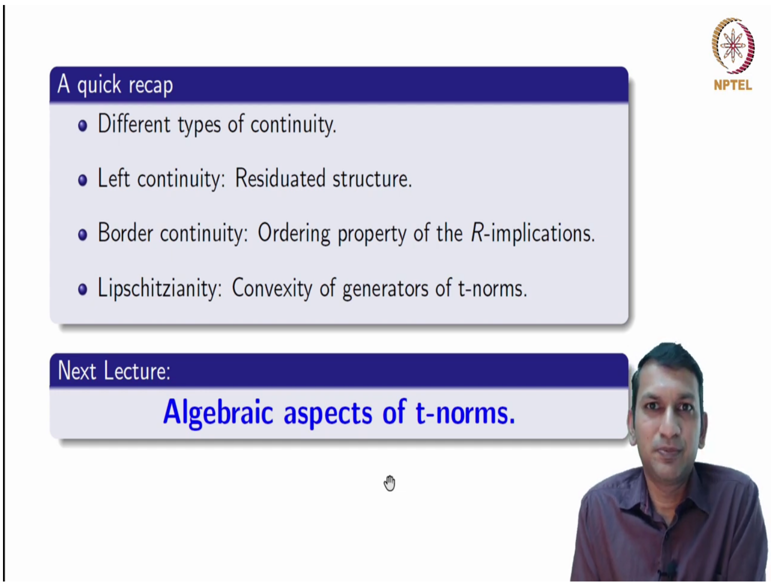

A quick recap of what we have seen in this lecture we have been introduced to different types of continuity that you can define about T-norms. But why did we choose this? We understand the normal concept of continuity, but does anything warrant introduction of these different types of continuities and why these continuities. We will see that the left continuity of a T-norm is very very important to lead us to a very important lattice in structure called the residuated structure.

The border continuity of a T-norm helps us in obtaining the ordering property of an implication that you can obtain from the T-norm which are termed as R implications, we will see this in the next week. And the Lipschitzianity of the T-norm is related to the convexity of generators of T-norms, once again we will see these moving forward.

What next? We have seen the analytical aspects of a T-norm essentially continuity and different types of continuity. In the next lecture we will look at the algebraic aspects of

T-norms both at the elemental level algebraic properties that an element with respect to a T-norm can have. And the algebraic properties that T-norms themselves can have or force on the 0 1 interval. Thanks once again for joining me in this lecture and hope to see you soon again in the next lecture.

Thank you.