Approximate Reasoning using Fuzzy Set Theory
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Lecture - 11
Triangular Norms

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Hello and welcome to the first of the lectures in this 3rd week, under the course titled Approximate Reasoning using Fuzzy Set Theory. A course offered and over the NPTEL platform.

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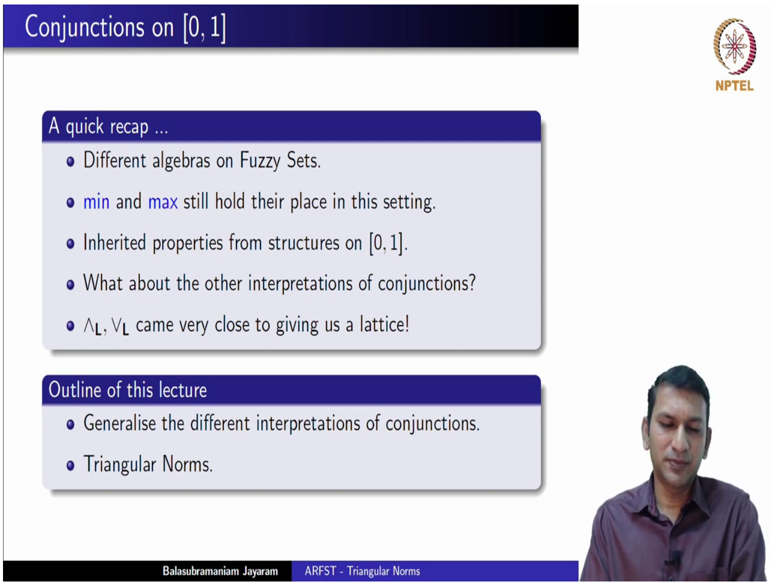

In the lectures during the last week, we have seen that we could introduce a few algebras on the set of fuzzy sets. We saw that min and max this combination of operations they played an important role.

Also, we should remember these were possible because of the properties, that we could inherit from the structures that were already available on the 0, 1 interval. However, we were left with the question as to what about the other interpretations of conjunctions, are they still useful? Note that the Lukasiewicz pair of operations they came very close to giving us a lattice.

In this lecture we will look at extracting the properties that are required of a fuzzy conjunction towards giving an axiomatic definition of what a one generalisation of fuzzy conjunction at least should be. This particular generalisation is called a triangular norm, it is a very well studied and most often used in the fuzzy logic community. And in this lecture we will look at triangular norms, their definitions some examples and some geometric perspectives.

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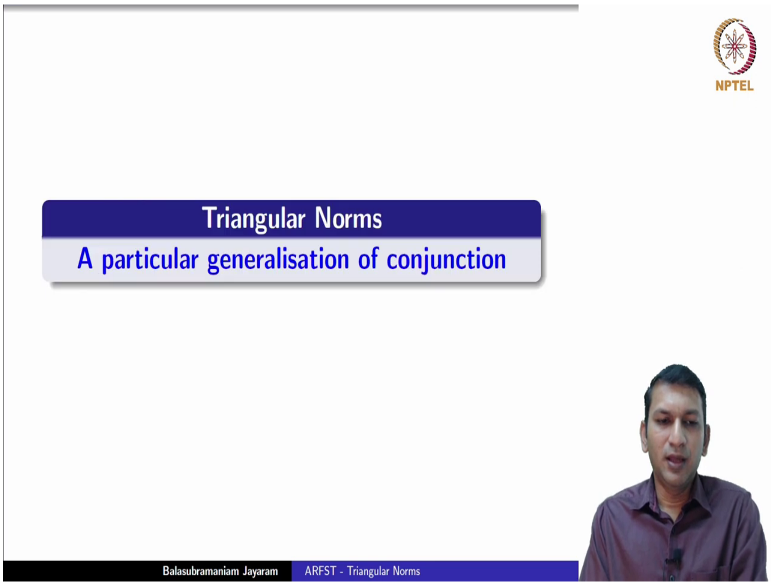

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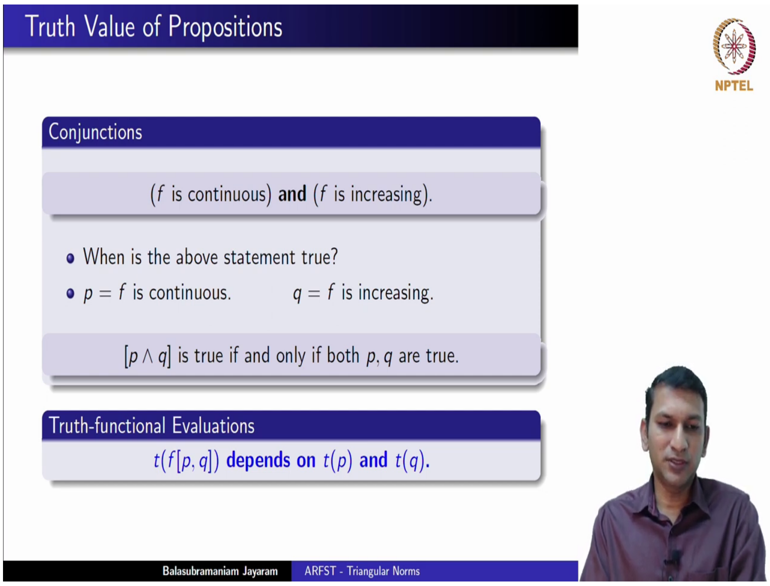

Towards this end we have seen that given two propositions p and q, a compound proposition could be made from them like this, f is continuous and f is increasing. Now, the question is when is the above statement true this compound statement which is a conjunction of two propositions p and q, when is this true. We have seen that this is true if and only if both the component propositions are true.

Now, how is this evaluation justified or valid? This is because we have this concept of truth functional evaluation or this assumption, where we say that the truth value of a function of propositions essentially depends on the truth values of the component propositions.

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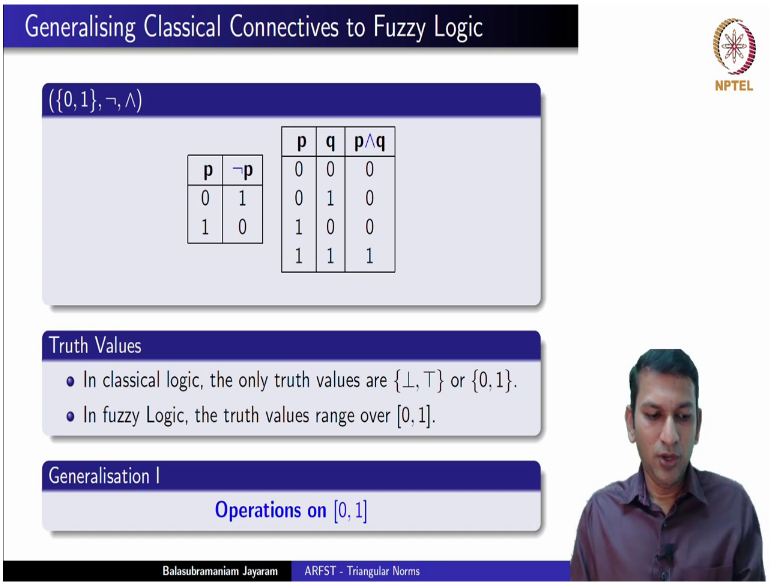

If you take the classical logic, where the truth values are just 0 and 1 and look at the corresponding connectives there.

So, negation it is the now familiar complementation. If you look at the truth table for the negation it is essentially converting 0 to 1 and inverting 1 to 0. Similarly, this is the corresponding truth table for the classical conjunction. Now we have seen that in fuzzy logic we are essentially extending this truth value set from just the 0, 1 to the entire interval 0, 1.

Now, we saw that if that were the case, then we could in propose operations on just the unit interval [0, 1], but having in mind that they should coincide in with the classical case when we are looking at the vertices. What do we mean by this?

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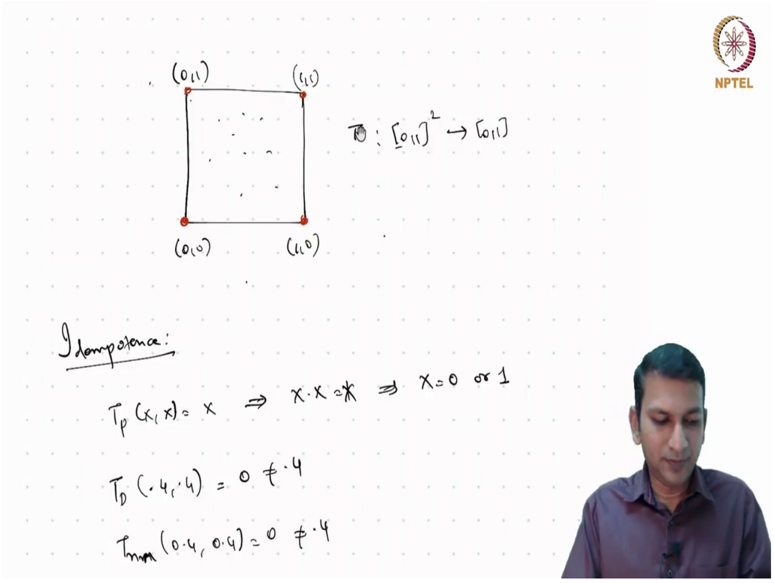

If you look at the unit square, the classical connectives, the classical connectives they only specify the values at these end points. The classical binary connectives and what we are interested in here is that we want to come up with a conjunction which soon we call it we will denote by the symbol T, which is not just from the set {0, 1} plus {0, 1} to {0, 1}, but from the entire interval [0, 1] to [0, 1].

That means, we need to generalize this to the entire domain to these points and also on the boundary in such a way that it coincides with the classical truth table for conjunction on the vertices.

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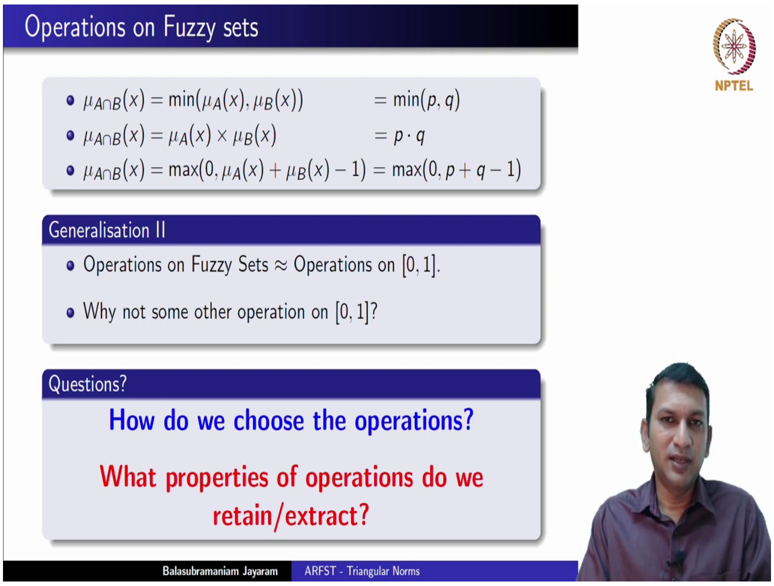

Now, we have seen that using this generalisation we have proposed a few operations as possible interpretations of the conjunction when it comes to dealing with pairs of fuzzy sets. We also ask the question why not other operations. So, this let us to first discuss this question it is how do we choose the operations and we have seen that, at least one way to look at them is looking at what are the theoretical structures that they could impose on the set of fuzzy sets themselves.

This is an approach that we have studied quite well in the series of lectures last week. Now, in this lecture we are going to look at on a other, but related question. What properties of operations do we retain or extract? This is not just limited only to conjunction, but this will be the question that we will deal with also with negations, implications and disjunctions. Now, let us examine this question a little deeper.

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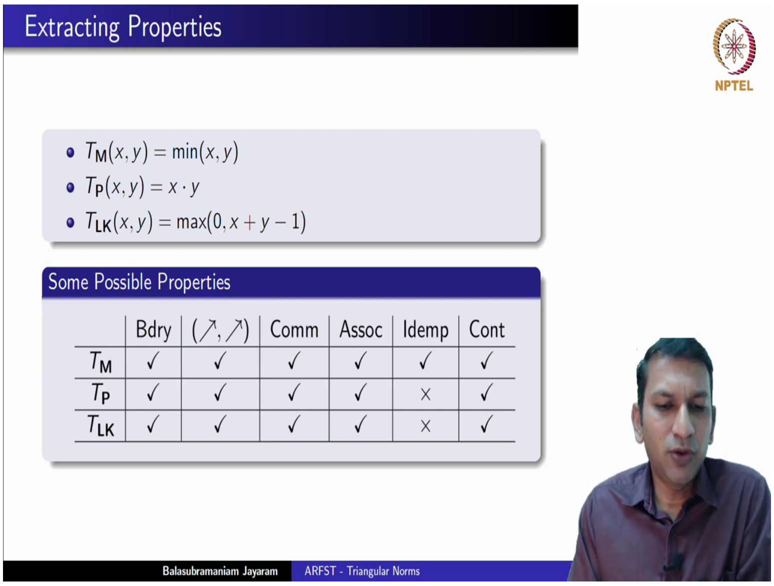

Now, these are the three operations that we are familiar with. Henceforth, we will use the symbol T, the nomenclature this the symbolism will become clear presently when we introduce triangular norms. Now, if you look at it, what properties do we want to extract what could be a good axiomatic definition for fuzzy conjunction? Towards helping us in that let us look at these 6 properties.

So, we need some boundary conditions to be satisfied that is clear and this is commutativity associativity idempotence continuity and increasing in the both these variables it is a binary function. So, by this symbol we indicate that it is increasing in both the variables.

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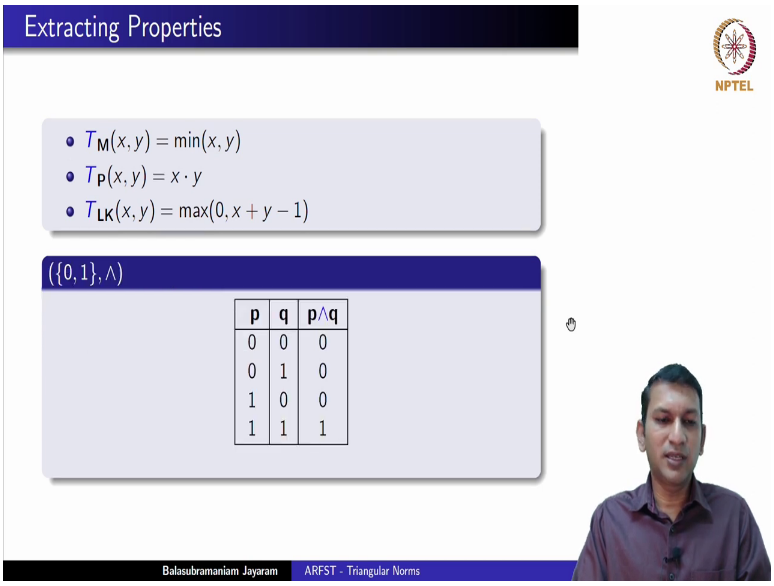

Now, how we justified in looking at it like this. Let us look at the corresponding classical truth table for this. So, the boundary conditions are clear. And we also see that it is commutative, we know that it is associative. What about idempotence? Yes, we have seen that also gives us leads us to a nice algebra extraction.

What about monotonicity? Increasing in the some both the variables. Now, let us look at this. Let us fix the first variable to be 1 and look at moving from 0 to 1. So, in this lattice we know 0 is smaller than 1. So, essentially when you fix the first variable and move from 0 to 1, we see that it is increasing.

So, perhaps it is not unjustified to ask for monotonicity. Now, why monotonicity in both the variables? Because it is commutative. So, if it is monotonic increasing here, then we want the same kind of monotonicity here also. So, these are the properties that we are looking at now. So, now, let us investigate each of these functions with respect to these properties. Now, it is clear that minimum satisfies the boundary property and if you fix y and increase x it is going to increase.

So, hence it is increasing in both the variables, it is also commutative and associative we know that min of x , x is x. So, it is idempotent and we know that it is also continuous; that means, it satisfies all these properties. Now, if you look at the product, we see that it is satisfying the boundary conditions, 1 times 0 is 0, 1 times 1 is 1 0 times 0 is 0. Once again it is increasing in both variables, commutative, associative it is continuous also. But what about idempotence?

So, now the idempotence we want the Tp (x) should be x, this implies x.x is x which implies x should either be 0 or 1. So, clearly, we know that it is not idempotent on (0, 1). This is what we have now. Consider the final conjunction which is the Lukasiewicz conjunction we know it satisfies the boundary condition if you fix y and increase x it is increasing. So, it is increasing in both variables, it is commutative associative we have seen this.

We have also know it is not idempotent and as a polynomial n x y it becomes continuous too. So, when we see these that, among the 6 properties that we have seen these three conjunctions, these three interpretations of conjunctions. They seem to satisfy almost all of them except for idempotence, but we should be little careful because when we are actually generalising and coming up with the properties we should ensure that we have a large catchment area of operation that we could consider for fuzzy conjunctions.

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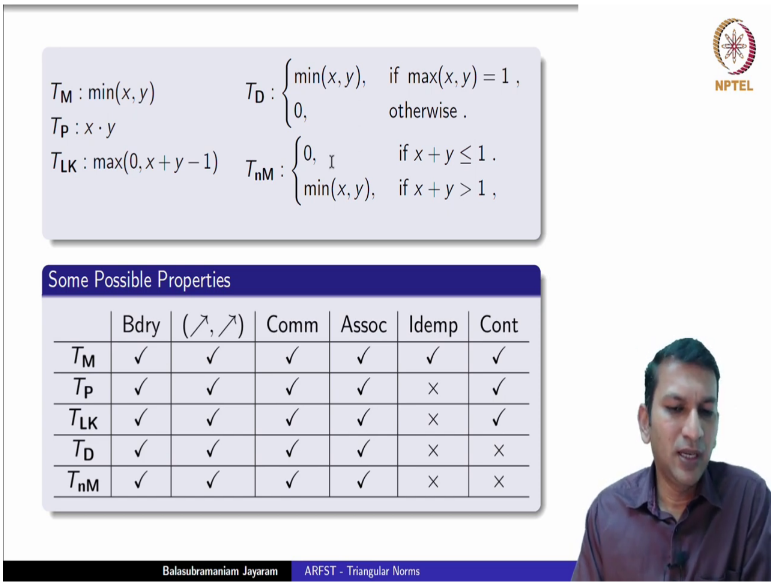

So, in this sense, let us look at continuity. Is it really warranted to help us in that let us introduce two more functions. Once again the symbolism will become clear presently by the end of this lecture. So, if we consider these two operations and let us see with respect to these properties that we have.

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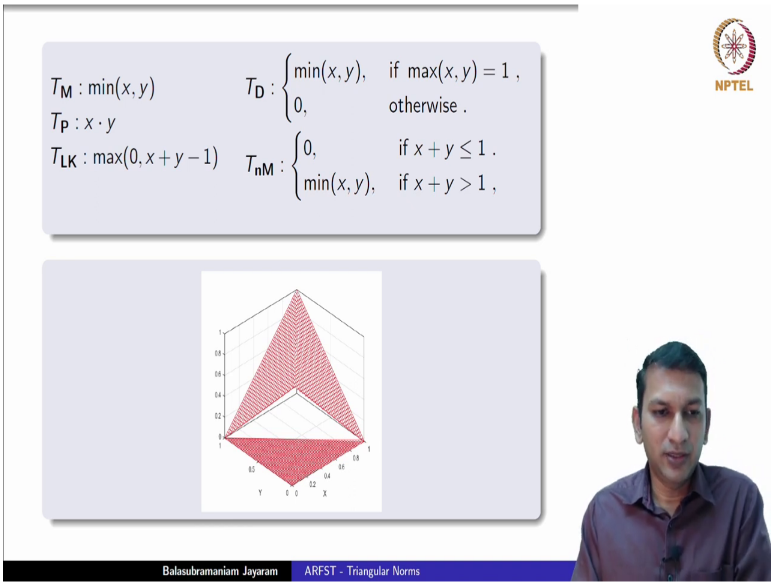

Now, to help us understand how this function T D looks like, if you were to actually look at the graph of this function this is how it looks like.

Now, notice that we are looking at functions from $[0, 1]^2$ to [0, 1]. So, the graph of this function will be contained entirely within the unit cube $[0, 1]^3$. And now what happens here is according to this definition, if either x or y is 1 then it is minimum. So, essentially it is the other value and that is what you see here this is y is equal to 1 as x varies this is what happens, this is x is equal to 1 as y varies this what happens and in the rest of the places it is 0.

So, clearly you see that this is not continuous. Is it idempotent clearly not? If you take T_D of say 0.4 , 0.4. Now, you see that 0.4 maximum of 0.4 , 0.4 is not 1 which means this is 0 and not equal to 0.4.

So, it is not idempotent on the entire interval. So, what it means is, that if you look at this it of course, it satisfies the boundary condition. In some sense it is increasing in both the variables, given a constant function 0 we do not ask for strict increase in this, it is commutative, associative, neither idempotent nor continuous.

Now, what about the last function that we have there? Let us look at its graph. This is how it looks like. Now, once again you will see that it is not continuous, essentially, we are seeing that it is it has gaps there and also that it is not idempotent. In fact, for the same value of 0.4, 0.4, if you look at it you see from the definition 0.4 plus 0.4 is less than or equal to 1. So, it is 0 which means it is 0 and not equal to 0.4.

So, once again we see that these two operations are neither idempotent nor continuous, which means perhaps idempotence and continuity are not the properties that you would like to put in an axiomatic definition of a fuzzy conjunction.

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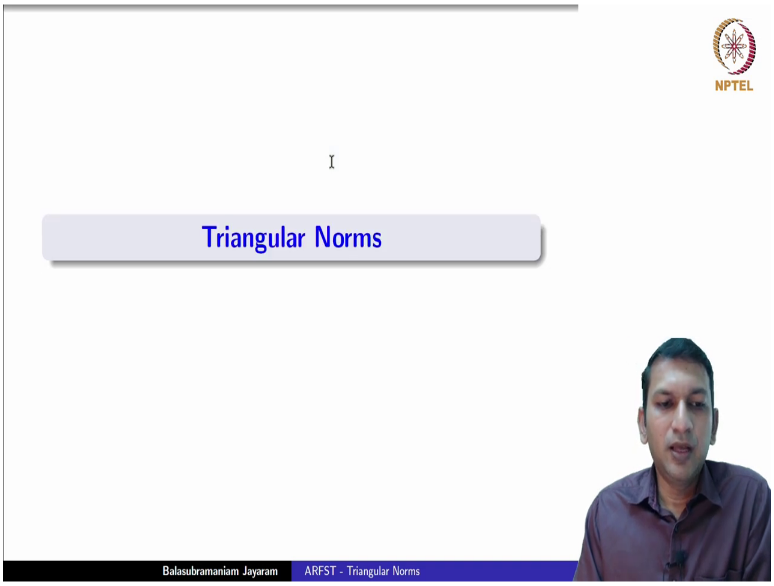

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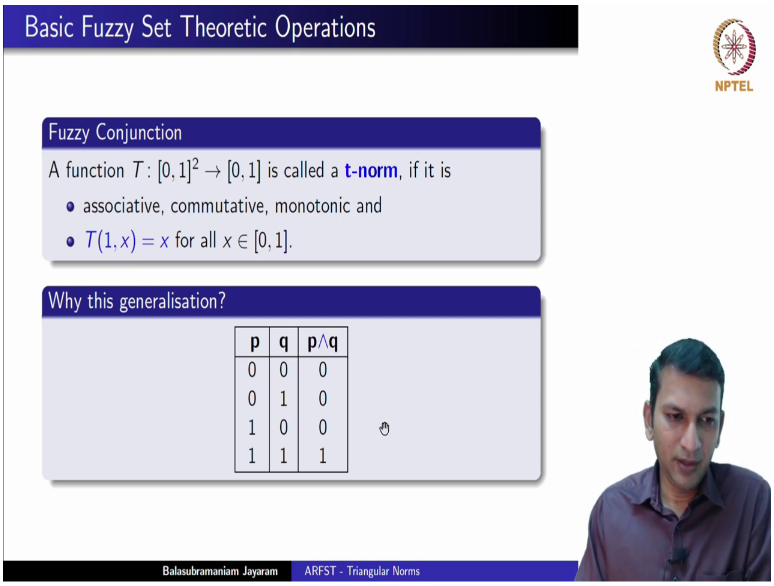

So, how is a triangular norm defined? So, as a function from $[0, 1]^2$ to [0, 1] and we will use the symbol T consistently to indicate a triangular norm. We will call it a t-norm for short means the abrogative.

A t-norm is a binary function on [0, 1] which is associative commutative monotonic and satisfies this condition. What is this condition? It essentially says that one should be the identity of this function. We do not have to worry whether it is left or right, because it is commutative. Now, is this warranted why this generalization. First of all if you look at the truth table for the binary conjunction, classical conjunction you see here 1, 0 is 0 and 1,1 is 1, which perhaps gives us this idea that we can ask for 1 to be an identity.

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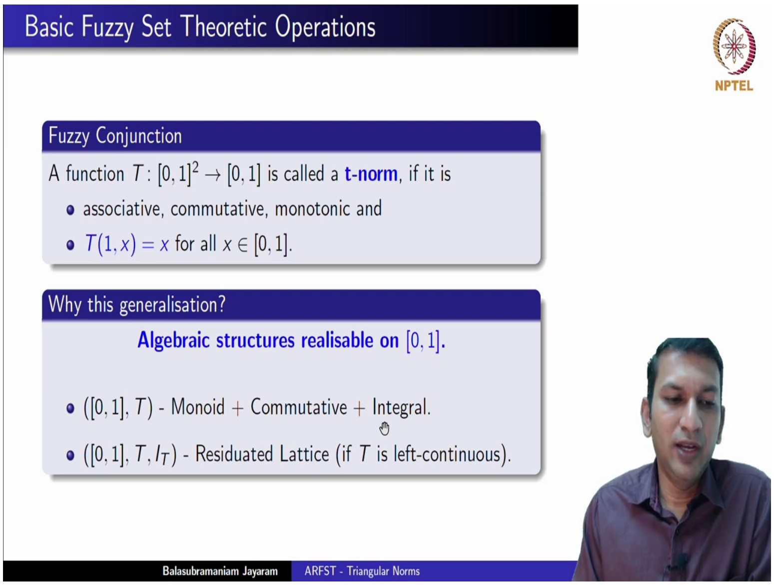

Another reason which perhaps will resonate better with you know is for the kind of a algebraic structures that we can realize on the 0, 1 interval with respect to T. For instance later on we will see that if you take this set 0, 1 along this function which is a binary operation of 0, 1 we can see it is actually a monoid commutative monoid. In fact, an integral commutative monoid. We will see these terms in detail in one of these lectures during this week itself.

In fact, we could also come up with another operation called the residual operation with respect to the schema with respect to which we get a very interesting lattician structure called the resituated lattice, ok. So, this is another reason why we could consider this as a valid or useful generalisation.

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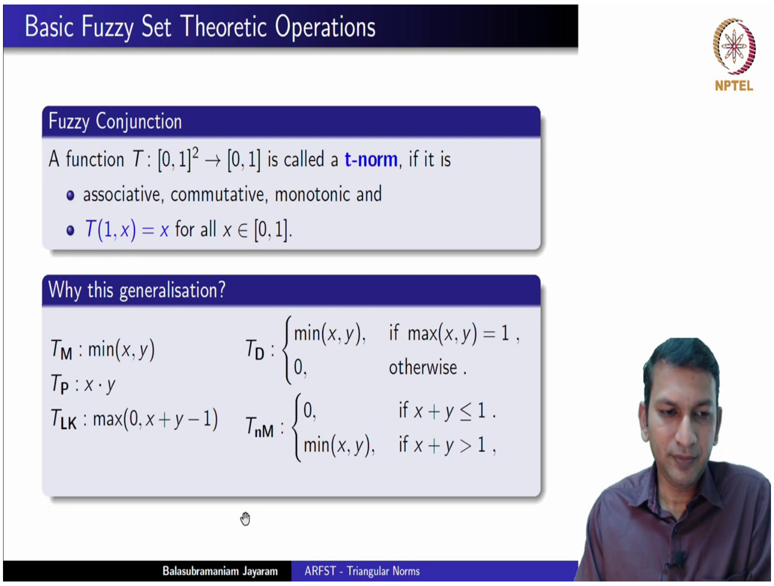

Finally let us look at these five interpretations that we have got, it is clear all of them satisfy this property also that is one is actually an identity for all these operations. So, perhaps we are not really unjustified and asking for this.

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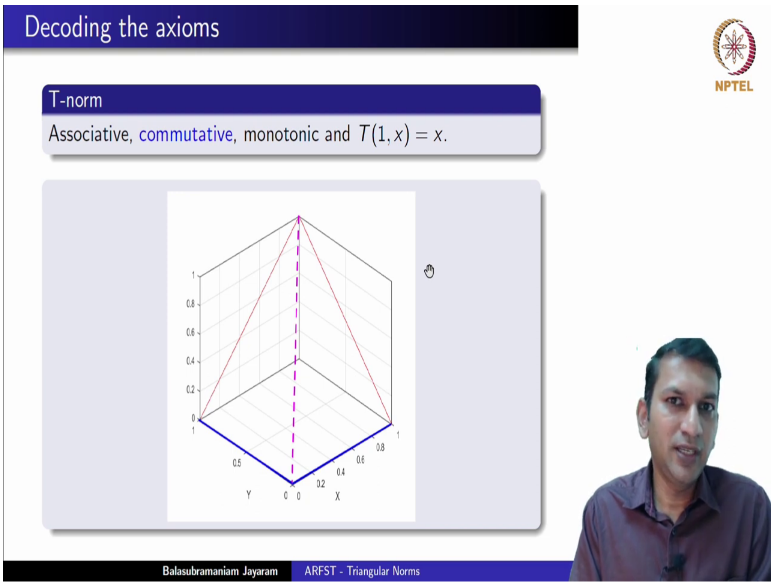

Now, let us try to decode these four axioms. And from a very geometric perspective so that we will have a visual idea of how these functions look like. Already we saw that the graph of the t-norm will be an embedded within the unit cube. Now what is T of 1 , x is equal to x min by commutativity whether x is 1 or y is 1, the other value is the value that the function T will take.

So, when x is equal to 1, you see here as y varies you get the identity function. It moves from 0 to 1. Similarly, when y is equal to 1, it moves as x varies it moves from 0 to 1. So, in the boundary when x or y is 1, this is the value that the function can take means the t-norm it appear it is an identity function when either x or y is equal to 1.

Now, along with the monotonicity let us look at what happens on the other boundary so, of this.

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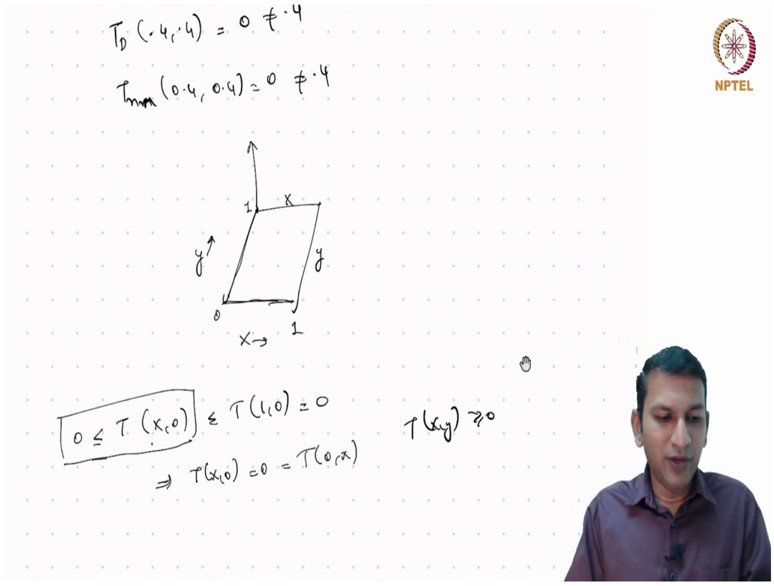

Now, when you look at this, we have this is the x axis this is the y axis this is 0, 1 now. We have seen that on these two, it acts like an identity function. What happens here? That means, what if one of them is 0, let us take this t of x , 0.

Now, we know by monotonicity this is less than or equal to T of 1 , 0 which is actually 0; however, T is a function from 0, 1 square to 0, 1. So, the smallest value of that will be 0 which means putting these two things together this implies T of x , 0 is 0, which is also the case with t of 0 , x by commutative.

So, we see here that any t-norm on the boundaries it is fixed when one of them is 0, it is 0 when one of them is one it is identity function. Now, what is commutativity? Commutativity says that it is symmetric about the diagonal. So, if you look at this diagonal, here I have drawn it in 3D, then we want the graph to be symmetric with respect to this axis. Now, what about associativity, is there a way to understand it geometrically?

Well, there are very well researched articles on that and some of them even claim that associativity can be thought of as 3-dimensional symmetry well. Now, let us look at some of these functions that we have.

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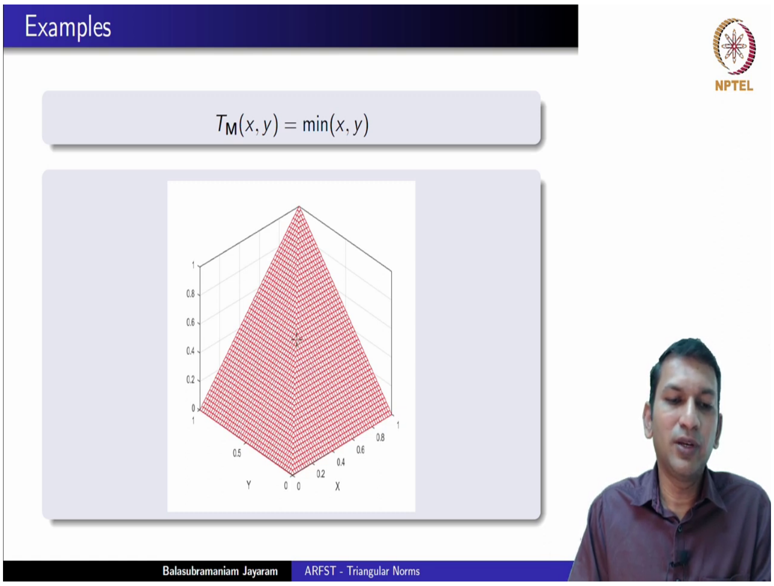

The first of them is the minimum function. Let us look at the graph of this minimum function and let us keep this boundaries of the skeleton graph for reference. So, if you look at the minimum function, the graph of that will look like this.

So, clearly it coincides on the boundary and what is interesting is exactly on this axis it makes because we know that it is idempotent, mean of x , x is x. So, along the diagonal when you move along the diagonal on the x y plane what you are going to get is the same value x on the z plane t-norm.

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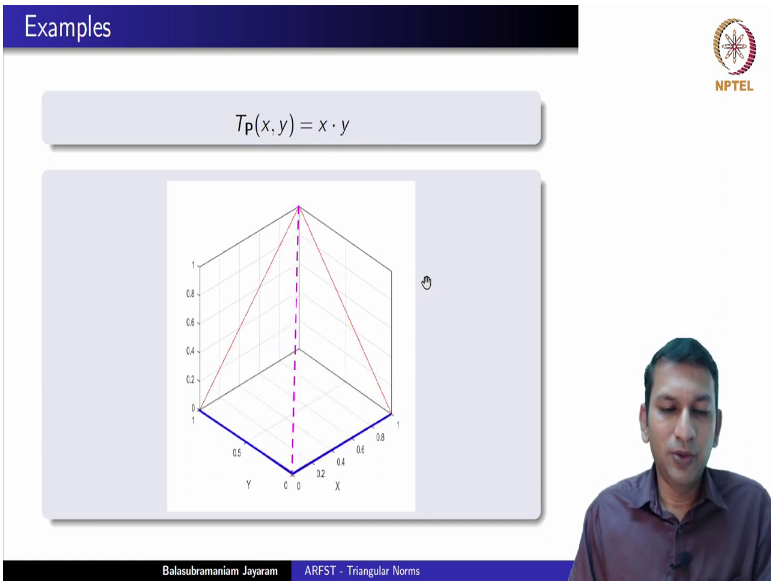

What about the product t-norm? How will it look like? This is how it looks like. So, if you imagine the line here, then you see that clearly it is symmetric with respect to that line there.

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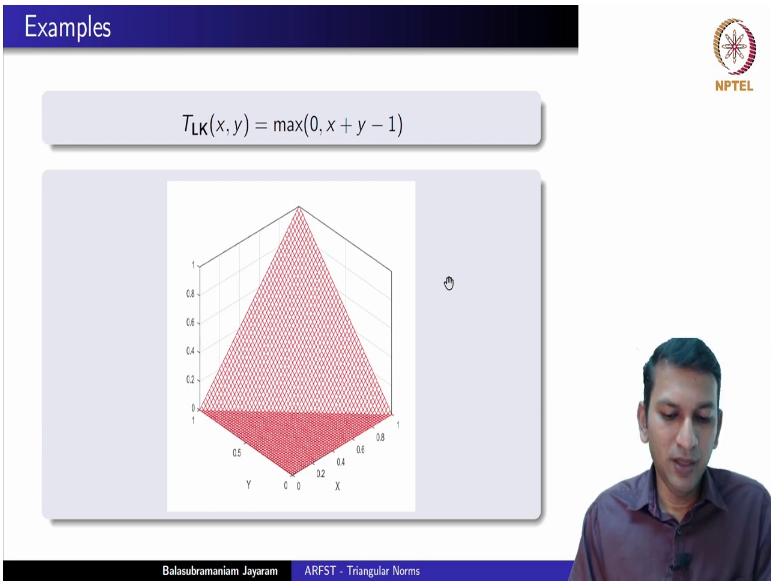

The Lukasiewicz triangular norm t-norm it looks like this. Once again you see it as symmetric and coincides on the boundary. But what you will also find is compared to product and minimum you see that, even when the arguments are not 0 x and y need not be 0; however, T LK of x y can be 0, there is a point, we will come back to in few lectures later.

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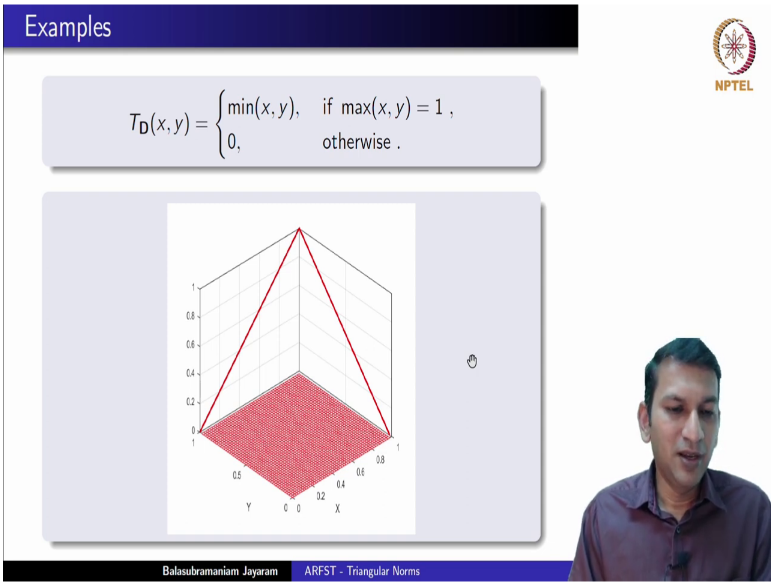

Now, if you look at this function which says that when one of them is 1, it is identity rest of the places it is 0. Now, once again trivially it is monotonic, it is associative and also commutative, but why this nomenclature of T D. If you look at it as I was shown the minimum value that any T can take is actually 0 and wherever possible this function has taken the value 0. It cannot take the value 0 on the boundaries of x is equal to 1 or y is equal to 1 because the definition demands that it should be identity, wherever else it is possible to take 0 it has taken.

So, in that sense this is really the smallest t-norm with respect to point wise ordering and in that sense, it was called the drastic t-norm and the symbolism that is called stuck to it is as T_D, D for the drastic word.

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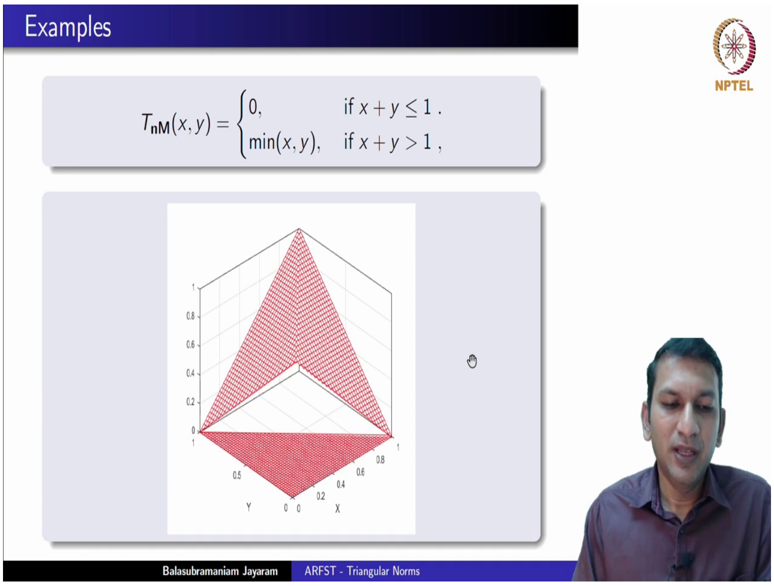

Now, if you look at this particular function, which is again a t-norm the graph of it looks like this. We have seen this a few slides earlier too, this is essentially this is your minimum t-norm.

All we have done is we have kind of put our foot on this and then smother it down to the floor. So, take the minimum t-norm and then cut it down to 0, that is exactly what we have done, but along the opposite diagonal or the anti-diagonal. So, this is how the graph of it looks like clearly it is not continuous, you see that these points coincide also with the min graph; that means, these are points on which idempotence property holds.

Of course, as I said we will see this in detail a little later. Now, we just wanted to have a graphical feel for these functions ok. As I said this is the minimum and all you are doing is smothering part of it on to the floor well.

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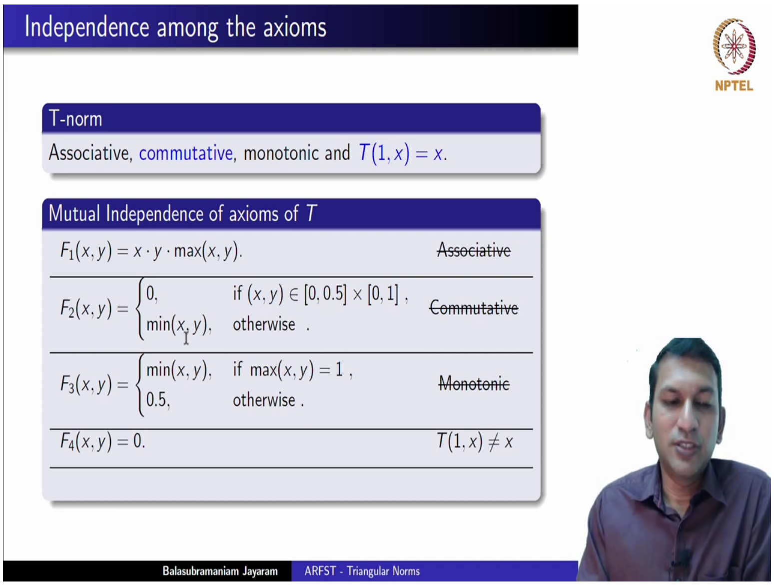

What about these axioms? Are they mutually independent? Well, whenever we come up with an axiom schema or axiom set we want to ensure that they are not only minimum, but they are also non-redundant. So, in that sense we ask the question are do any of these imply the other or do any combination of these imply the other.

We will see that is not true. For example, if you take this function x y into max of x y, you will see that it in fact satisfies all the other properties of it is commutative, monotonic, increasing and also if you take one becomes the identity. Now, so here we see that this function does not satisfy associativity, it is neither commutative it is, but it is commutative and monotonic.

And it also satisfies the identity property if you take y to be 1, it is x into 1 into 1 which is x thus it satisfies all the 3 properties of commutativity monotonicity and identity, but not associativity and if you look at this function it can be shown it is associative, clearly when it is not commutative because when you take the value.

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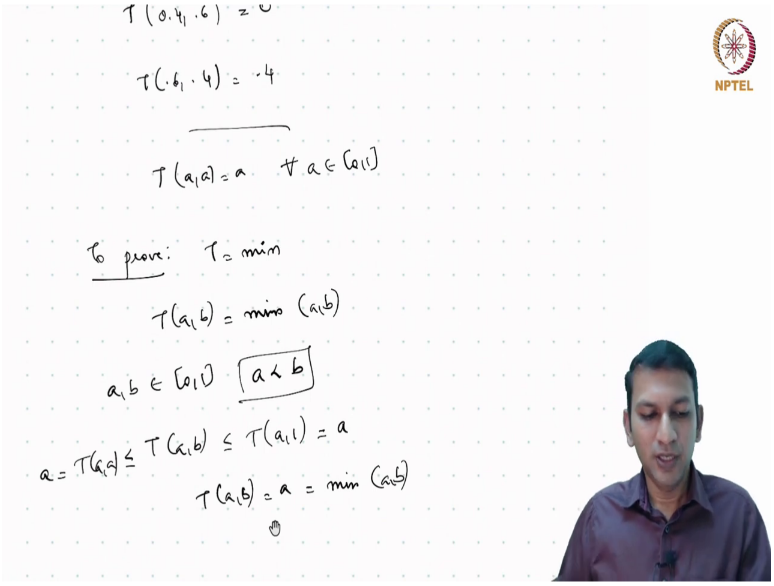

Let us look at for this particular function. Let us look at the 0 0.4 , 0.6. So, it falls in this regime here it becomes 0; however, if you look at T 0.6, 0.4 and we see that it falls under this regime where it is minimum and you get it to be 0.4. Thus, it is not commutative of course, it is monotonic that can be seen. Now, let us look at this function. It is essentially the drastic t-norm only that we have not smothered it onto the floor, but we have raised it to the level 0.5 clearly because of this it is not going to be monotonic.

For instance, if you look at T of 1 , 0.4, it will be 0.4 somewhere and suddenly it increases to 0.5 and so on and so forth. So, it can be seen that it is not monotonic. Finally, if you look at this function which is just the 0 function everywhere, constant 0 function vacuously it is associative commutative and monotonic, but; obviously, one does not play the role of an identity.

Thus, all these four properties are in fact, independent of each other ok, mutually independent.

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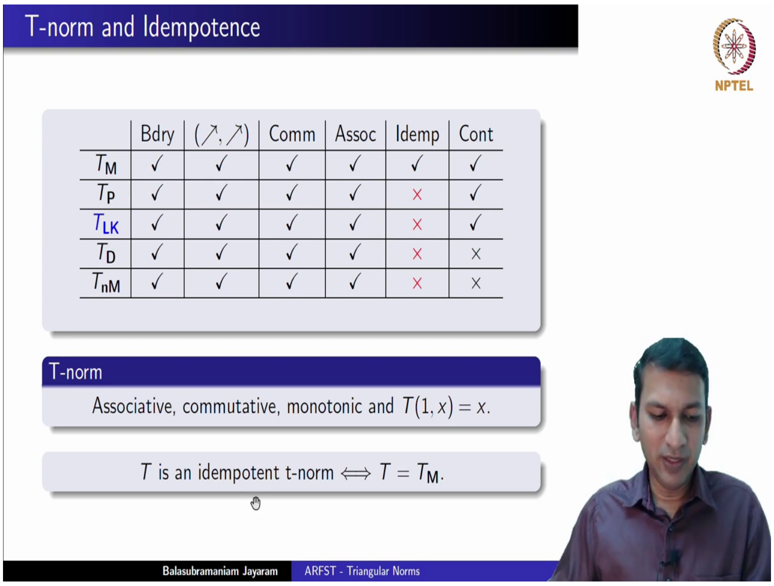

Now, we have seen that among all these six properties that we considered initially, we saw there were many functions t-norms which are continuous and non continuous term. However, we have seen that there is only one example of a t-norm we have seen so far, which is idempotent. Are there other t-norms which are idempotent?

In fact, when you look at it can be proven that if T is an idempotent t-norm, then T really has to be only the minimum t-norm. So, essentially it says that minimum is the only idempotent t-norm. Perhaps firstly, this justifies why we did not ask for idempotents in the axiom set. Now, to prove this is very simple the converse is clear because if minimum is idempotent. So, minimum is idempotent.

But let us now look at let us been given an idempotent t-norm so; that means, we have a T such that T of a , a is equal to a for all a in the close to 0, 1 interval. Now, we need to prove that this T is in fact, minimum; that means, we need to show that for any a, b T of a, b is actually min of a, b. So, let us consider arbitrarily two values from 0, 1. Now, on we know that 0, 1 is a totally ordered set which means there is always an ordering between a and b.

Let us assume that a is strictly less than b. Now, what could be the value of T of a, b? Now, we know that by monotonicity this is greater than T a, once again by monotonicity this is smaller than T of a, 1.

But T of a, 1 because 1 is identity is actually a and by idempotence we see that this is a, which means T of a, b is sandwiched between a on either side, means T of a b is a. But what is this essentially? By our assumption a is less than b there is nothing but minimum of a , b and the commutativity if b is smaller than a, similar argument can be given to show that T of a, b is actually b so; that means, T is essentially the minimum t-norm.

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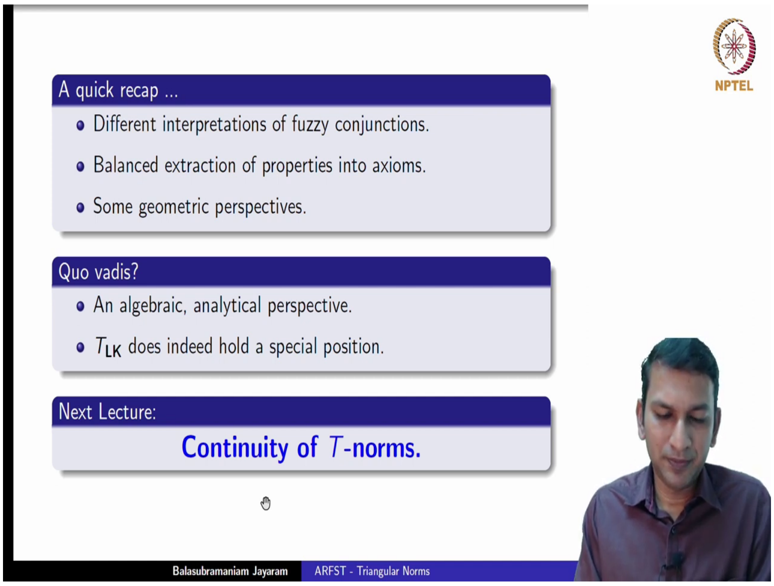

A quick recap of what we have seen in this lecture. We had seen different interpretations of fuzzy conjunctions. What we have done is a balanced extraction of properties into axioms, which we have seen we do not ask for continuity, we do not ask for idempotence. Also we did not just ask for the boundary conditions, instead we asked for one to be the identity. And the boundary conditions are automatically satisfied because we had one to be the identity and also because of the monotonicity condition.

Finally, we have seen some geometric perspectives. Some visual illustrations, because it is always good to have a visual idea of how these functions look like and what these properties mean visually. Now, where do we go from here? In the rest of this week we will look at t-norms from a little more algebraic and analytical perspective. We look at some algebraic and analytical properties which are very important for us going forward especially in the applicational setting that we will be discussing.

And we will also justify that the Lukasiewicz t-norm that we had seen earlier also which almost came to giving us a lattice along with the dual operation, its fair. We see that it still holds a very special position. In the next lecture we will start with the analytical properties of t-norm different types of continuities that are available on t-norms.

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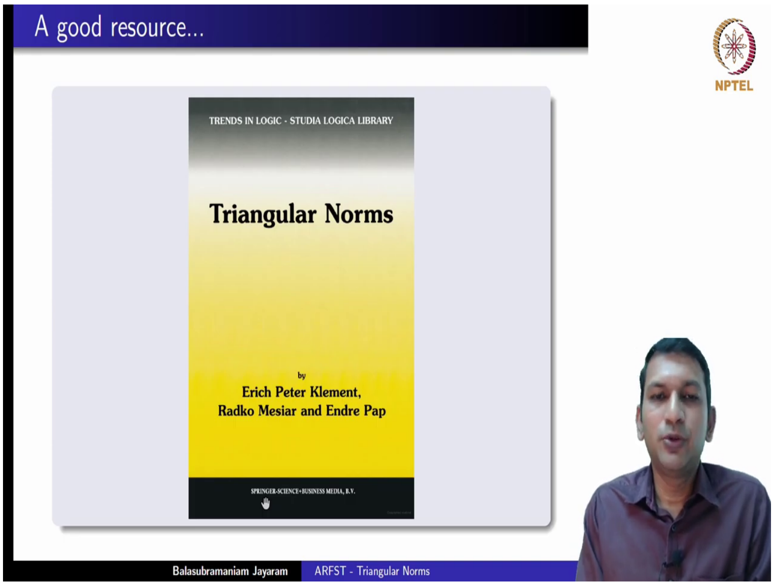

And for this lecture a very good source, if you want to turn to would be the book by Klement, Mesiar, Pap title Triangular Norms.

Thank you once again for joining in this lecture and hope to see you soon in the next lecture.

Thank you.