Approximate Reasoning using Fuzzy Set Theory
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Lecture - 10
Algebras on fuzzy Sets

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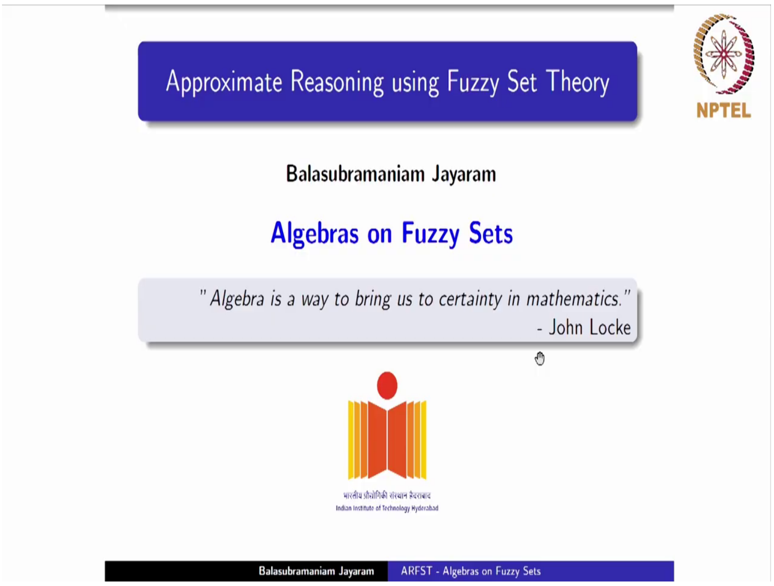

Hello and welcome to the last of the lectures of this week, under the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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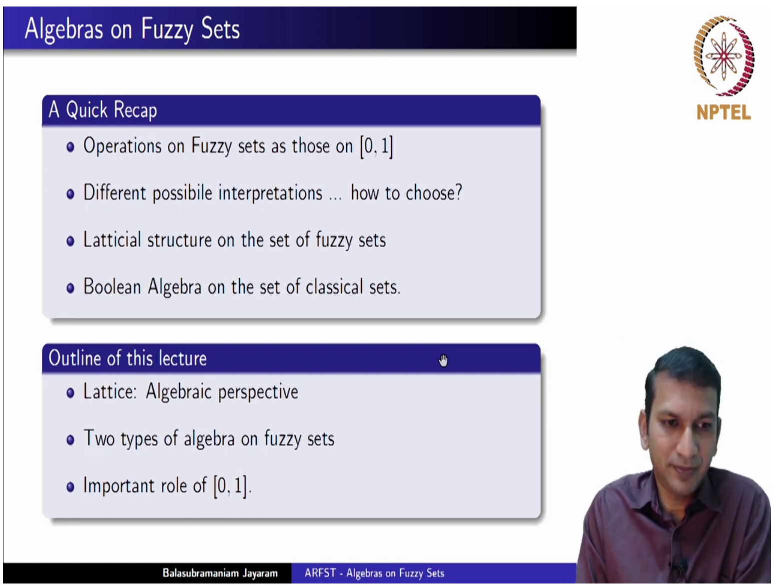

Over the course of the last two weeks we have seen how operations on fuzzy sets can be thought of as those being performed on the unit interval [0, 1]. Note that the unit interval [0, 1] comes here because it forms the co-domain of the corresponding membership functions.

We have seen that there are different possible interpretations for these operations and we were left wondering how to choose among them. As was already mentioned we could either look at the theoretical structures that they lead to in terms of the richness of the theoretical structures or look at how useful they are in the application context.

Presently we are taking the first approach in due course of time we will also look at the second approach once we introduce fuzzy inference systems. Along this approach we have seen that it was possible to introduce a latticial structure on the set of fuzzy sets. In fact, we have introduced a complete lattice on the set of fuzzy sets and when we looked at the corresponding structures that are available on the set of classical sets, we saw that we could. In fact, go towards Boolean algebra.

In this lecture, we will continue looking at lattices from an algebraic perspective. We will show that at least two different types of algebras are possible to be imposed on the set of fuzzy sets. And once again we will see the important role played by the unit interval [0, 1].

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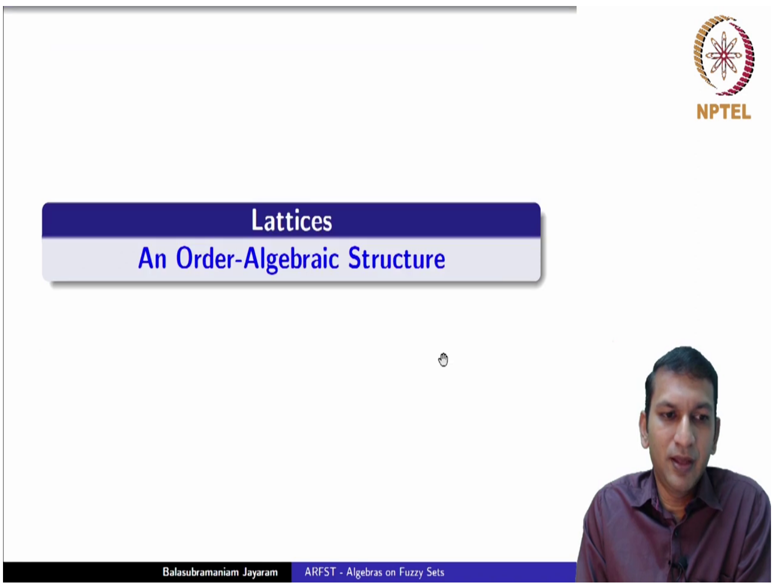

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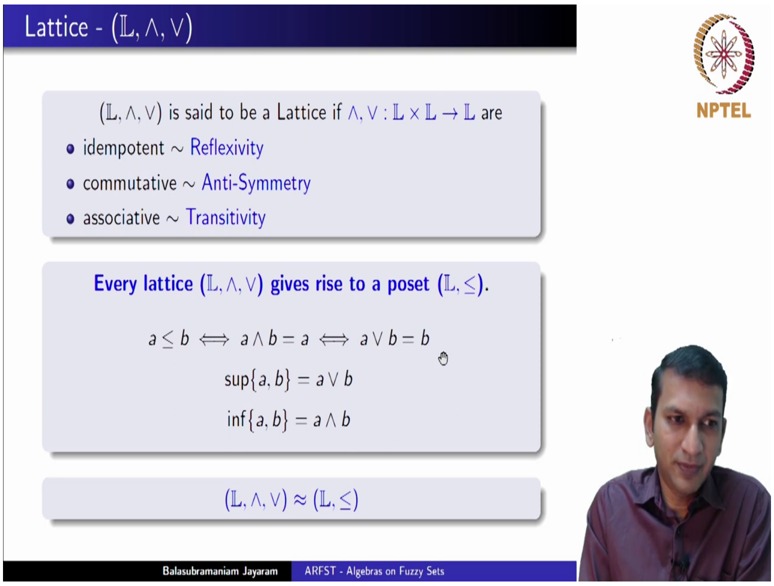

We have seen lattices from an order theoretic perspective and also from an algebraic perspective. We have seen that lattice can be thought of as a set with two binary operations that are closed; that means, these are operations from L cross L to L having these three properties of idempotence, commutativity and associativity. And we have seen that with these three properties, it was possible to obtain a partial order and order relation on the set L. And we have seen that how these three properties of idempotence, commutativity and associativity relative relate to the corresponding reflexivity anti-symmetry and transitivity of the binary relation binary order relation that we have defined now.

So, essentially we are also seeing that the what we call as the meet and join operations from an algebraic perspective on the lattice L. They actually turn out to be the supremum, the infimum and the supremum. The meet is the infimum and the join is the supremum of a pair of elements.

In that sense we have seen that lattice can be looked at from purely order theoretic point of view or from an algebraic perspective.

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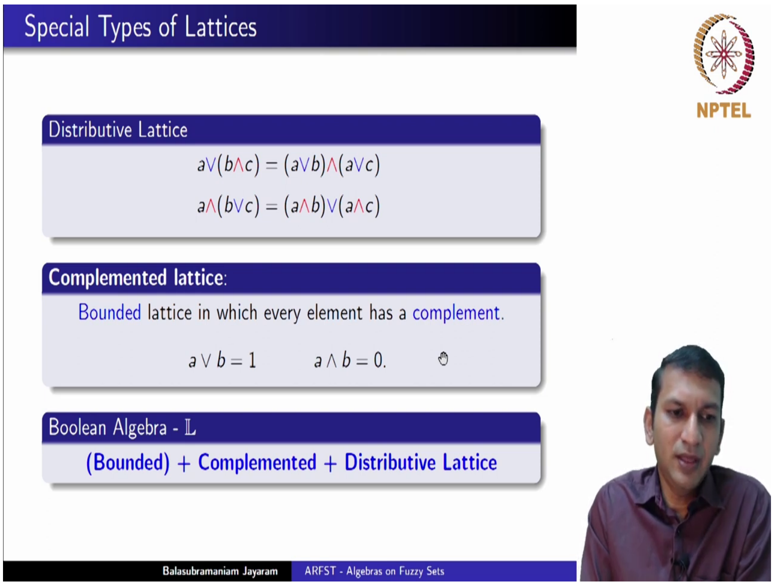

We also seen a few special types of lattices first among them from an algebraic perspective is that of a distributive lattice, where the joint and meet both these operations distribute over each other. We have also seen what a complemented lattice is to begin with we need a bounded lattice and given an element a we say it has a complement in b if these two identities hold; that means, a join b should be 1 and a meet b should be 0.

If this happens for every element in the lattice L we say it is a complemented lattice. Finally, in the last lecture we have seen what a Boolean algebra is it is a complemented distributive lattice note that complemented lattice means we already have a bounded lattice underneath.

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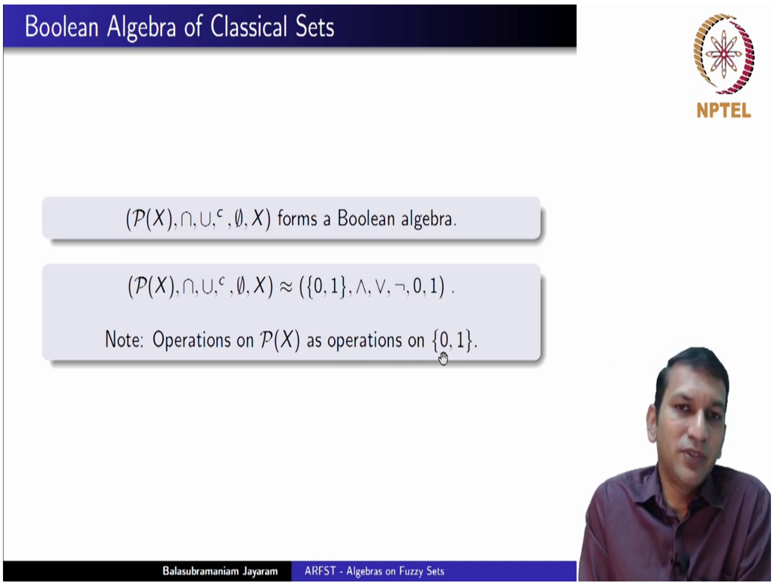

Now if you consider a non empty set X and the set of all subsets of X the classical subsets of X, along with its usual intersection union and complementation operation with empty set and the set itself acting as its lower and upper bounds. We know that it forms an algebra.

And note that this was also readily translatable and noticeable in terms of the underlying co domain of the corresponding characteristic functions, that is on the set with just the 0 and 1 as the elements. So, once again it highlights that operations on classical sets can be thought of as performing operations on the co-domain of the corresponding characteristic functions.

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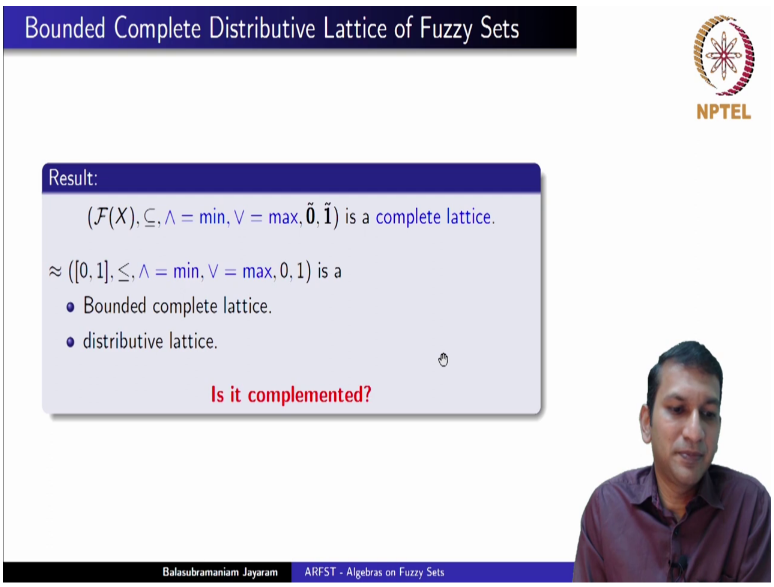

Now, we also have seen that if you take a non empty set X and consider the set of all fuzzy sets that you can define on them. That means, functions from X to [0, 1] with a usual point wise ordering between pairs of fuzzy sets and considering the minimum operation for the meet and the maximum operation for the join and the constant 0 and constant 1 functions acting as the lower and upper bounds.

We have seen that this structure is actually a complete lattice, once again we have inherited the properties from the unit interval 0 1 with the usual order and the meet and join being given by the min and max operations on them and 0 1 0 and 1 acting as the bonds. We have seen that this structure is a bounded complete lattice, in fact it is also a distributed lattice.

Now, what prevents this from becoming a Boolean algebra you know Boolean algebra is a complemented distributive lattice. So, we have a bounded distributive lattice the only question that we need to ask ourselves is this structure also complemented.

Let us look at this.

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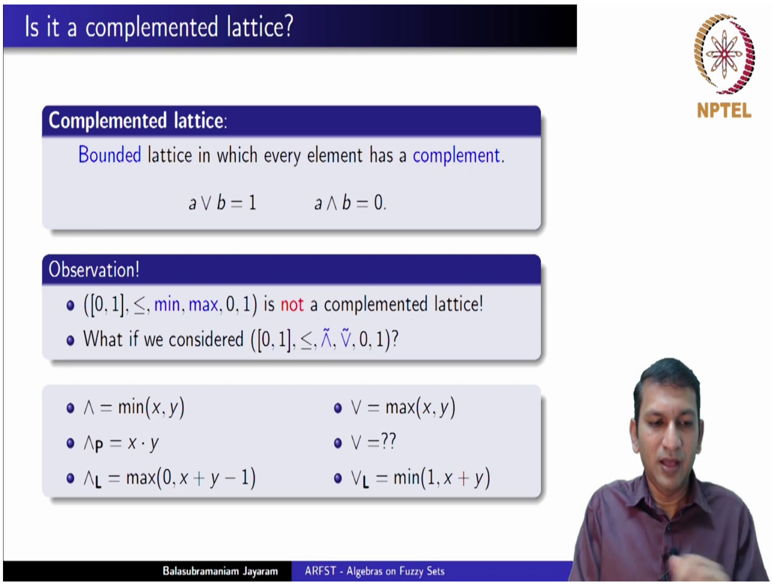

Now, what is a complemented lattice a bounded lattice in which every element a has a complement b, which means satisfies these two identities. Now this is what we actually observe why do we say? So, let us look at the structure.

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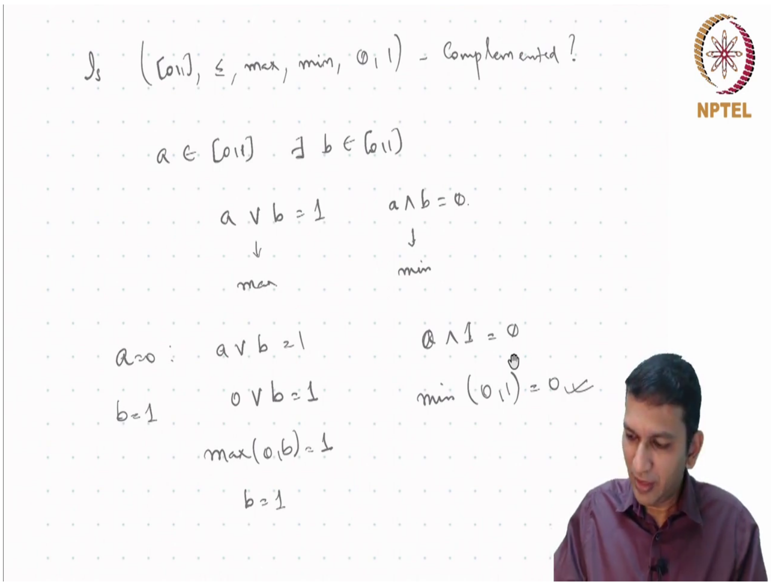

We are asking the question is this structure complemented. Now for complementation we have seen that if you take any element a in it, in which case it is [0, 1] there should exists a b again then (Refer Time: 07:52) minimum should be element of 0 1 such that a join b is 1 and a meet b should be phi.

So, now you are going to take, for join we are going to take max and for meet we are going to take min. Now if a is 0 we are looking at a join b to be 1 this means 0 or b should be equal to 1. Now we are interpreting this join as max; that means, its max of 0 comma b should be equal to. It is clear that in this case; that means, b has to be 1.

So, a is 0 then its complement b is 1. Now, let us verify whether this is true also for the second identity. That means, we are asking the question is 0 need 1 equal to 0 is nothing but minimum of 0 comma 1 we know that this is 0, so this identity goes.

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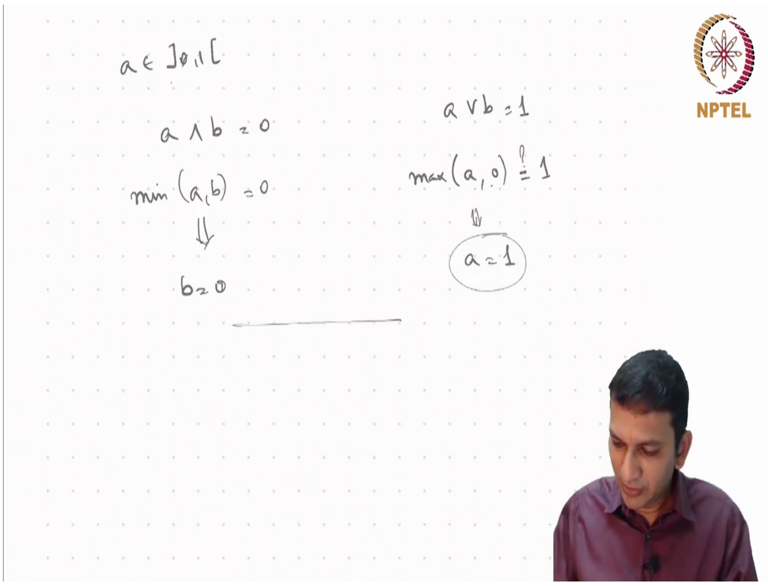

Let us look at for elements coming from the open interval 0 1 so; that means, a is neither 0 nor 1. Now what we want is once again a b such that a meet b 0 and a join b is equal to 1 this essentially minimum of a comma b should be equal to 0.

Now, we know that a is not 0. So, if this has to be 0 from here we see that b has to be 0; that means, for every a element of 0 1 b is the its complement has to be 0. But now let us look at this.

So, now, this is translating translates as max of a comma b we know is 0, we are asking the question is it equal to 1. Now this can be 1 only if a is 1 because this is already 0 and a comes from open 0 1 which means a is greater than 0. So, now this tells us that a has to be 1.

However, we know by our assumption a comes from open 0 1 which means this is not true this tells us that with min and max we are not able to obtain a complement for every a in the (0, 1) interval, 0 and 1 yes they are complementary their complements are they are complements of each other. So now, where does this leave us? Well not all is lost, why not consider other interpretations of conjunctions and disjunctions.

Now, what are the interpretations that we have known so far we know min we have looked into it, we also know that we could consider product for the conjunction and this particular operation which we will soon have seen as the operation which conjunction. So, we have seen that with min and the join interpreted as the maximum function, we have not been able to show that it is a complemented lattice even though it is a lattice. Now, let us look at the case for product.

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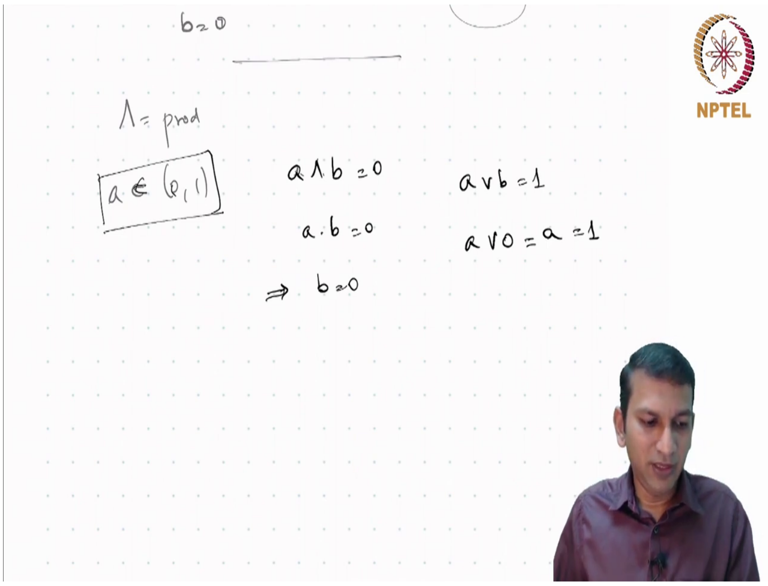

So, we asking the question. If we interpret meters product do we get a complement for every element of the set 0 1. Once again let us directly consider the case where a belongs to the open 0 1 interval. So, now, we are looking for a b we are looking for a b such that a and b should be 0 that in this case a dot b should be 0 note that a is not 0. So, a dot b is 0 implies b is 0 ok.

Once again we are seeing that for every a in open 0 1 the complement b has to be 0. Now the question is fine what would be the corresponding join operation. But perhaps this is a mood question, because what we want is that a joint b should be 1 and in this case it has to be a joint 0. But we know that 0 is actually the identity for the joint; that means, a joint 0 will be a and if it has to be equal to 1 then a has to be equal to 1, but we know that a actually comes from the open interval 0 1.

Which means no matter what join operation that you are considering we are not going to get the, show that the identity a join b is 1 where b is the complement of a. This leaves us with the last of the operations let us look at the Lukasiewicz conjunction.

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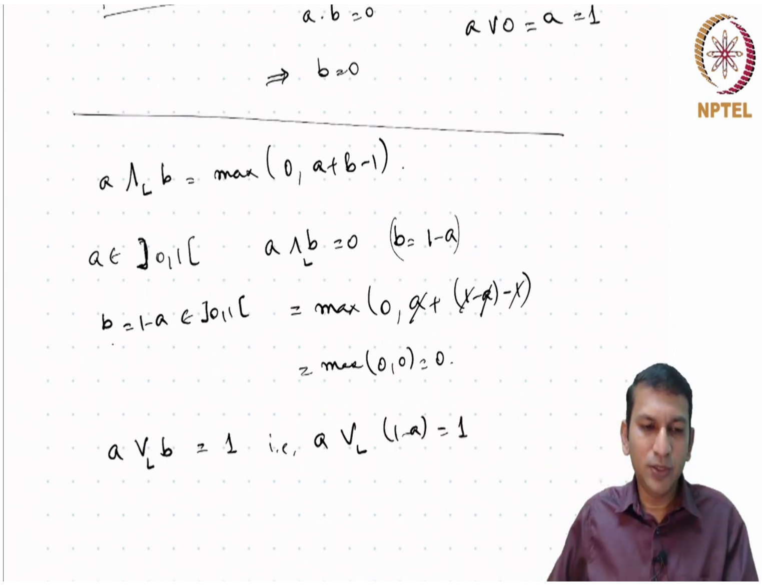

So, it has given as a b is maximum of 0 comma a plus b minus 1. So, now, once again consider a coming from open 0. So, that is the question does there exist a b such that a will be 0, why not try the option b is equal to 1 minus a.

So, you see here if b is 1 minus a then this is actually equal to maximum of 0 comma a plus 1 minus a minus 1 is nothing but max of cancelling maximum (Refer Time: 13:45) maximum 0 0 is 0. So, for the first time we are seeing that whenever a belongs to 0 1, b is equal to 1 minus a seems to be a possible complement of a. Note that if a comes from open 0 1 then b also comes from open 0 1.

Now, the question is does there exist a join operation related to Lukasiewicz conjunction such that for any a here with the corresponding b as 1 minus a. That means, a some join 1 minus a should be equal to 1 this is the question we are asking, why not try this particular function? The function x y minimum of 1 comma x plus y.

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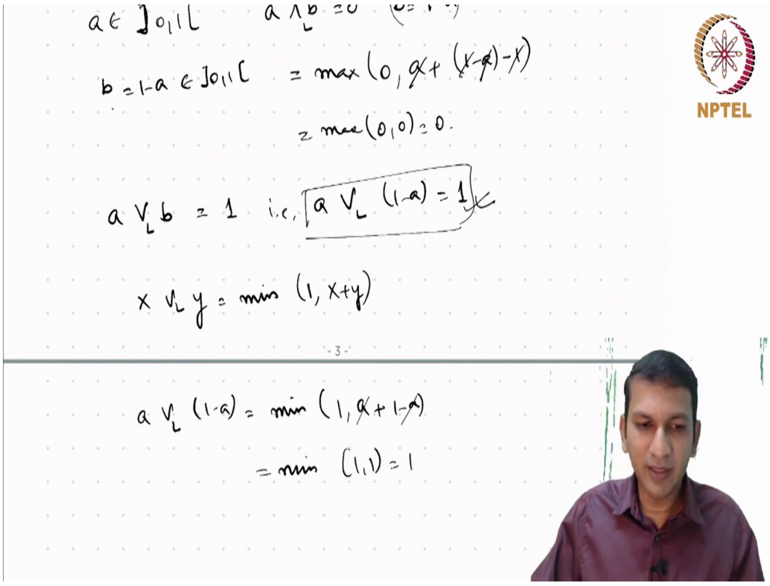

So, now, we need to verify if this identity holds true. That means, again 1 minus a is equal to minimum of 1 comma a plus 1 minus a. You see that this is nothing but minimum of 1 comma 1 which is 1. So that means, this identity is valuable.

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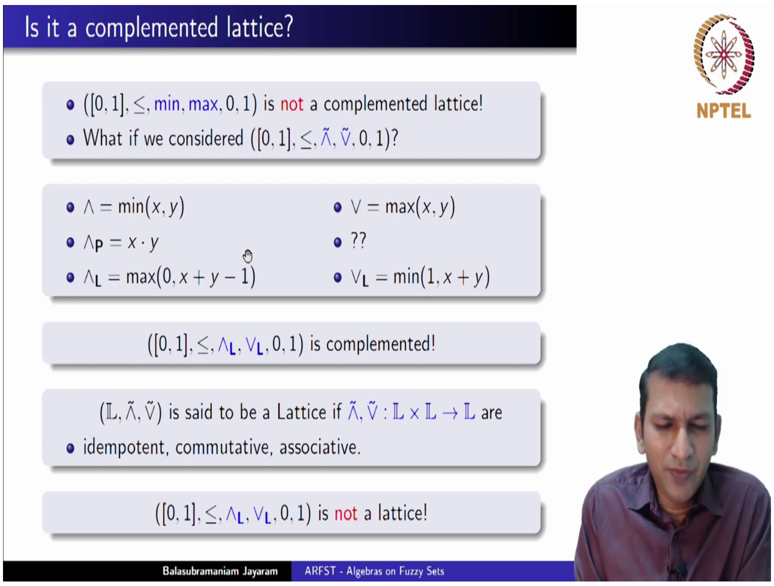

So, what we see here is, if you consider among these three pairs of operations we see that with this pair of operations the Lukasiewicz conjunction and the corresponding operation in the 1 comma x plus y as the join we see that it is complemented.

Note that we have the set with the model and these two operations which are closed and now we see that they actually give rise to compliment. But the question is do they form a lattice? For lattice we need that these two binary operations close though they are they should satisfy the conditions of idempotence, commutativity and associativity.

Commutativity easily you can see instead of x plus equal to y plus x similarly y plus x here plus x commutative. Associativity, also follows you can quickly verify that. What about idempotence? Right, let us look at that.

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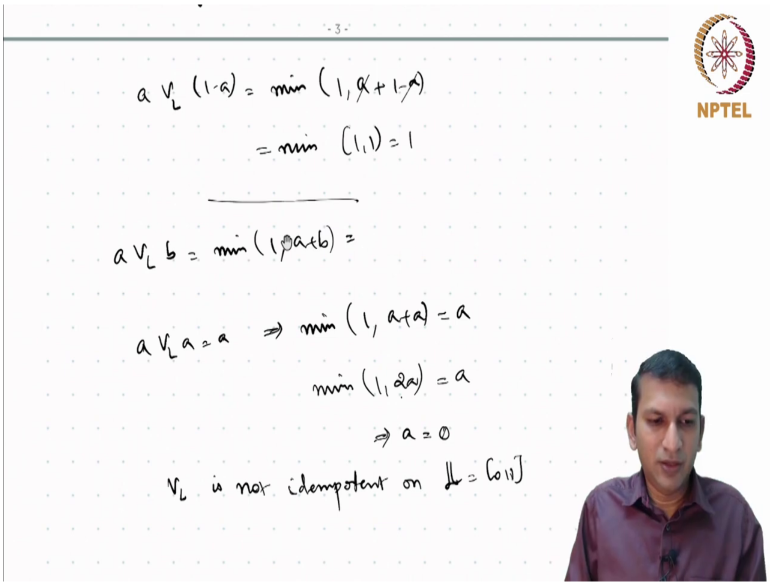

So, we know that a join b if it is given as this 1 comma a plus b. We want this to be idempotence means we want this to if a join a should be equal to this implies minimum of 1 comma a plus a should be equal. Now this is minimum of 1 comma 2 now the left hand side could either be 1 or 2 a if it is 1 then a is 1 if it is 2 a then 2 a is a implies a actually has to be 0.

So; that means, the only idempotent elements are 0 or 1 means join l is not idempotent on the (Refer Time: 17:00) 0. While it is both commutative and associative it is not idempotent, similarly you can show that that the Lukasiewicz conjunction is also not idempotent.

So, we see that it is not a lattice though it gave rise to a complement operation, it was not it is not a lattice where does this leave us?

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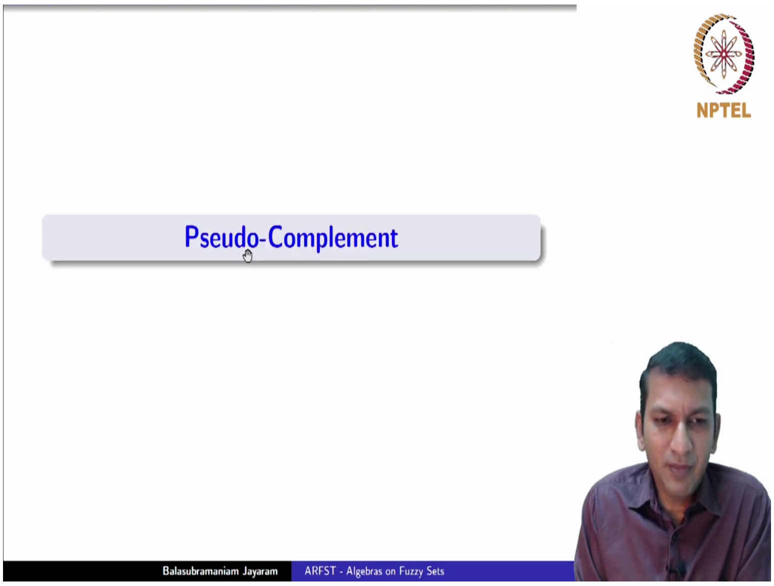

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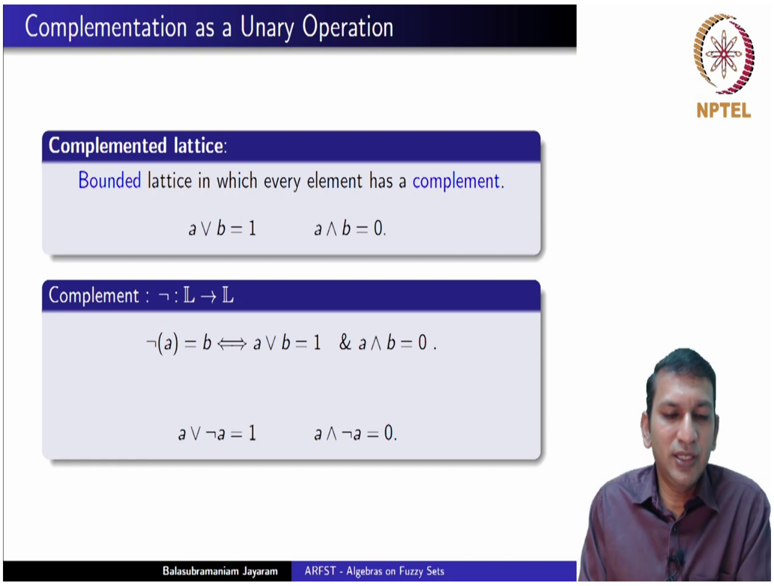

Let us look at complementation as a unary operation what is a complement. So, we start with the bounded lattice for a complement for any given element a we only need another element b such that these two identities are valid. Now that means, it takes an element a and maps it to another element b, how does that do that? Let us denote the complement in terms of the symbol negation.

Now if you take a negation of a we say is b, if and only if a join b is 1 and a meet be is 0. Now, if you were to write b as negation a this is how the above identities will look like a join negation is 1 a meet negation is 0.

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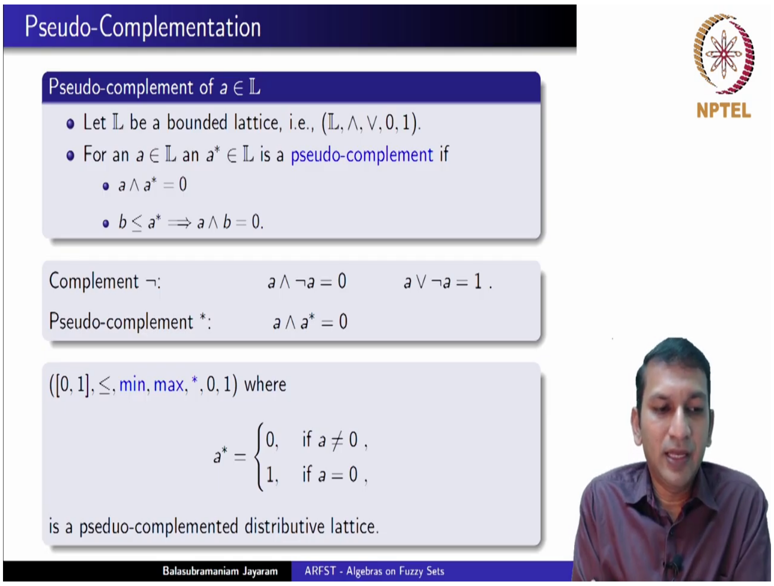

Let us define what is a pseudo complement of the element, in the lattice. Once again we need a bounded lattice we say an a star is a pseudo complement of an element a and l. If two things are; one a meet a star should be 0 that is the first condition, the second condition is whenever b is smaller than a star then a meet b should also be 0.

Note that what this says is the, a star that we are choosing should be such that it is the largest element among all elements of l, such that a meet a star is 0. If there is a b which is smaller than a star then a meet b also should be 0 in that sense we are seen that a pseudo complement is unique because it is the largest one in the lattice the largest element such that a meet a star is 0.

However, notice that there are also other elements such that a meet the meet of a with those elements also are 0. So, in short a pseudo complement of an element a is the largest element such that a meet that element is 0. Now, let us compare the concepts of complement and pseudo component for pseudo complement we have only the first condition not the second one.

Consider this no familiar lattice that 0 1 in the unit interval with the usual order and min and max are taken for the meet and join operations and consider this function. Note that now we are looking at complement as a function, clearly a star is a function on 0 1. It takes the value 0 whenever a is not 0 and takes the value 1 when it is 0.

Now the question is it a pseudo complement on this lattice? Fortunately yes, it is pseudo complement on this lattice. How do we know this? Let us look at this.

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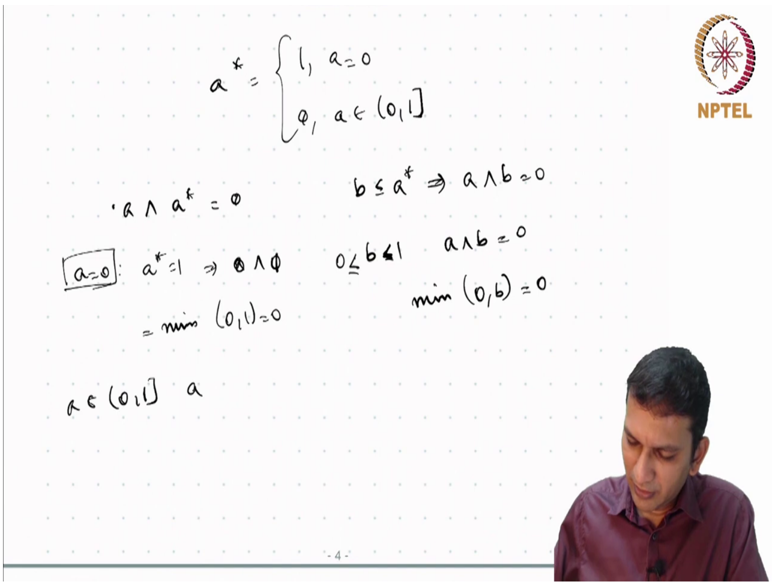

So, what we have? We have given an a star is 1 if a is equal to 0 0, if a is not equal to 0; that means, a belongs to o open 0 close to 1.

Now, what we need to show is this a and a star is actually equal to 0. Also if there is a b less than or equal to a star then this should imply that a and b is 0. Now if a is equal to 0 we see that a star is 1, which means a need 0 and 1 is nothing but minimum of 0 comma 1 because 0.

So, this valid now since a star is 1 b could be anything less than 1. So, let us assume that b is strictly less than 1, now what we need to show is a and b should be equal to 0, but note that a is 0. So, this is interpreted as z minimum of 0 comma b we know is actually, 0 because b if we take it to the between this it is 0. So, for a is equal to 0 we see that both these properties are valid.

Let us take a to b from open interval open 0 close to 1, then by this function we know that a star is 0.

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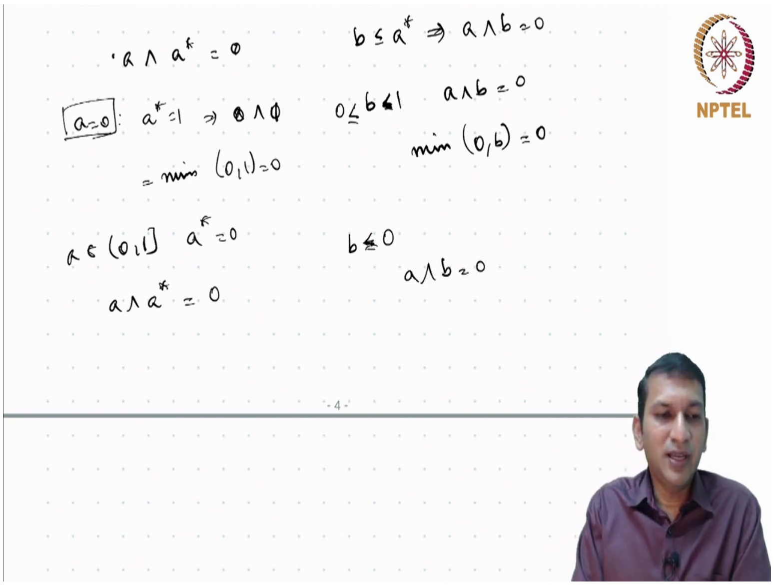

Clearly, now a mean a star is minimum of these two which is 0 and since a star is 0 the only b that can be smaller than the, smaller than or equal to 0 is itself there is no other element which is strictly less than 0. So, then essentially b is 0 and we have already proven that for that a and b is 0.

So, we see that this operation that we have here gives you a pseudo complement on this lattice and hence this lattice becomes a pseudo complemented distributive lattice.

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Let us not stop with that, we discussed and said that we should try and get as rich as structure as possible.

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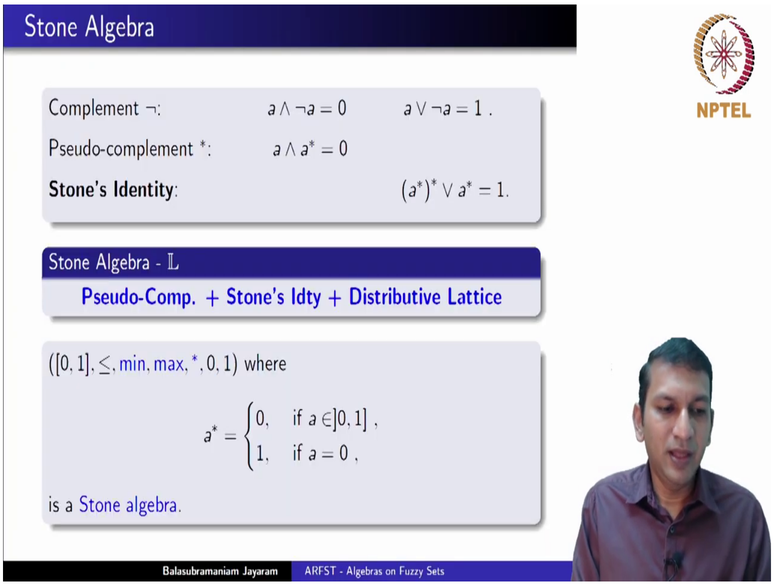

Now, once again compare complementation and pseudo complementation. We see that pseudo complementation differs in that the second identity is not valid. Stone introduced a particular identity which is given like this you would see the close relationship or resemblance with respect to the second identity which is a joining negation this one. However, notice that negation a which is the complement of a is replaced by the pseudo complement of a star and a itself is replaced by double pseudo complementation a double star.

Now, a pseudo complement may or may not satisfy stones identity. But if a pseudo complementation exists on a lattice cell and if it satisfies stones identity and if that lattice happens to be a distributive lattice note that even pseudo complementation is defined only on a boundary that is. So, if a lattice L is distributive has a pseudo complement which also satisfies the stones identity we call it as Stone algebra.

And what is interesting is what we have seen as a pseudo complemented distributive lattice under the same pseudo complement it can be shown that it is a Stone algebra. That means, this particular function actually satisfies the stones identity, once again this is not very difficult to see.

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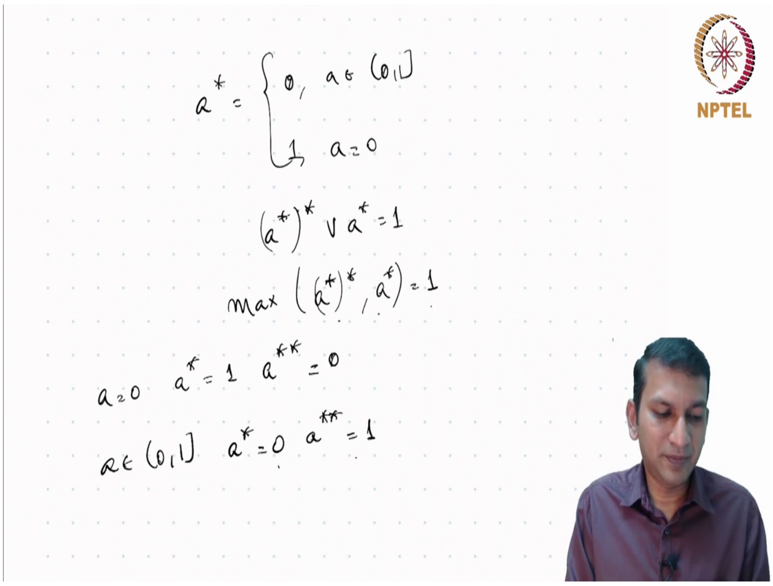

Now, what do we have, for any a star is given like this it is 0 if a belongs to open 0 close to 1 and 1 if a is equal to 0. And what we need to show double star a join b star is equal to 1 this translates as max of a plus star comma a star which is equal to 1 a is 0 and a star is 1 a double star is 0.

So, we substitute here we see that this is 0 this is 1 max of these two is 1. If a belongs to open 0 close to 1 then a star is 0 and a double star, now is actually 1. Once again max of these two is 1. So, clearly this pseudo complement operation satisfies the stone identity. So, it becomes a Stone algebra.

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Well, let us see whether we can get yet another algebraic structure, we are all familiar of this structure the DeMorgan algebra.

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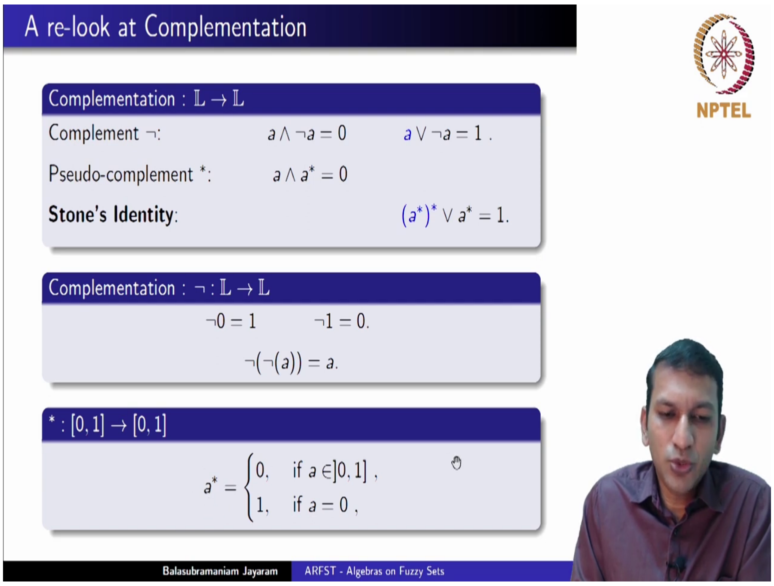

But before that let us look at complementation itself. We have now two different types of complementation complement and the pseudo complement and we also investigated a little bit on the stones identity.

So, we have said that the stones identity differs in the sense that instead of a it asks for double pseudo complementation. Now we said we could look at complementation itself as a unary function from L to L, it is clear that it inverts 0 to 1 and 1 to 0. But if you also insist on this would look like this double negation of a is here this is normally called the involution of the evaluative property.

So, if you have such a compliment operation, then these two are essentially same for a instead of a double stack you could as well use a itself. Of course, here we are talking about complementation it could also be solo complement. Now, since we are the L is fixed as 0 1 the complementation function and we looked at as a unary function from 0 1 to 0.

We have seen that this is a star is a pseudo complement. Now this does not enjoy the property that a double star is a. Why not consider introducing another function, which may enjoy this evaluative property.

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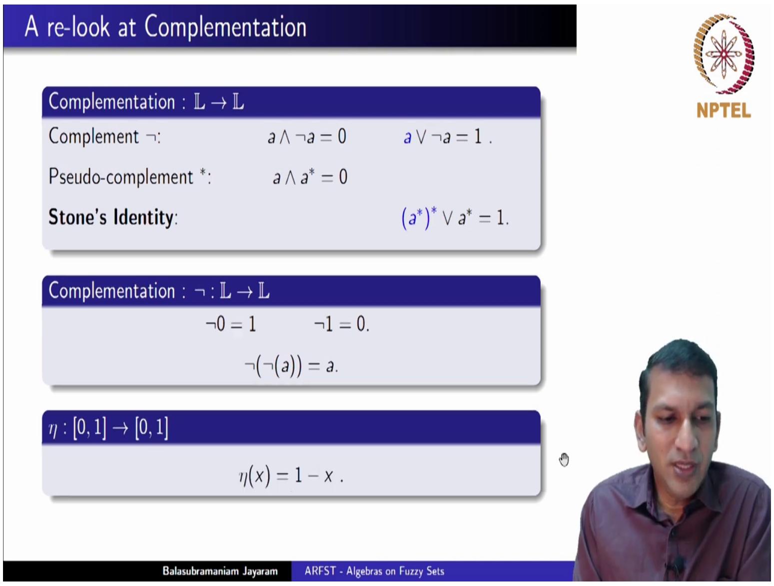

One such function is this eta function which is defined as 1 minus x eta of x is 1 minus x. Note that clearly eta of 0 is 1 eta of 1 is 0 and eta of 1 minus x is 1 minus 1 minus x which is equal to x. So that means, eta of a is actually a.

So, this seems to be a good candidate to be considered for complementation operation with involution.

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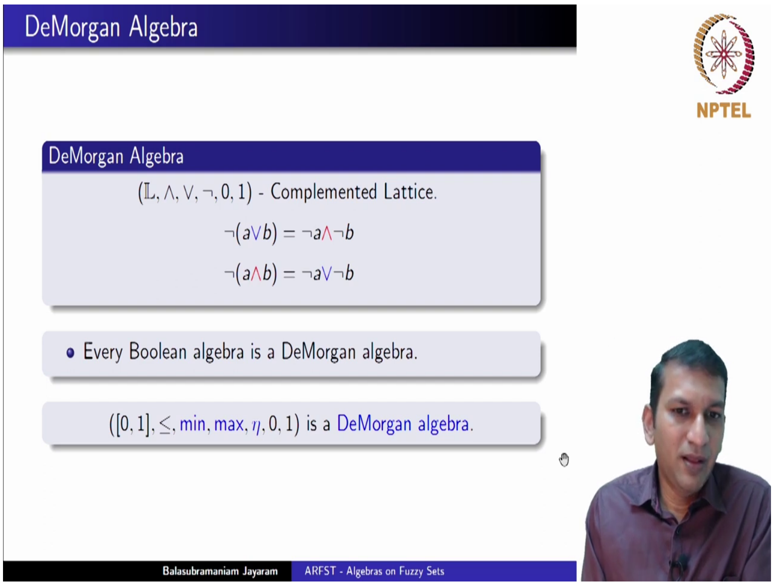

Now what is a DeMorgan algebra? Once again it is a bounded that is a complemented lattice, such that these two identities hold. Now, this is perhaps very common to us from our school days, if you were to think of it in terms of sets if a and b are sets all we are saying is a union b complement is a complement intersection b complement and a intersection b complement is equal to a compliment union b.

Now once again this is also known to us that every Boolean algebra is a DeMorgan algebra we just looked at an example from in the sense of looking at you know pairs of classical sets. So, every Boolean algebra is a DeMorgan algebra.

Now can we at least come up with a DeMorgan algebra on the set of fuzzy sets which means we want to look at the corresponding underlying o domain of the membership function which is once again the 0 1 interval. Let us look at the usual now familiar lattice, but instead of using the pseudo complement as we defined earlier let us use this eta function which is 1- x.

Now, the question is with respect to eta does it satisfy the DeMorgan laws; it can be easily verified that yes with eta both min and max do satisfy this do DeMorgan laws and hence it becomes a DeMorgan algebra.

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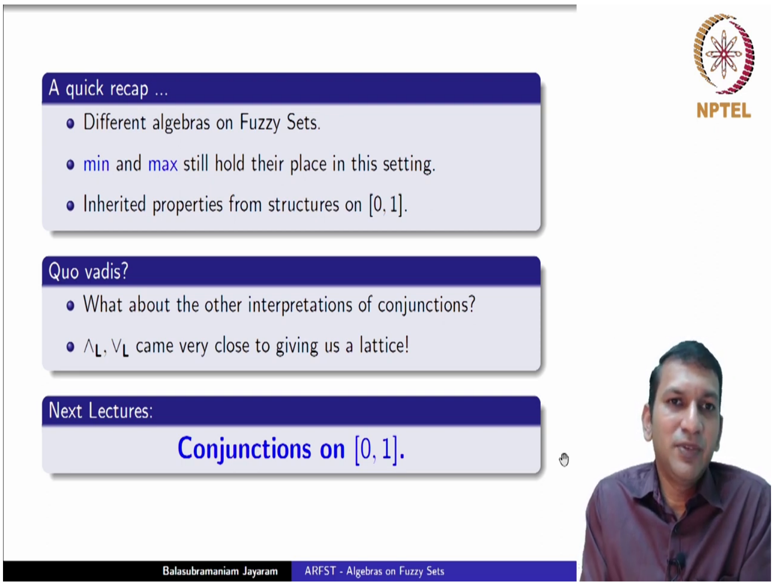

In this lecture we have seen at least two different algebras on the set of fuzzy sets Stone algebra and a DeMorgan algebra. Once again we have seen that min and max still enjoy their place in the setting and many of these results were possible, because we were inheriting the properties from structures that are available on [0, 1].

Where do we go from here what about other interpretations of conjunctions. Note that the Lukasiewicz pair of conjunction and disjunction they came very close to giving us a lattice. They enjoyed complementation they were both commutative and associated it was only at idempotence. They fail, because they were not giving us a lattice should we discard them well definitely not.

They will help us going forward and what kind of structures do they give. So, this becomes the topic for the lectures that will be delivered in the next week, wherein we will specifically look at conjunctions on the unit interval 0 1 which also are conjunctions on the set of fuzzy sets. Even a pair of fuzzy sets the conjunctions that are possible the different interpretations.

And we will look at one particular generalization of a conjunction from the classical to valid logic. And discuss the algebraic analytic and order therapy aspects and come up with results which will be useful later on when we deal with applications or in the theoretical section of setting up applications.

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This book could be a good resource for you if you would like to check out more on the topics that we have touched upon in this lecture. Thank you once again for being with me through this lecture and hope to see you soon in the next lecture.

Thank you.