Measure and Integration

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Lecture-9

2.4 Lebesgue measure: The Ring

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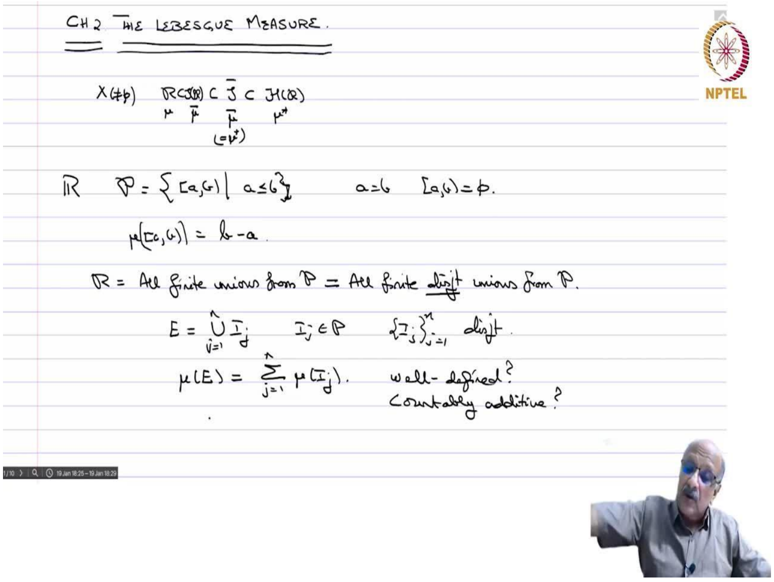

So, we will now start a new chapter: Chapter 2: The Lebesgue measure:

This is the most important measure which we will study in this course. And therefore, we will do it in a little detail. So, remember the Caratheodory thing. So, you start with a non-empty set and then you have a ring and the measure on it, then you go to the hereditary sigma ring, which has a natural outer measure, and then you have the set of all mu star measurable sets and you define the measure on it, which is the same as mu star and then of course, you show that in fact, this contains both R and S of R and therefore, mu bar is an extension.

So, this S bar with $\bar{\mu}$ that gives you a complete measure, and this is the process which you are going to play. So, we are going to start with $\mathbb{R}$ so, we will do everything in $\mathbb{R}$ but you will see immediately how to extend it to $\mathbb{R}^N$ so, in one go, we will finish everything. So, if you take $\mathbb{R}$ recall that we had this example $P = \{[a, b): a \leq b\};\ for\ a = b,\ [a, b) = \phi$.

So, and so, if a equal to b then you say a b is same this is the definition the convention which we are doing and we are going to define on this

$$\mu([a, b)) = b - a$$

and then if you take $R =$ all finite unions from P = all finite disjoint unions from P.

So, if you take $E = \cup_{i=1}^{n} I_j$, $I_j \in P$, $\{I_j\}_{j=1}^{n}$ mutually disjoint.

So, then it is almost obvious because measure has to be countably additive and in particular additives therefore, $\mu(E) = \sum_{j=1}^{n} \mu(I_j)$. But then any set E which is a finite union you may be able to write in more than one way as a finite union of disjoint intervals like this and therefore, we have to show that this is well defined and then we must of course show this is countably additive. If we know these 2 things, then mu will be a measure on the ring and then the Caratheodory method will take over and whatever complete measures we get there that will be called the Lebesgue measure.

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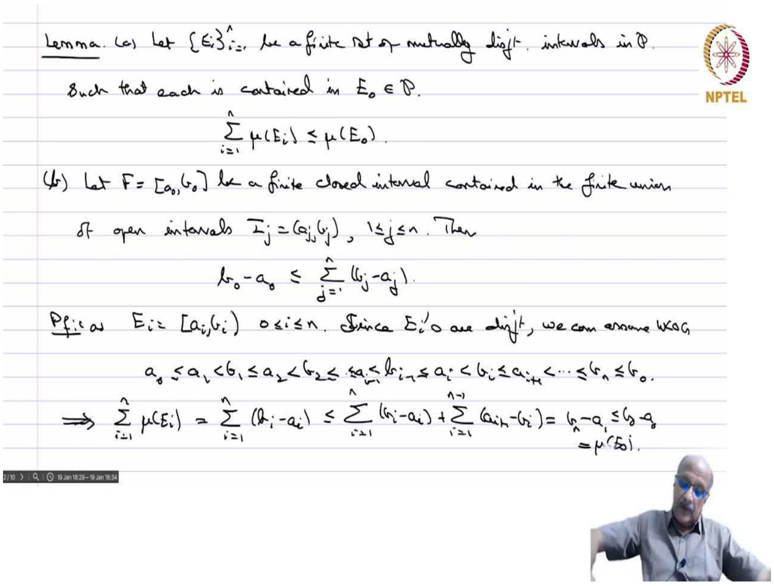

So, a series of technical results which we need to prove are all more or less obvious. So, we will have to write them down carefully, especially the first few results.

Lemma 1: (a) let $\{E_i\}_{i=1}^{n}$ be a finite set of mutually disjoint intervals in P such that each is contained in $E_0 \in P$. Then $\sum_{j=1}^{n} \mu(E_j) \le E_0$.

(b) Let $F = [a_0, b_0]$ closed interval contained in the finite union of open intervals

$I_j = (a_j, b_j),\ 1 \le j \le n$. Then $b_0 - a_0 \le \sum_{j=1}^{n} (b_j - a_j)$.

Proof: (a) So, you write $E_i = [a_i, b_i),\ 0 \le i \le n$. Since E_ii is disjoint we can assume without loss of generality

$$a_0 \le a_1 < b_1 \le a_2 < b_2 \le .... \le a_{i-1} \le b_{i-1} \le a_i < b_i \le a_{i+1} <... \le b_n \le b_0 .$$

So, this implies that

$$\Rightarrow \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n} (b_i - a_i) \le \sum_{i=1}^{n} (b_i - a_i) + \sum_{i=1}^{n+1} (a_{i+1} - b_i) = b_n - a_1 \le b_0 - a_0 = \mu(E_0).$$

So, that proves (a).

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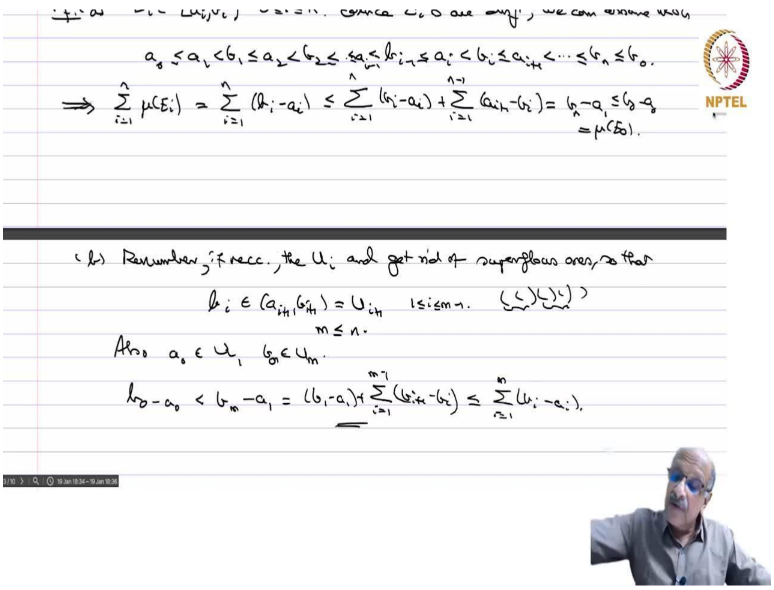

(b) So again renumber, if necessary, the U_i and get rid of superfluous ones, so that you have

$$b_i \in (a_{i+1}, b_{i+1}) = U_{i+1}\ ,\ 1 \le i \le m - 1,\ m \le n.$$

So, we have got rid of some extra ones which may be the repetitions of the sets and so on and that only adds to the right-hand side of this equation here and therefore, getting rid of any set proving it for a smaller number is not a problem that still proves the same thing.

So, and also $a_0 \in U_1$ *and* $b_0 \in U_m$. So again

$$b_0 - a_0 < b_m - a_1 = b_1 - a_1 + \sum_{i=1}^{m-1} (b_{i+1} - b_i) \leq \sum_{i=1}^{m} (b_i - a_i).$$

So, I am writing a (())(11:03) sum and that is less than or equal to sigma i equals 1 to m of bi minus ai because you have that bi is contained in ai plus 1. So, that is why you have this you get this and therefore, that proves this theorem. So, this is the first technical lemma which is really nothing just intuitively if you drew pictures, the proof will be obvious.

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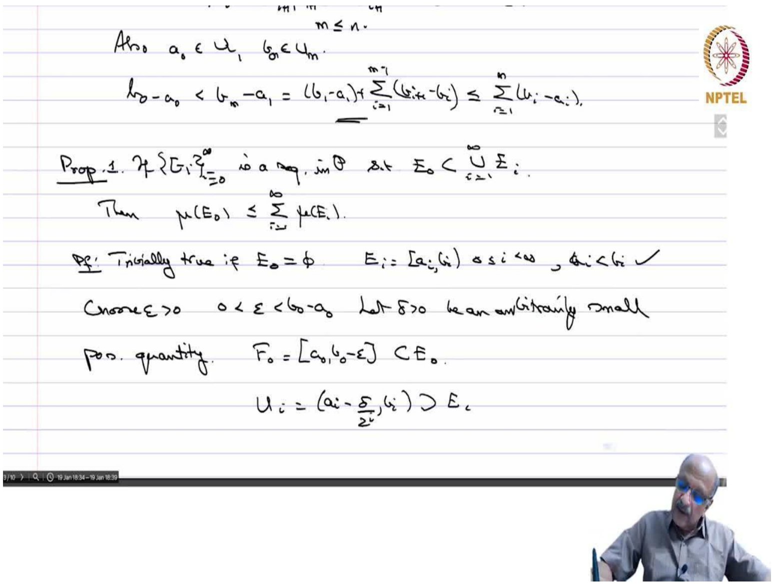

Proposition 1: If $\{E_i\}_{i=1}^{\infty}$ is a sequence in P such that $E_0 \subset \cup_{i=1}^{\infty} E_i$. Then

$$\mu(E_0) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

proof: Trivially true if $E_0 = \phi$. So let us take $E_i = [a_i, b_i)$, $1 \leq i < \infty$, $a_i < b_i$. So, all the intervals are disjoint and you choose $0 < \epsilon < b_0 - a_0$. Let $\delta > 0$ be arbitrarily small. So, then $F_0 = [a_0, b_0 - \epsilon) \subset E_0$ and you define $U_i = (a_i - \frac{\delta}{2^i}, b_i) \supset E_i$.

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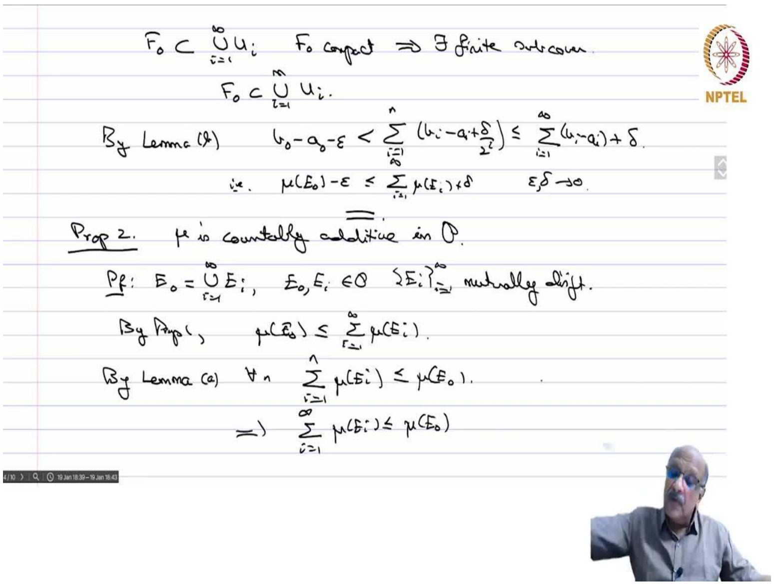

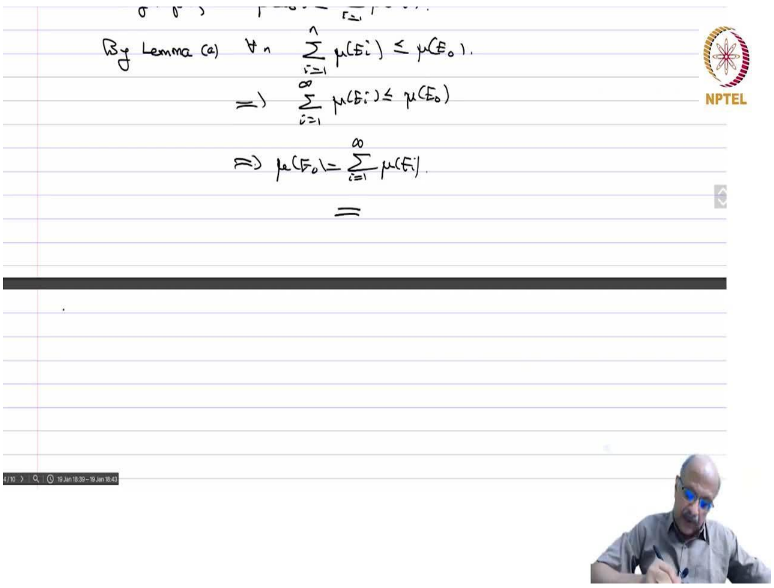

And therefore, $F_0 \subset \cup_{i=1}^{\infty} U_i$ and F_0 is compact and so there exists a finite subcover. So, let us assume that $F_0 \subset \cup_{i=1}^{m} U_i$.

So, Lemma (b), of course, you will get

$$b_0 - a_0 - \epsilon < \sum_{i=1}^{n} (b_i - a_i + \frac{\delta}{2^i}) \leq \sum_{i=1}^{\infty} (b_i - a_i) + \delta.$$

$$i.e.,\ \mu(E_0) - \epsilon < \sum_{i=1}^{\infty} \mu(E_i) + \delta.$$

Letting ϵ, $\delta \to 0$, you get the proposition.

Proposition 2: μ is countably additive in P.

proof: $E_0 = \cup_{i=1}^{\infty} E_i$, E_0, $E_i \in P$, $\{E_i\}_{i=1}^{\infty}$ mutually disjoint.

So, by Proposition 1 we have $\mu(E_0) \le \sum_{i=1}^{\infty} \mu(E_i)$. By Lemma (a) for every n, we have

$$\sum_{i=1}^{n} \mu(E_i) \le \mu(E_0) \Rightarrow \sum_{i=1}^{\infty} \mu(E_i) \le \mu(E_0) \Rightarrow \mu(E_0) = \sum_{i=1}^{\infty} \mu(E_i).$$

So, that proves that it is countably additive in P.

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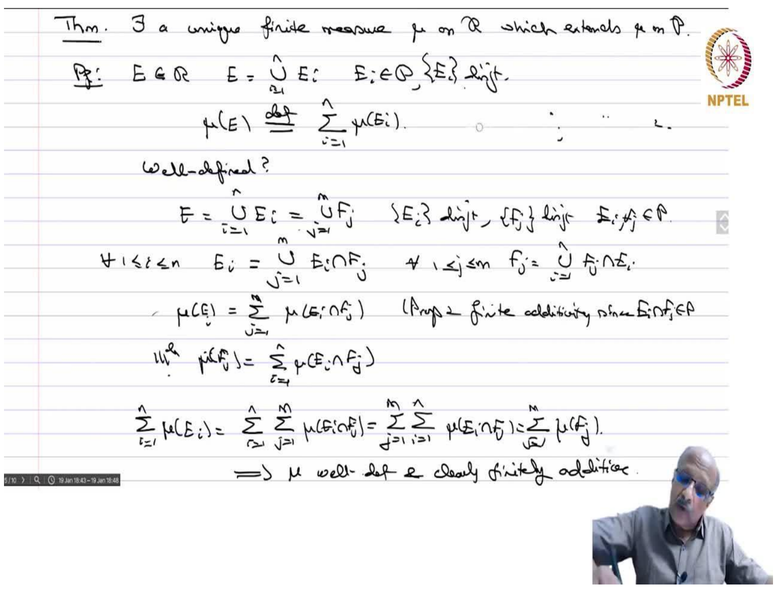

Theorem: There exists a unique finite measure μ on R which extends μ on P.

proof: Let $E \in R$. So, $E = \cup_{i=1}^{n} E_i$, $E_i \in P$, $\{E_i\}_{i=1}^{n}$ is disjoint.

So, we are more or less forced to define so, $\mu(E) = \sum_{i=1}^{n} \mu(E_i)$.

Only we have to check that this is well defined because it may have a different decomposition.

So, we assume so, well defined so, we take $E = \cup_{i=1}^{n} E_i = \cup_{j=1}^{m} F_j$, $\{E_i\}_{i=1}^{n}$, $\{F_j\}_{j=1}^{m}$ are disjoint, E_i, $F_j \in P$.

So, for all $1 \leq i \leq n$, we have $E = \cup_{j=1}^{m} (E_i \cap F_j)$, $\forall\, 1 \leq i \leq m$, $F_j = \cup_{i=1}^{n} (F_j \cap E_i)$.

So, $\mu(E_i) = \sum_{j=1}^{m} \mu(F_j \cap E_i)$. Similarly, $\mu(F_j) = \sum_{i=1}^{n} \mu(F_j \cap E_i)$. So,

$$\sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(F_j \cap E_i) = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(E_i \cap F_j) \sum_{j=1}^{m} \mu(F_j).$$

$\Rightarrow$ μ is well defined and clearly and clearly finitely additive.

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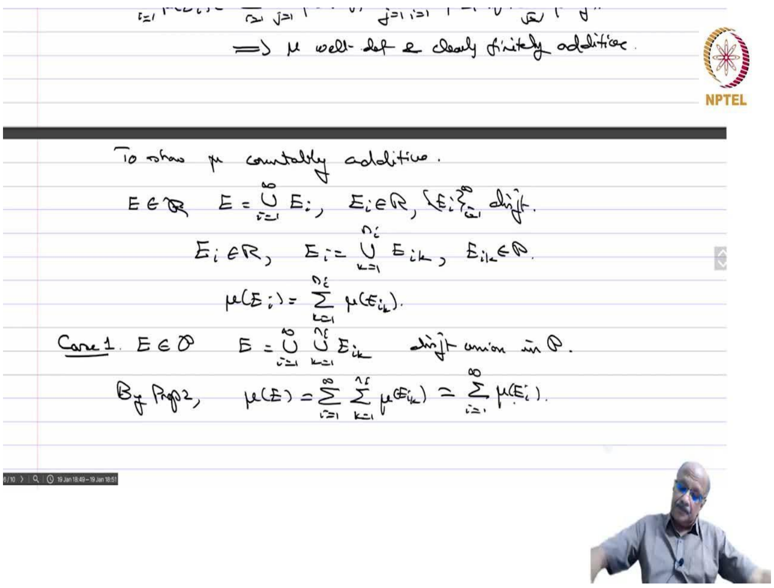

So, now to show mu is countably additive that is all we need to show, so we take $E \in R$ and

$E = \cup_{i=1}^{\infty} E_i$, $E_i \in R$, $\{E_i\}_{i=1}^{\infty}$ disjoint. So, each Ei belongs to R. So,

$E_i = \cup_{k=1}^{n_i} E_{ik}$, $E_{ik} \in P$ and $\mu(E_i) = \sum_{k=1}^{n_i} \mu(E_{ik})$.

case 1: $E \in P$, then of course $E = \cup_{i=1}^{\infty} \cup_{k=1}^{n_i} E_{nk}$, disjoint union in P.

And therefore by proposition 2 we have $\mu(E) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \mu(E_{ik}) = \sum_{i=1}^{\infty} \mu(E_i)$.

Case 2: $E = \cup_{j=1}^{n} F_j$, $F_j \in P$, $\{F_j\}_{j=1}^{m}$ disjoint

Then $F_j = \cup_{i=1}^{\infty} (F_j \cap E_i)$.

Fj equals union i equals 1 to infinity of Fj intersection Ei, so this of course belongs to R each of these belongs to R and this belongs to P. So, by case 1, you have

$$\mu(F_j) = \sum_{i=1}^{\infty} \mu(E_i \cap F_j).$$

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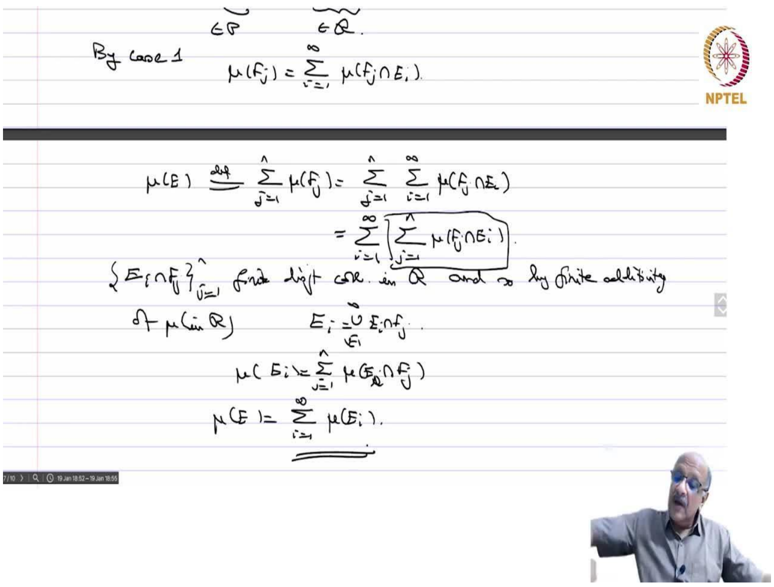

So, $\mu(E) = \sum_{j=1}^{n} \mu(F_j) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \mu(F_j \cap E_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \mu(F_j \cap E_i)$.

mu of E and this is by definition sigma j equals 1 to n mu of Fj equals sigma j equals 1 to n sigma i equals 1 to infinity mu of Fj intersection Ei and now, this is finite sum this is infinite sum everything is non-negative no problems. So, sigma i equals 1 to infinity sigma j equals 1 to n mu of Fj intersection Ei. Ei intersection Fj j equals 1 to n is finite disjoint collection in R and so, by finite additivity of mu in R we have already said clearly finitely additive.

So, by finite additivity in R you have that and you also have Ei equals union J equals 1 to infinity Ei intersection Fj and therefore, you have that mu of Ei equal to sigma j equals 1 to n mu of Ei intersection Fj and that is exactly this quantity here and therefore, you have

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

So, this proves completely that we have a measure on this.

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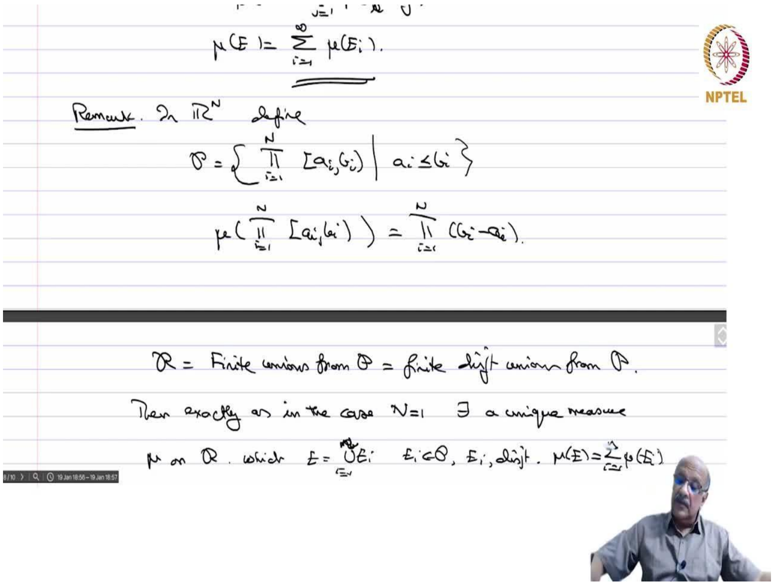

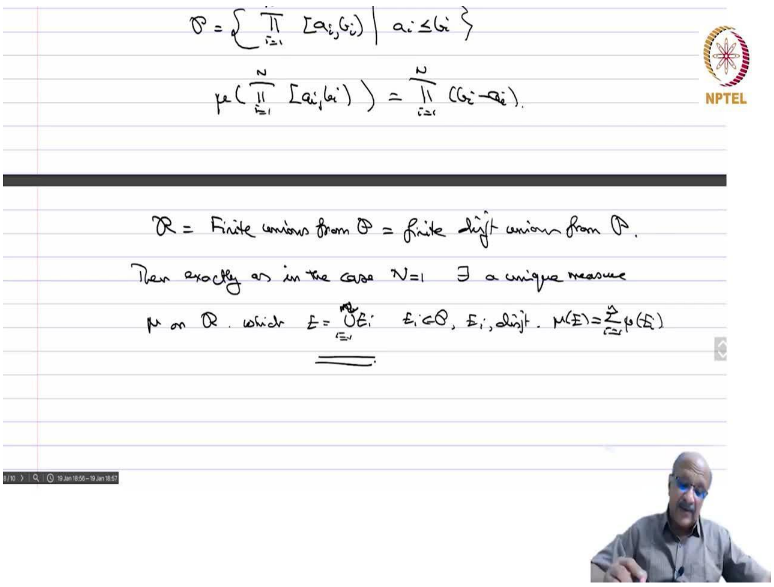

Remark: In $\mathbb{R}^N$, define

$$P = \{\Pi_{i=1}^{n} [a_i, b_i): a_i \le b_i\},\ \mu(\Pi_{i=1}^{n} [a_i, b_i)) = \Pi_{i=1}^{n} (b_i - a_i).$$

And then you will define R= finite unions from P= finite disjoint unions from P.

Then exactly as in the case only your notations competitions are a little messy, but there is no ideological no difference from the arguments or the ideas involved. So, exactly as the case N=1, there exists a unique measure μ on R.

So, this is the sum of all the which is if $E = \cup_{i=1}^{n} E_i$, $E_i \in P$, $\{E_i\}$ disjoint and then you have $\mu(E) = \sum_{i=1}^{n} \mu(E_i)$.

So, now we have a ring, we have a measure and we are ready to apply Caratheodory's method to construct the Lebesgue measures and as I said what we have done in case n equals 1 also we can do similarly in any general dimension and therefore, we will to construct the Lebesgue measures in all dimensions next time.