Measure and Integration
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Lecture- 8
2.3 - Exercises

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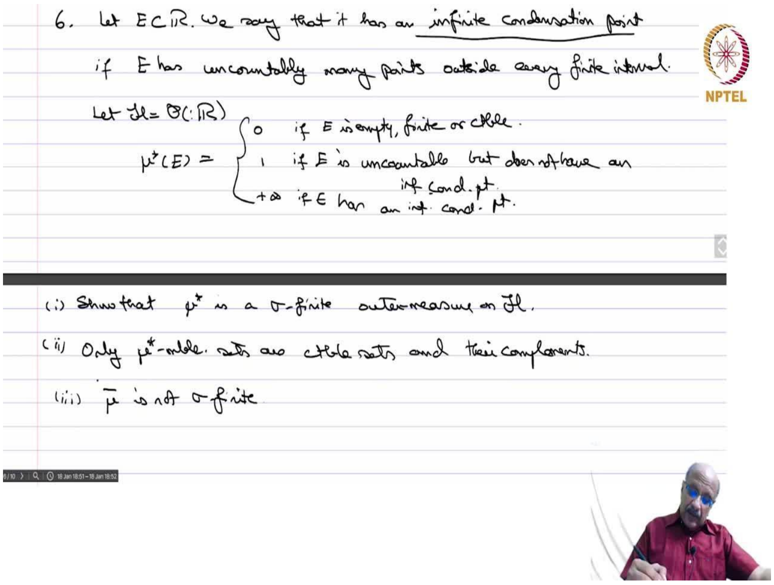

(6) So, let $E \subset \mathbb{R}$. We say that it has an infinite condensation point if E has uncountably many points outside every finite interval.

So, now, let $H = P(\mathbb{R})$ and you define

$\mu * (E) = 0,$ if E is empty finite or countable,

$= 1,$ if E is uncountable but does not have an infinite condensation point,

$= + \infty,$ if E has an infinite condensation point.

(i) show that μ * is σ-finite outer measure on H.

(ii) only μ * measurable sets are countable sets and their complement.

(iii) $\bar{\mu}$ is not σ-finite.

So you are taking outer measure on the power set and these are the only mu star measurable sets and you know that you have an outcome measure and mu star measurable sets, the mu stuff becomes a measure on that set and that measure we want to show is not sigma finite. So, this is contrast to what we did earlier we started with a ring which is sigma finite and went through the Caratheodory reconstruction that if the ring were sigma finite, everything is sigma finite here we do not be directly about measures on the sigma ring and it does not mean necessarily even if it is sigma finite, it does not mean that $\bar{\mu}$ will be sigma favorite.

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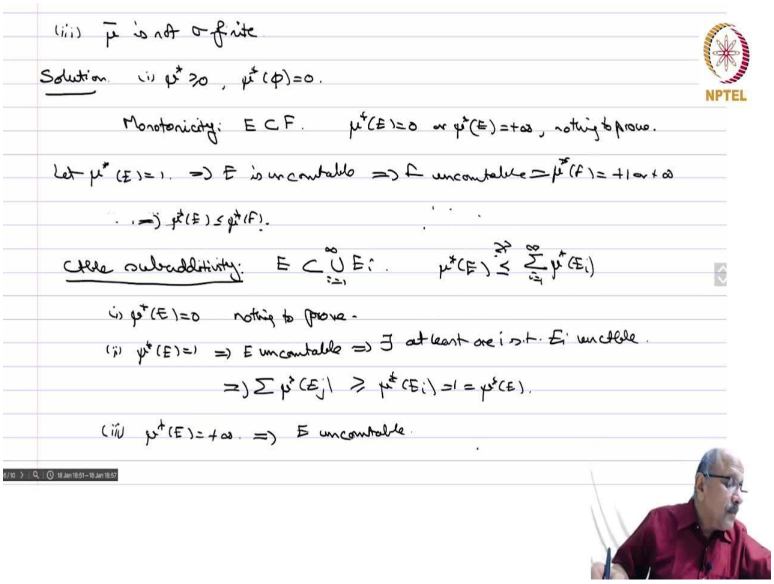

Solution: (i) $\mu^* \geq 0,\ \mu^*(\phi) = 0.$

Monotonicity: $E \subset F$. If $\mu^*(E) = 0$ *or* $\mu^*(E) = +\infty$, nothing to prove !

So, let us assume that $\mu^*(E) = 1 \Rightarrow E$ is uncountable $\Rightarrow F$ is also uncountable $\Rightarrow \mu^*(E) = 1$ *or* $+\infty$.

$$\Rightarrow \mu^*(E) \leq \mu^*(F).$$

So, now, we have countable subadditivity. So, let us take $E \subset \cup_{i=1}^{\infty} E_i$.

(i) If $\mu^*(E) = 0$, nothing to prove!

(ii) If $\mu^*(E) = 1 \Rightarrow E$ is uncountable$\Rightarrow$ there exists at least one i s.t. E_i is uncountable.

$$\Rightarrow \sum_j \mu^*(E_j) \geq \mu^*(E_i) = 1 = \mu^*(E).$$

(iii) Suppose $\mu^*(E) = +\infty \Rightarrow E$ is uncountable. In particular, it means much more and so, we are to look at various possibilities.

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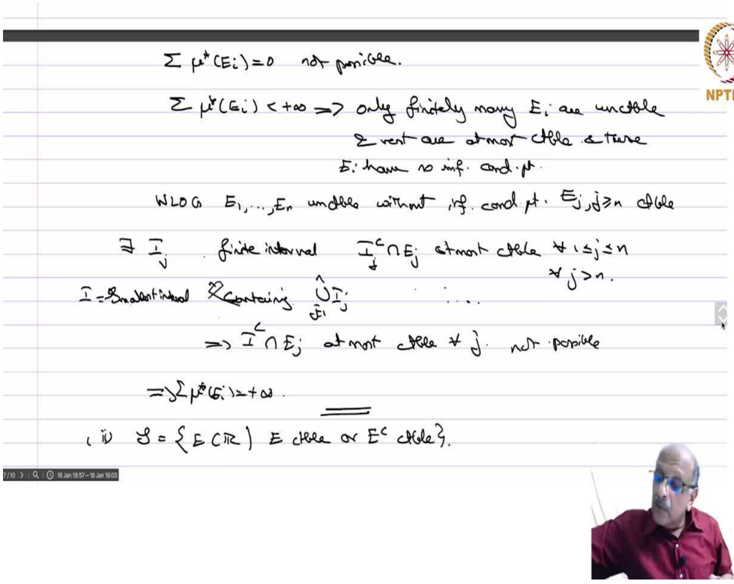

$\sum \mu * (E_i) = 0$, not possible ! because all the $\mu * (E_i)$ will be 0 that means all the E_i are at most countable and therefore, the union will be countable which is not possible because it contains an uncountable set.

$\sum \mu * (E_i)\backslash < \infty \Rightarrow$ only finitely many E_i are uncountable and rest are at most countable and this E_i have no infinite condensation point. So, let us assume without loss of generality $E_1, E_2, ..., E_n$ uncountable without inf condensation point and E_j, $j \geq n$ is countable. So, there exists finite interval I_j such that $I_j^c \cup E_j$, $\forall\, 1 \leq j \leq n$, is countable. So, now you are taking $\cup_{j=1}^{n} I_j$ and then take I = smallest interval containing union $\cup_{j=1}^{n} I_j$.

$\Rightarrow I^c \cup E_j$ is at most countable for all $j \geq n$ and this is not possible.

$\Rightarrow \sum \mu * (E_i) <+ \infty$.

So, second part: we want to show that $S = \{E \subset \mathbb{R}\text{: } E \text{ or } E^c \text{ is countable}\}$.

So, we want to show that these are the only mu star measurable sets.

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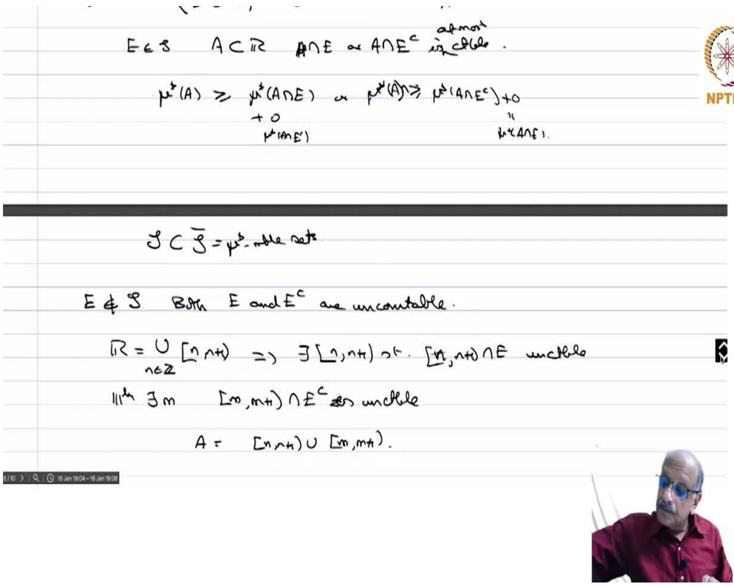

So, if $E \in S$ and $A \subset \mathbb{R}$, then $A \cup E$ *or* $A \cup E^c$ is at most countable and therefore,

$$\mu*(A) \geq \mu*(A \cup E) \text{ or } \mu*(A) + 0 \geq \mu*(A \cup E^c) + 0.$$

So, $S \subset \overline{S} = \mu$ *-measurable sets.

Suppose $E \notin S$, that means both E and E^c are uncountable. So, you take

$$\mathbb{R} = \cup_{n \in \mathbb{Z}}[n, n+1) \Rightarrow \exists\, [n, n+1)\ s.t.\ [n, n+1) \cap E \text{ uncountable.}$$

Similarly, there exists an m such that $[m, m+1) \cap E^c$ is uncountable. So, let

$$A = [n, n+1) \cup [m, m+1) \subset \text{a finite interval.}$$

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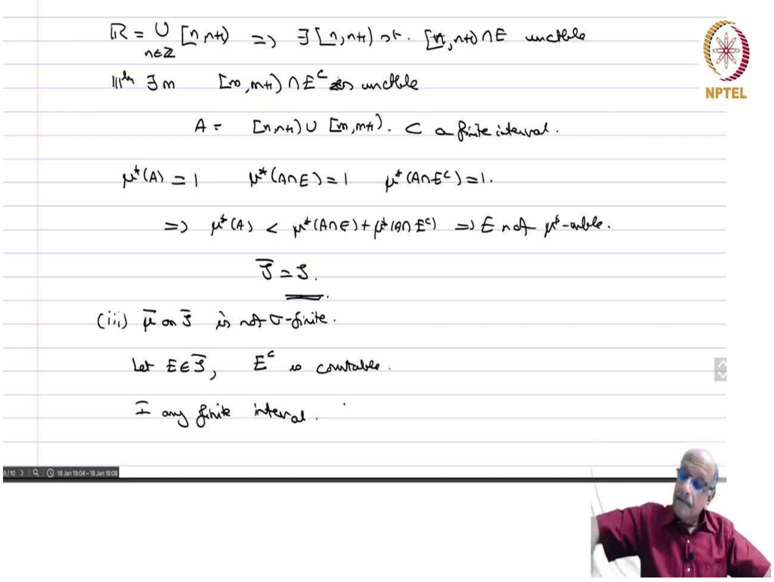

Then mu star of A is contained in a finite interval again which is the bigger of the two if you like or I mean take the union and then you put them in n to m plus 1 for the instance if n is smaller, so, that this is contained in a finite interval and therefore, it is an uncountable set but it does not have an infinite condensation point.

So, $\mu * (A) = 1$, but $\mu * (A \cap E) = 1$ and $\mu * (A \cap E^c) = 1$.

$\Rightarrow \mu * (A) < \mu * (A \cap E) + \mu * (A \cap E^c)$

$\Rightarrow E$ is not μ *-measurable.

$\Rightarrow \bar{S} = S$.

(iii) $\bar{\mu}$ on $\bar{S}$ is not σ-finite. Let $E \in \bar{S}$ such that E^c is countable. Let I be any finite interval.

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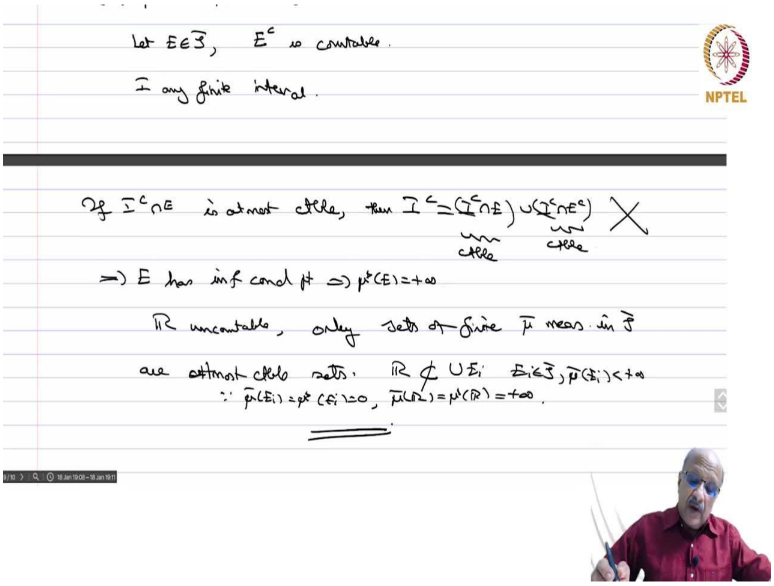

So, if $I^c \cap E$ is at most countable, then $I^c = (I^c \cap E) \cup (I^c \cap E^c)$, so you have a contradiction. So, therefore, E has an infinite condensation point $\Rightarrow \mu * (E) = + \infty$.

Therefore, for example, if you take ℝ, which is uncountable, only sets of finite $\overline{\mu}$ measure in $\overline{S}$ are at most countable sets because any infinite set which is a complement of a countable set that will have infinite measure and therefore, ℝ cannot be written as union $\cup_{i=1}^{\infty} E_i$, $E_i \in \overline{S}$, $\overline{\mu}(E_i) < + \infty$.

All of these will only be 0 this side will be infinity. So, $\overline{\mu}(E_i) = \mu * (E_i) = + \infty$.

So, you cannot have a cover because this side is infinity, this side is 0 that is not possible by Monotones. So, with that we will conclude this section. And so next time we will start the construction of the Lebesgue measure.