Measure and Integration
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Lecture No-79
12.5 – Change of variable

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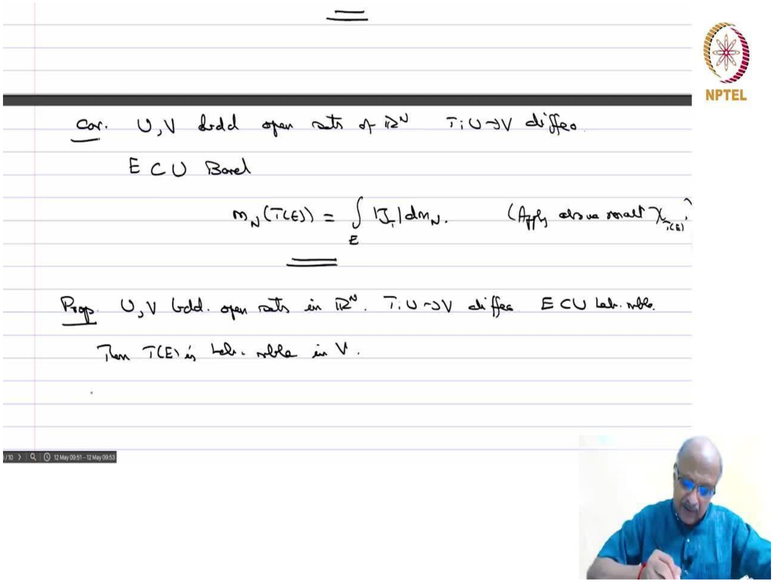

Proposition: U, V bounded open sets in $\mathbb{R}^N$, T: $U \rightarrow V$ diffeomorphism, $E \subset U$ Lebesgue measurable, then $T(E)$ is Lebesgue measurable in V.

So, this is now, converse is obviously true U apply it to T inverse, so, if something is Lebesgue measurable in V then its inverse image will be Lebesgue measurable in U. So, we did it for Borel sets we knew. So, now, we are proving Lebesgue measurable sets.

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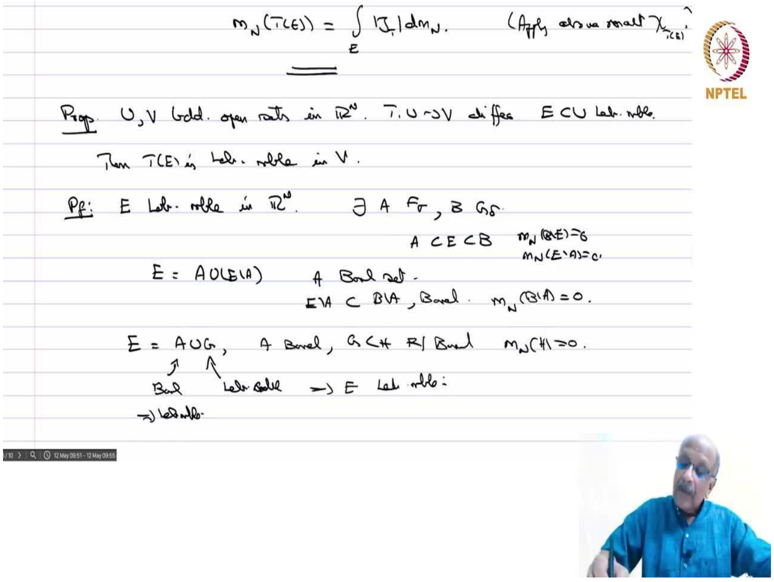

Now, E Lebesgue measurable in R N, so, what do you mean by this. Then there exists A F_{σ}, B G_{δ}, so, set $A \subset E \subset B$ and $m_N(B \setminus E) = 0$, $m_N(E \setminus A) = 0$. And therefore, you can write $E = A \cup (E \setminus A)$ and this A is of course a Borel set and E minus A is contained in B minus A which is Borel and mN of B minus A is equal to 0.

Now, conversely, if you have $E = A \cup G$, where A Borel and $G \subset H$, H Borel and $m_N(H)$ =0. Then A is of course, Borel, so Lebesgue measure Borel implies Lebesgue measurable. And G is a subset of a set of measures 0 and therefore, this is Lebesgue measurable by completeness. Therefore, E is Lebesgue measurable.

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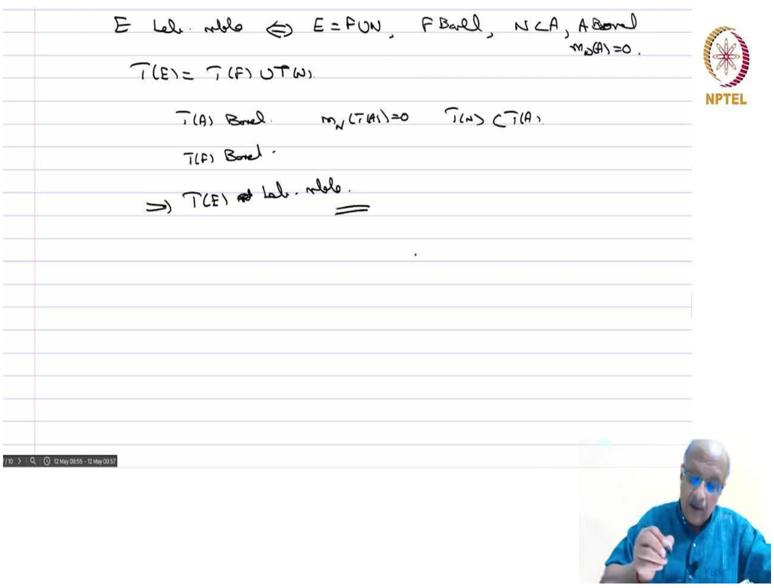

So, another characterization of Lebesgue measurable sets. So, E Lebesgue measurable if and only if $E = F \cup N$, F Borel, $N \subset A$, A Borel, $m_N(A) = 0$. So, it is just a Borel set plus a subset of a set of measure 0, then these are all Lebesgue measurable sets. So now, so you write E in this fashion, then $T(E) = T(F) \cup T(N)$. Now, $T(A)$ is Borel and we have already seen that $m_N(T(A)) = 0$, that corollary we have already proved because A has measures 0 and $T(N) \subset T(A)$. So, T(F) is Borel. So, this $T(E)$ union of a Borel set and a subset of measure 0, so this implies $T(E)$measurable Lebesgue measure.

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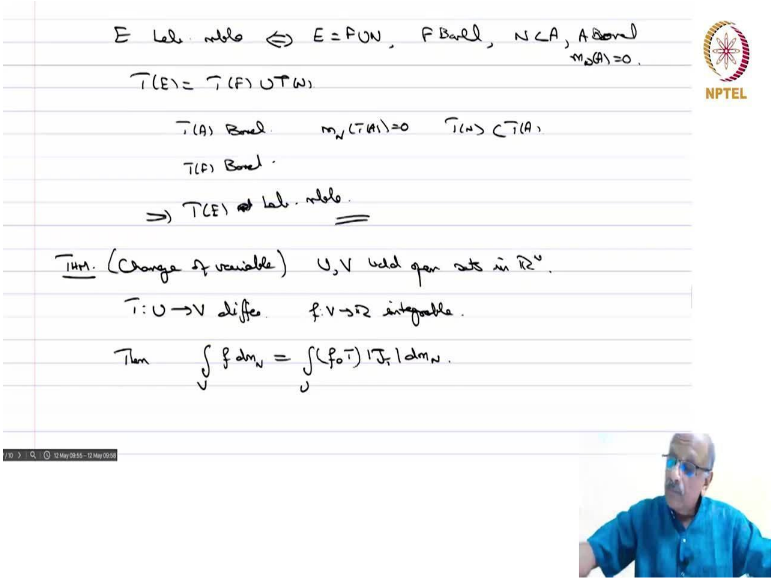

So, finally, we have this following theorem.

Theorem: (change of variable) U, V bounded open sets in $\mathbb{R}^N$, T: $U \rightarrow V$ diffeomorphism, $f: V \rightarrow \mathbb{R}$ integrable. Then

$$\int_V f dm_N = \int_U (f \circ T)|J_T|\, dm_N.$$

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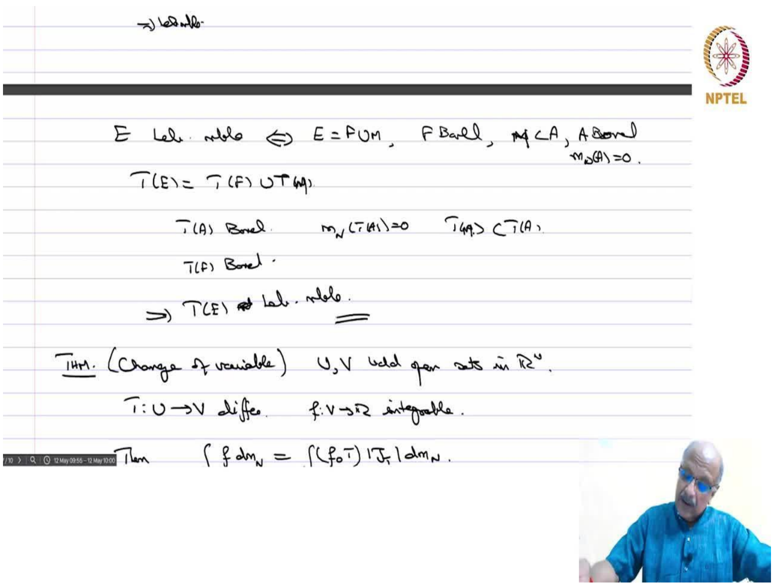

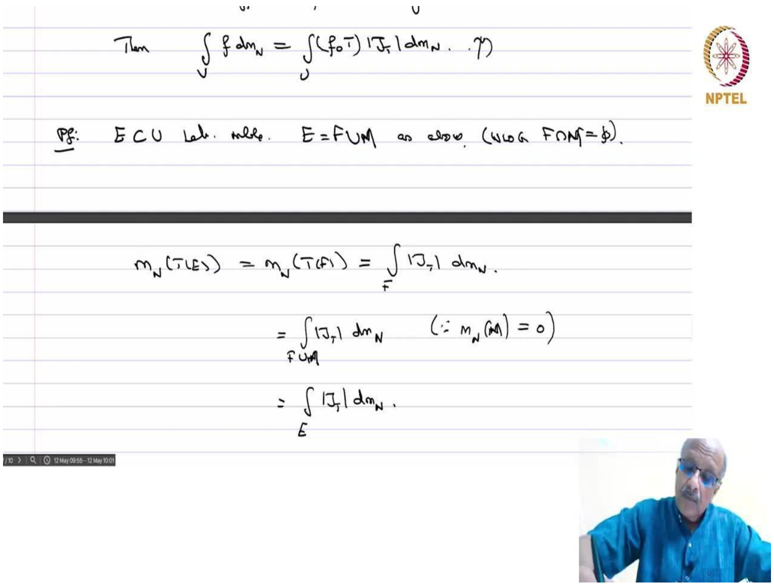

Proof. So, $E \subset U$ Lebesgue measurable, but we write $E = F \cup M$ as above, so, F is Borel and this N is a subset of a Borel set of measure 0 as above. And without loss of generality you can assume F intersection N is empty. So,

$$m_N(T(E)) = m_N(T(F)) = \int_F |J_T|\, dm_N = \int_{F \cup M} |J_T|\, dm_N = \int_E |J_T|\, dm_N.$$

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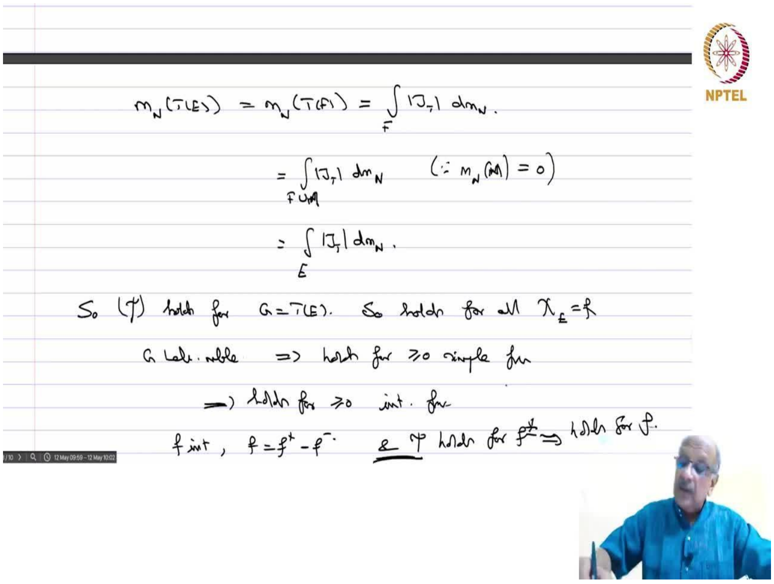

So, dagger holds for, so $G = T(E)$. So, holds for all chi G equals to f, G Lebesgue measure, implies holds for non-negative simple functions, implies holds for non-negative Lebesgue measurable functions, non negative integrable functions. And then f integrable you write

$f = f^{+} - f^{-}$ and dagger holds for f plus minus, implies holds for f. So, that gives you the change of variable formula for this.

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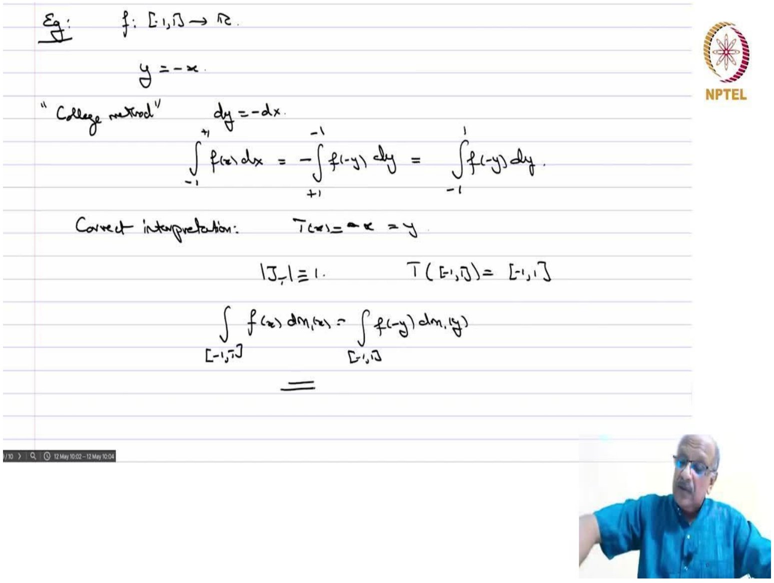

Examples: So, the first example is something which we do in college, so $f: [-1, 1] \rightarrow \mathbb{R}$. So, we change the variable y=-x, then the interval remains the same. Now how would you do in college? So, college method. So, you write dy equals minus dx and therefore, you write

$$\int_{-1}^{1} f(x)dx = -\int_{+1}^{-1} f(-y)dy = \int_{-1}^{1} f(-y)dy.$$

But the correct interpretation is: So, $T(x) = -x = y$, $|J_T| = 1$, $T([-1, 1]) = [-1, 1]$.

And therefore, $\int_{[-1,1]} f(x)dm_1(x) = \int_{[-1,1]} f(-y)dm_1(y)$.

So, this is how we write it, and so, this is the correct way of interpreting this thing. So, the Jacobian is plus 1 and so, this thing is that all ad hoc rules when you change the limits of the integral to be the minus sign and so on.

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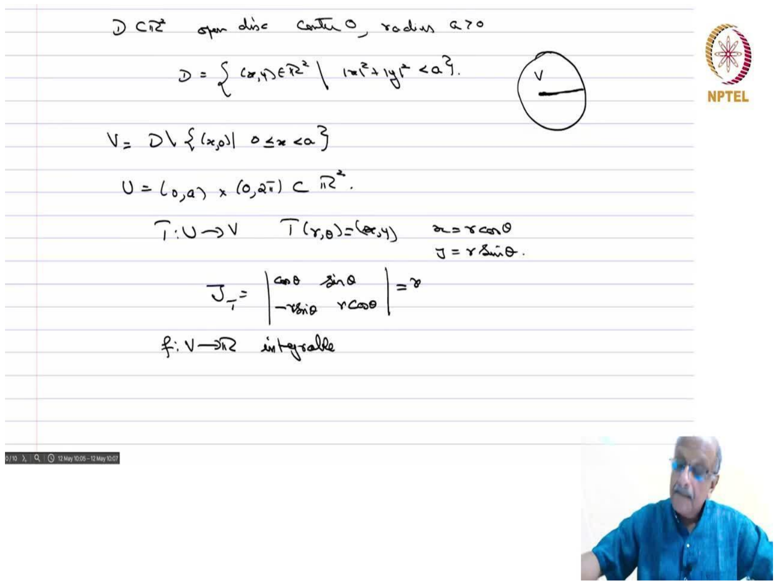

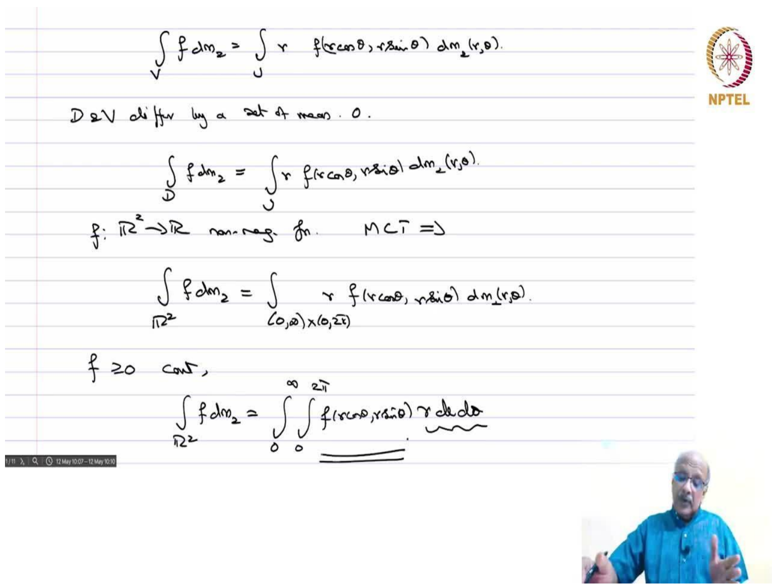

Example: these are about the polar coordinates. So, $D \subset \mathbb{R}^2$ open disc, center 0, radius say a positive. So, $D = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < a\}$. Now, you take $V = D \backslash \{(x, 0) : 0 \leq x < a\}$. So, you are taking the disc here and you are removing this radius here and that is the meaning set V, $U = (0, a) \times (0, 2\pi) \subset \mathbb{R}^2$. Then T from U to V, so, $T(r, \theta) = (x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. So, this diffeomorphism between the 2 sets and then you have JT is the determinant of cos theta and then sine theta when minus r sine theta and r cos theta and that gives you r.

So, if f from V to R integrable, then you have

$$\int_V f dm_2 = \int_U r f(r\cos\theta, r\sin\theta) dm_2(r,\theta).$$

Now, D and V differ by a set of measure 0 and therefore,

$$\int_D f dm_2 = \int_U r f(r\cos\theta, r\sin\theta) dm_2(r,\theta).$$

So, now, if f from R 2 to R, so, non-negative function then monotone convergence theorem implies, $\int_{\mathbb{R}} f dm_2 = \int_{(0,\infty)\times(0,2\pi)} r f(r\cos\theta, r\sin\theta) dm_2(r,\theta)$.

So, in particular $f \geq 0$ and continuous then you have

$$\int_{\mathbb{R}} f dm_2 = \int_0^{\infty}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r dr\, d\theta.$$

This is the familiar change of variable formula which you know and that comes precisely because the Jacobean has set it. And then if you have an integrable function you apply f plus and f minus and U too.

So, this is about how the change varies. Similarly, you can try your hand at writing down the expression for the you will see that you will get a new right to polar coordinates in 3 dimensions cylindrical or spherical, you will get precisely whatever you have been doing in your calculus courses, because the Jacobian will be exactly what you have there. So, that is why there is no mystery as to why you get these numbers $rdr\, d\theta$Why do you write that, there is no mystery, because of the Jacobean. So, with this, I conclude this course. I hope you enjoyed it and I hope it was a good learning experience for you. Thank you very much.