Measure and Integration
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Lecture No-78
12.4 – Change of variable

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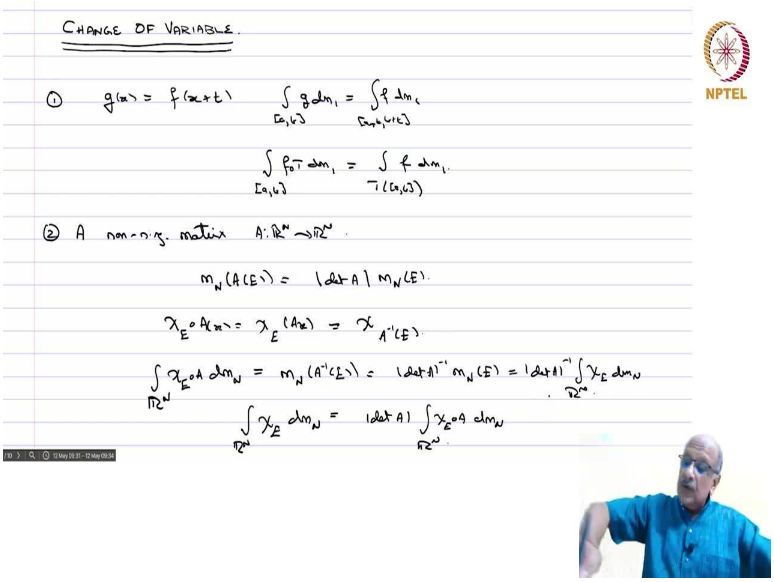

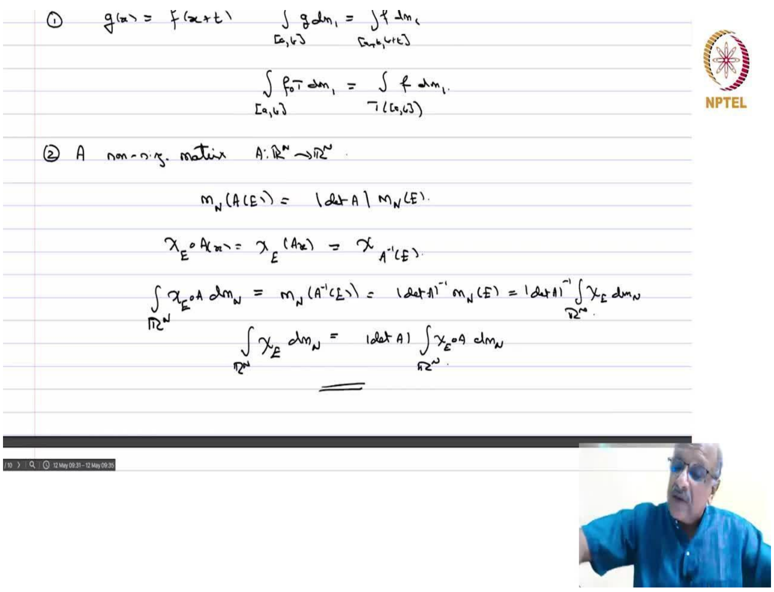

The last topic I want to discuss is the change of variables. For lack of time, I cannot give you all the details, but I hope I will give you a sufficient amount and give you references for how to do it. So, let us take for example.

(1) Take $g(x) = f(x + t)$, where t is a fixed term. Then you will know that

$$\int_{[a,b]} g\,dm_1 = \int_{[a+t,b+t]} f\,dm_1.$$

Now, this is the same as saying that $\int_{[a,b]} f \circ T\, dm_1 = \int_{T([a,b])} f\,dm_1$.

The other example, which we have already seen is when A is a nonsingular matrix or nonsingular linear transformation. So, A defines a linear transformation R N to R N, and then you saw that $m_N(A(E)) = |\det(A)| m_N(E)$.

So, now, $\chi_E \circ A(x) = \chi_E(A(x)) = \chi_{A^{-1}(E)}$. And therefore,

$$\int_{\mathbb{R}^N} \chi_E \circ A dm_N = m_N(A^{-1}(E)) = |\det(A)|^{-1} m_N(E) = |\det(A)|^{-1} \int_{\mathbb{R}^N} \chi_E dm_N.$$

$$\Rightarrow \int_{\mathbb{R}^N} \chi_E dm_N = |\det(A)| \int_{\mathbb{R}^N} \chi_E \circ A dm_N.$$

And using this of course, you can now proceed by for simple functions and then non negative functions and integrable functions and so on and we can rewrite the write the formula so, our aim in this section is to in this chapter is to generalize this to arbitrary transformations of course, not all arbitrary transformations are possible. So, we will see what.

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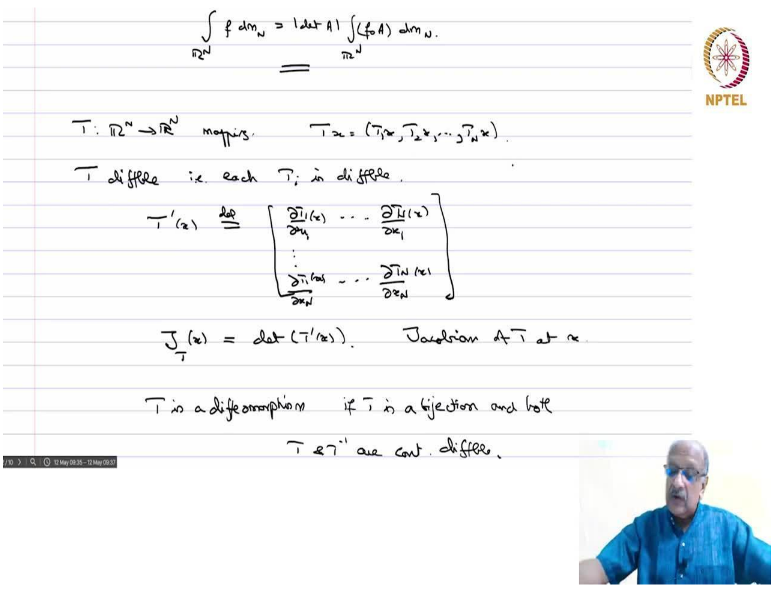

So, the previous one as I said, So, integral R N of f d mN will be equal to a mod determinant of A times integral over R N f composed with A d mN. So, this was how.

So, we want to generalize this to arbitrary. So, let us take $T: \mathbb{R}^N \to \mathbb{R}^N$ mapping. So,

$T(x) = (T_1(x),....., T_N(x))$. So, if T is differentiable that is each Ti is differentiable when we say

$$T'(x) = \begin{matrix} \frac{\partial T_1(x)}{\partial x_1}, ..., & \frac{\partial T_N(x)}{\partial x_1} \\ \frac{\partial T_1(x)}{\partial x_N}, ..., & \frac{\partial T_N(x)}{\partial x_N} \end{matrix}$$

And then we say JT aTx this is called the determinant of T prime x, and this is called the Jacobian of T and x. So, T is diffeomorphism if T is a bijection and both T and E inverse are continuously differentiable. So, if Ti is C1 that means continuously differentiable, that is the derivative is continuous then we say that function is continuously differentiable.

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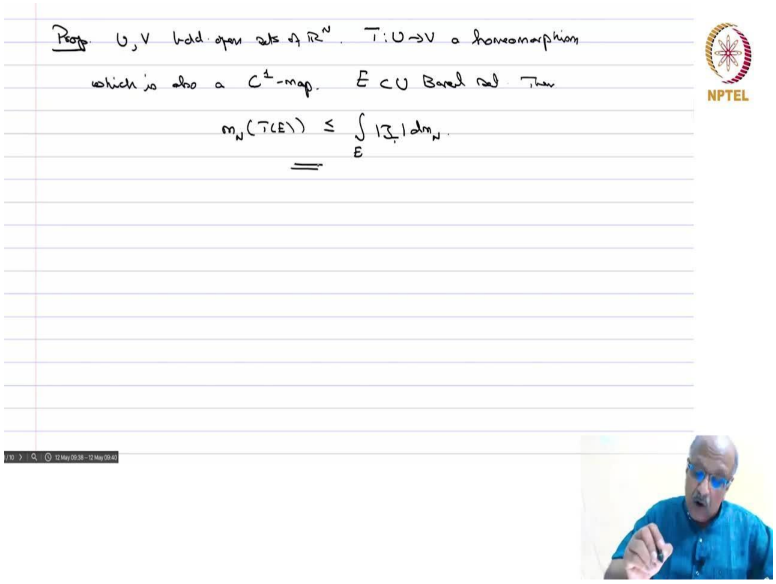

Proposition: So, U, V bounded open sets of $\mathbb{R}^N$, $T: U \to V$ a homeomorphism which is also a C^1 map that means, $E \subset U$ Borel set, then

$$m_N(T(E)) \leq \int_E |J_T| dm_N.$$

So, if you see in the case of linear transformation mod derivative of a linear transformation is the same matrix itself and you will have mod determinant of A which is the matrix which will come out and you get integrally and M and E, integrals over E d mN is nothing but A mN of E. So, this is exactly the generalization only you have now less than or equal to sign because of this.

So, this is the proposition from which we will start. The proof of this is somewhat TDS, very technical and very delicate also and long and therefore, we have been omitted.

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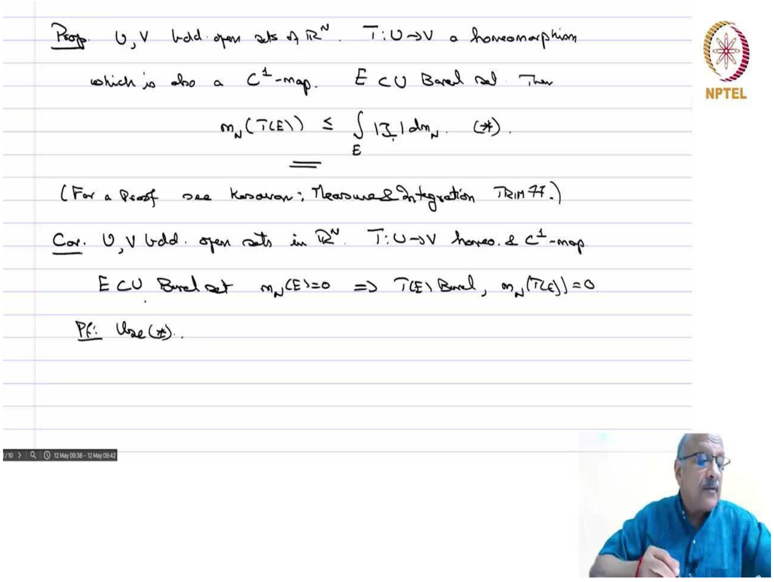

So, for proof see my book Kesavan: Measure and integration Trim 77, so you can see a full proof of this inequality there.

Corollary: So, U and V bounded open sets in $\mathbb{R}^N$, $T: U \rightarrow V$ homeomorphism and C^1 map. E contained in U Borel set, $m_N(E) = 0 \Rightarrow T(E)\ Borel\ and\ m_N(T(E)) = 0$.

proof: Use (*).

So, that is the corollary which you have here.

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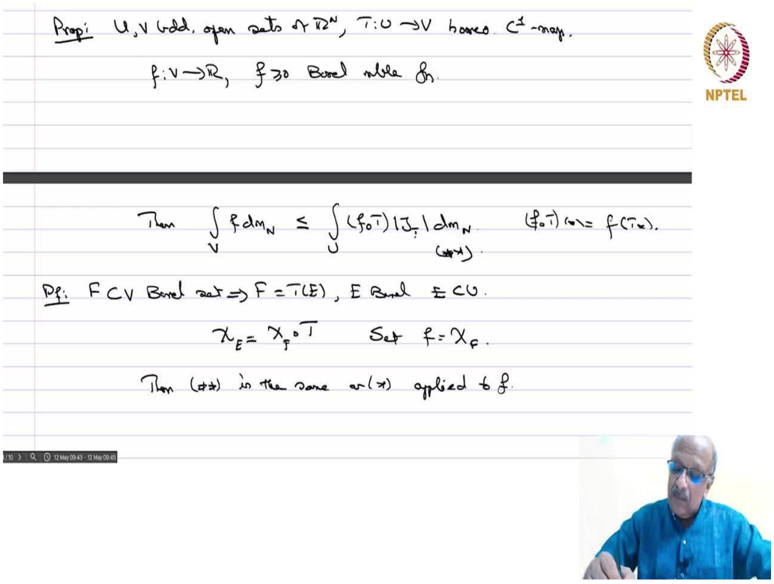

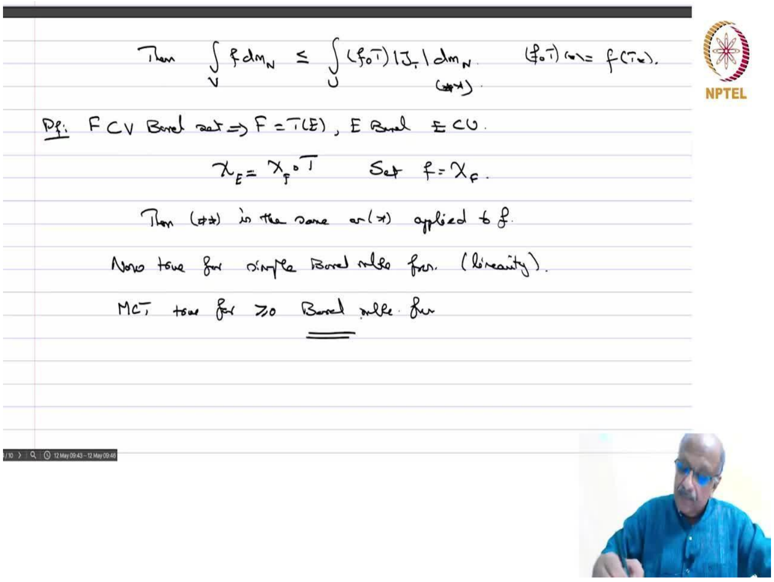

Proposition: U, V will be bounded open sets of $\mathbb{R}^N$, $T: U \to V$ is a homeomorphism and C^1 map, $f: V \to \mathbb{R}$ non-negative Borel measurable function then

$$\int_V f dm_N \leq \int_U (f \circ T)\, |J_T| dm_N.$$

So, if composed with T aTx you know is f of T of x. So, this is the so, we call this double star.

proof: F contained in E Borel set, then F will be equal to TE, E Borel E contained in Y. Then chi E we have just seen is nothing but chi f composed with T, because chi f of Tx that means Tx belongs to F that means x belongs E, so that means chi E is equal to chi f composed with T.

And then set f equal to chi E, then double star is the same as star applied to f, we have f of E mN of E is less than or equal to mN of T is the integral TE JT of mN. So, this is exactly the relationship which we have here. So, f is equal to chi of E and therefore, this is and then f composed with T, but not satisfied this is applied to chi of F. So, applied to chi of F, so, then you have mN of F which is mN of T of E is less than equal or equal to f composed T which is chi of E.

So, integral over E and then mod JT d mN. So, double star is the same as the star applied to chi of F. So, now true for simple Borel measurable functions then the monotone convergence theorem is true for non-negative Borel measurable functions, this is just linearity of the integral and therefore this is true.

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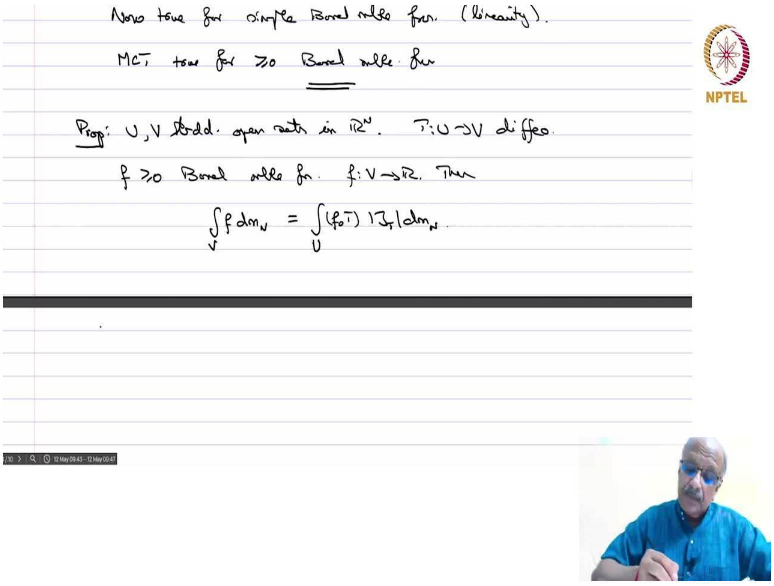

Proposition: U, V bounded open sets in $\mathbb{R}^N$, $T: U \to V$ diffeomorphism, $f \geq 0$ Borel measurable function. Then

$$\int_V f\, dm_N = \int_U (f \circ T)\, |J_T|\, dm_N.$$

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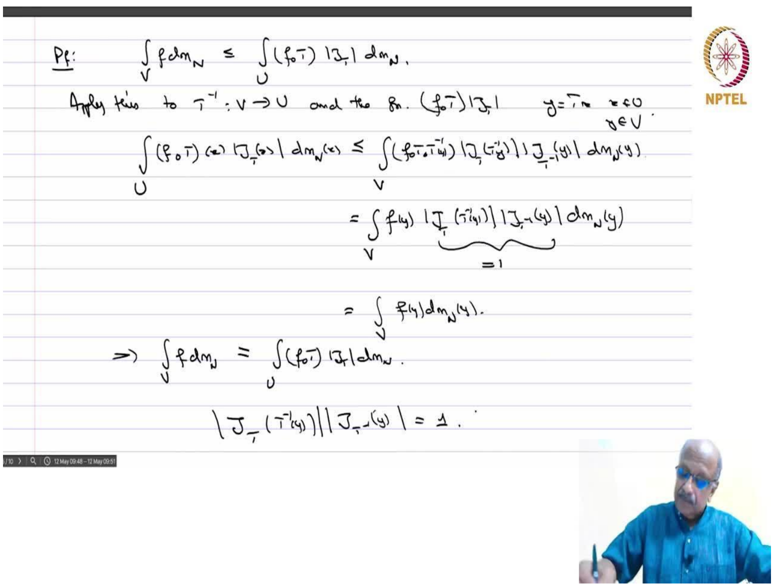

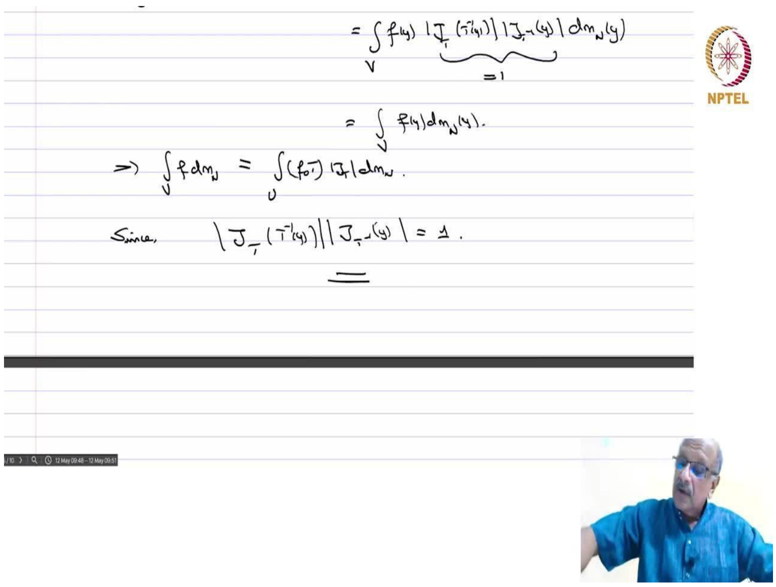

proof: We already know that

$$\int_V f dm_N \leq \int_U (f \circ T)\, |J_T|\, dm_N.$$

Now apply this to $T^{-1}: V \rightarrow U$ and the function $(f \circ T)\, |J_T|$. So, what do you get? So, you put y equals T of x, x is in U, y is in V. So, you have that

$$\int_U (f \circ T)(x)\, |J_T(x)|\, dm_N(x) \leq \int_V (f \circ T \circ T^{-1}(y)) |J_T(T^{-1}(y))| |J_{T^{-1}}(y)| dm_N(y)$$

$$= \int_V f(y)\, |J_T(T^{-1}(y))| |J_{T^{-1}}(y)| dm_N(y) = \int_V f(y)\, dm_N(y).$$

$$\Rightarrow \int_V f dm_N = \int_U (f \circ T)\, |J_T|\, dm_N \ \ , \ since \ |J_T(T^{-1}(y))| |J_{T^{-1}}(y)| = 1.$$

So, that proves the result.

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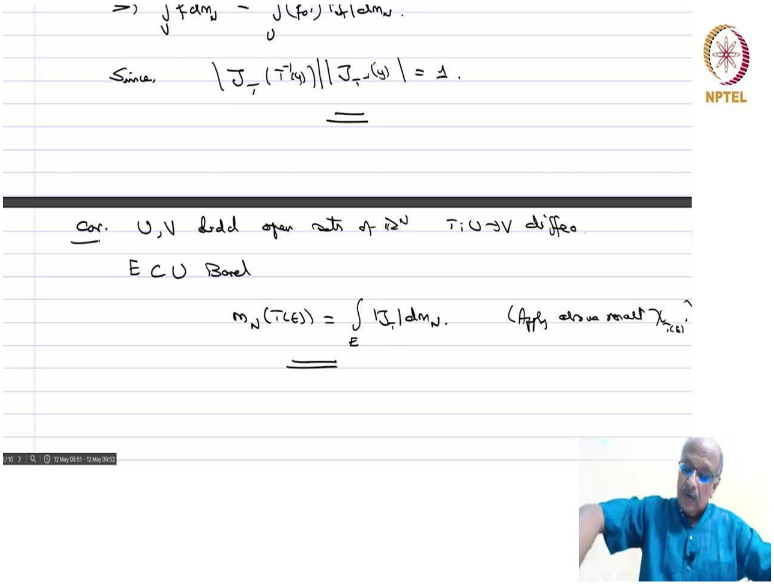

Corollary: You have U, V bounded open sets of $\mathbb{R}^N$, T: $U \rightarrow V$ diffeomorphism, $E \subset U$ Borel, then you have $m_N(T(E)) = \int_E |J_T|\, dm_N$.

(so apply the above result to $\chi_{f(T)}$)

So, we have already done that, you have this corollary (())(21:15).