Measure and Integration
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Lecture No-77
12.3 – Exercise

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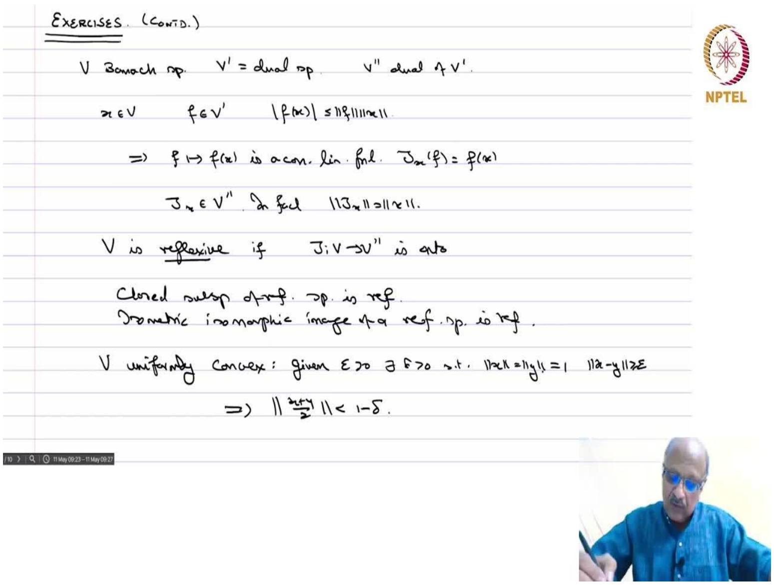

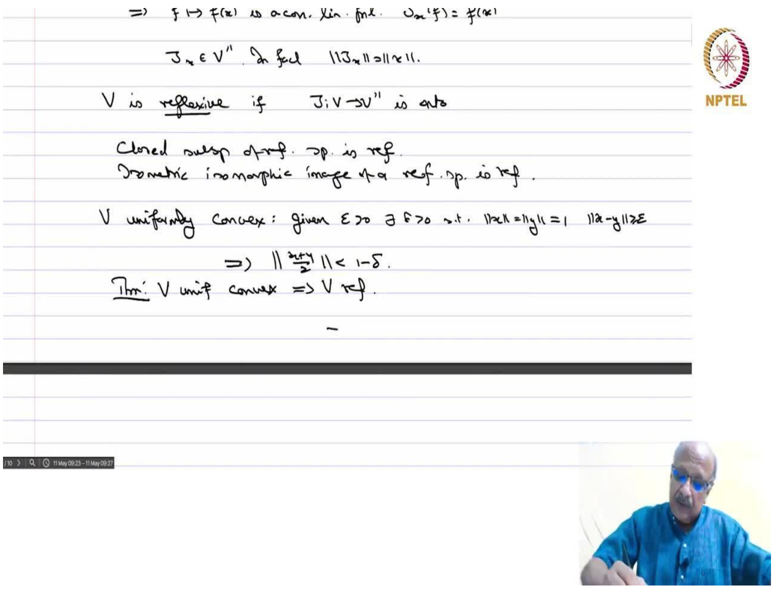

We continue on exercises, so we will before I start the set of exercises. This is another proof of the duality for the L^p spaces, $1 < p < \infty$. So, that requires some notions from function analysis which I briefly recall. So, if V is Banach space then V' prime equals dual space that means, the space of all continuous linear functionals. So, then V'' is again a dual of V'. So, if you take x in V then f in V prime then $|f(x)| \leq ||f|| \, ||x||$, This f going to f of x is a continuous linear functional.

So, we will call this $J_x(f) = f(x)$. So, $J_x \in V''$. In fact, the norm of Jx is the same as norm of x. So, V is reflexive if J going to be V double prime is onto, that means every continuous linear functional on V dash occurs only in this fashion namely it should be able to form Jx then it is supposed to be reflexive space and then close subspace of reflexive space is reflexive. Isometric isomorphic images of a reflexive space are reflexive.

Now, V is said to be uniformly convex, that means, the unit ball bulges uniformly. If you take the ball in R2 then it is in Rn with the Euclidean norm then it is a nice round ball, but if you take it with the one norm or the infinity norm then it will have lots of flat portions. So, it is said to be uniformly convex if it bulges uniformly in all directions. So, this geometric property of the norms given epsilon positive there exists a delta positive such that $||x|| = ||y|| = 1$, $||x - y|| \geq \epsilon \Rightarrow ||\frac{x+y}{2}|| < 1 - \delta$.

Theorem: V is uniformly convex ⇒ V is reflexive.

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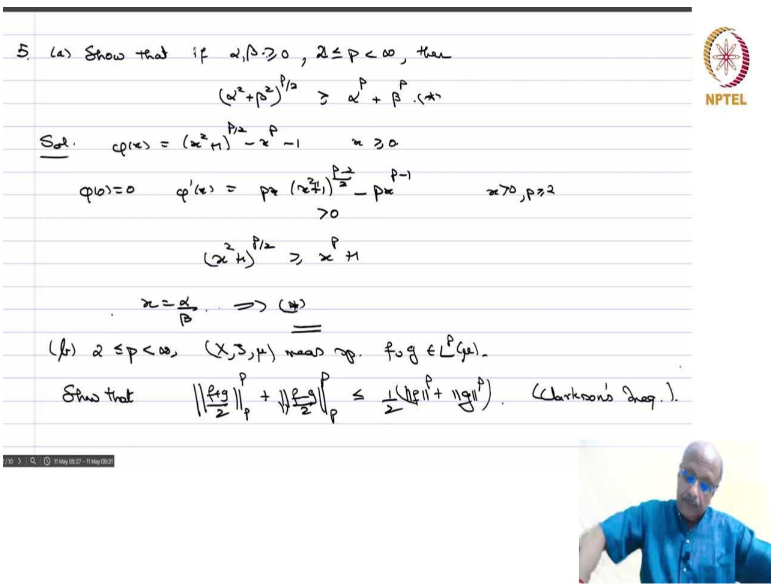

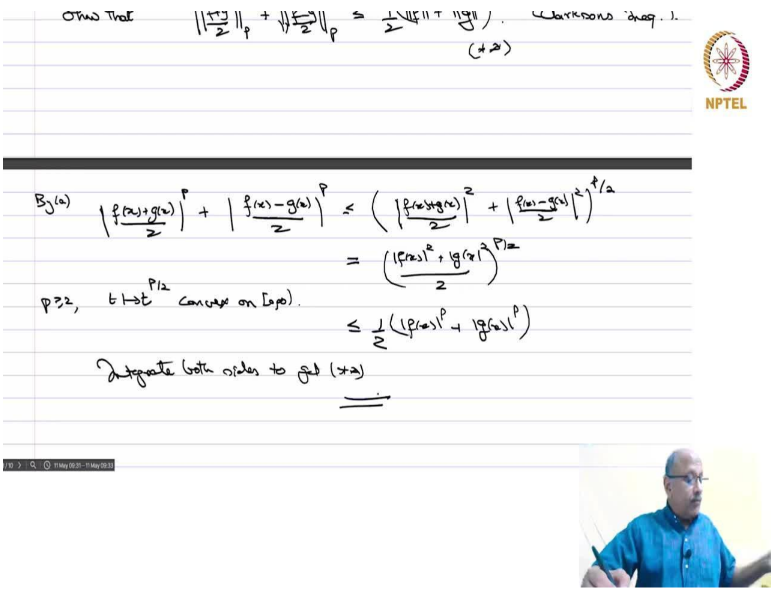

5 (a). So, show that if $\alpha, \beta \geq 0,\ 2 \leq p < \infty$, then $(\alpha^2 + \beta^2)^{\frac{p}{2}} \geq \alpha^p + \beta^p$ —-- (*).

Solution: so, we look at the function $\phi(x) = (x^2 + 1)^{p/2} - x^p - 1,\ x \geq 0$. Then phi of 0 is of course 0 and

$$\phi'(x) = px(x^2 + 1)^{\frac{p-2}{2}} - px^{p-1} > 0\ if\ x > 0,\ p \geq 2.$$

So, this means that phi is an increasing function, so, you have $(x^2 + 1)^{p/2} \geq x^p + 1$.

Now, you put if beta or alpha not 0 if one of them is 0 there is nothing to prove. So, you take x equals alpha by beta, then you will get from this whatever the inequality, so that is the solution of this.

(b). $2 \leq p < \infty$, and (X, S, μ) measure space, $f, g \in L^p(\mu)$, show that

$$||\frac{f+g}{2}||_p^p + ||\frac{f-g}{2}||_p^p \leq \frac{1}{2}(||f||_p^p + ||g||_p^p).\ \text{—-- (**)}$$

(this is called Clarkson's inequality).

Solution: So, mod fx plus gx by 2 power p plus mod fx minus gx by 2 power p is less than or equal to. So, A power p alpha p plus beta power p is less than equal to alpha square plus beta square power p by 2.

So, by a, this is less than or equal to mod fx plus gx by 2 the whole square plus mod fx minus gx by 2 whole square power p by 2. And inside if you simplify this becomes mod fx square plus gx square by 2 whole power p by 2. Now, if p is greater than or equal to 2, then you have T going to p by 2, p power p by 2 is convex, take the second derivative that is positive because p is bigger than 2 therefore, this is a convex function. So, by definition of convex functions something in the midpoint is less than or equal to the average. So, this is one half of mod x square power p by 2 which is mod fx power p plus, similarly mod gx. Now, integrate both sides to get a (**).

So, then this is Clarkson's inequality.

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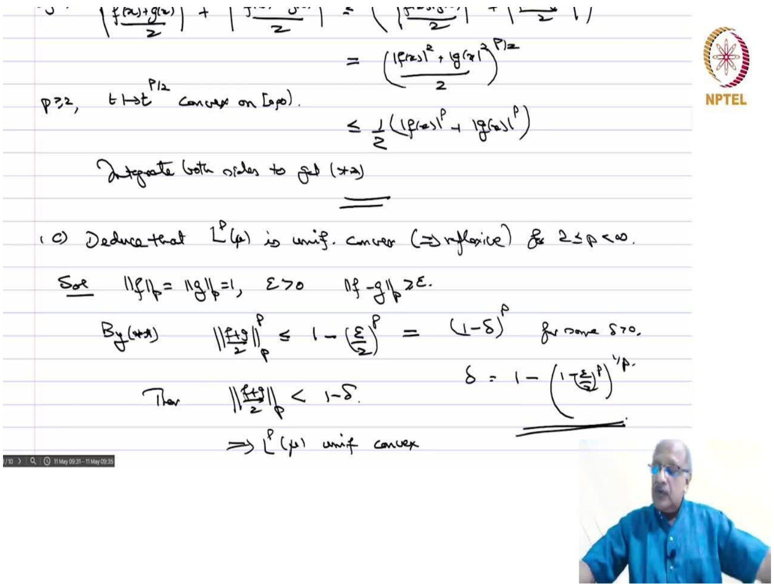

(c) Deduce that $L^p(\mu)$ is uniformly convex for $2 \leq p < \infty$.

solution: $||f|| = ||g|| = 1$, $\epsilon > 0$, $||f - g||_p \geq p$. Then by (**), by Clarkson's inequality, you have $||\frac{f+g}{2}||_p^{\ p} \leq 1 - (\frac{\epsilon}{2})^p = (1 - \delta)^p$ for some $\delta > 0$.

Therefore, $||\frac{f+g}{2}||_p \leq 1 - \delta$. Hence, $L^p(\mu)$ is uniformly convex.

So, what is delta? Delta is equal to 1 minus 1 minus epsilon by 2 power p whole power 1 by p. So, this is the number which you have for delta.

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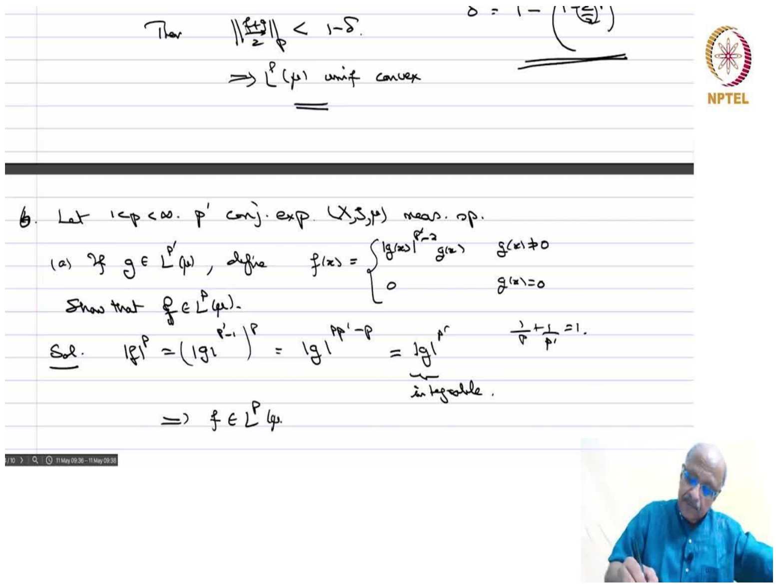

(6). let 1 less than p less than infinity p dash conjugate exponent and X, S, mu measure space. a, if g belongs to L p dash must define fx to be equal to we have done this kind of thing for mod gx to the power of p dash minus 2 gx, if gx is not 0, and 0 if gx equal to 0. Why am I doing this, because if p dash minus 2 may be a negative exponent, we do not want things blowing up, so that is why we are showing we are defining it this way. Then show that f belongs to L p mu, so that is easy.

So, mod f power p equals mod g power pp dash, when you take mod f then you get mod g to the p dash minus 1 whole power p is equal to mod g to the pp dash minus p which is equal to mod g power p dash, because 1 by p plus 1 by p dash equal to 1, so, there they call that. Now, this is integrable and therefore, you have that f belongs to L p.

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(b). $g \in L^{p'}(\mu)$. $T_g(f) \stackrel{def}{=} \int_X fg\, d\mu \quad \forall f \in L^p(\mu)$.

Then $T_g \in (L^p(\mu))'$ & $\|T_g\| = \|g\|_{p'}$.

$|T_g(f)| \le \|f\|_p \|g\|_{p'}$ (Hölder)

$\Rightarrow T_g \in (L^p(\mu))' \quad \|T_g\| \le \|g\|_{p'}$.

But $f \in L^p(\mu)$ as in (a) $\quad fg = |g|^{p'}$

$$T_g(f) = \int_X fg\, d\mu = \int_X |g|^{p'} d\mu.$$

$|T_g(f)|$

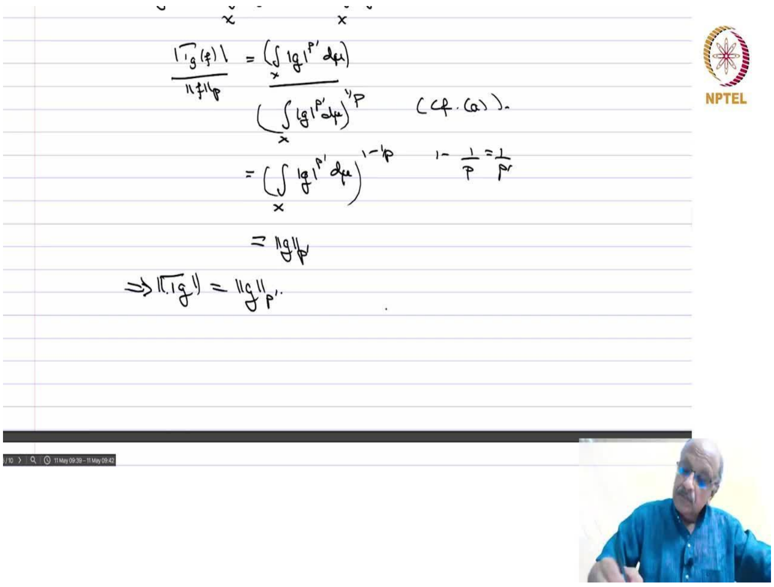
But $f \in L^p(\mu)$ as in (a) $\quad fg = |g|^{p'}$

$$T_g(f) = \int_X fg\, d\mu = \int_X |g|^{p'} d\mu.$$

$$\frac{|T_g(f)|}{\|f\|_p} = \frac{\left(\int_X |g|^{p'} d\mu\right)}{\left(\int_X |g|^{p'} d\mu\right)^{1/p}} \quad \text{(cf. (a))}.$$

$$= \left(\int_X |g|^{p'} d\mu\right)^{1-1/p} \qquad 1-\frac{1}{p} = \frac{1}{p'}$$

$$= \|g\|_{p'}$$



b, g in L p dash mu Tg of f definition is integral f g d mu is over X for every f in L p, then Tg belongs to L p mu that dash and norm Tg is equal to norm g p dash. Remember we did this with lots of lemmas in case of finite measure spaces, so now we are doing it in the general case. So, we have the Tg mod Tg f is less than equal to norm f p norm g p dash this is Holder's inequality. And therefore, this implies Tg belongs to L p mu dash and norm Tg is less than equal to norm g p dash.

But, f in L p mu as in a, then Tg of f equals integral fg d mu over x but fg equals mod g to the power of p dash. So, this is equal to integral over X mod g to the p dash d mu. So, now, mod of Tg f by norm f p is equal to integral mod g to the p dash dm over X by, norm f p is what? Norm f is the integral of mod f p power 1 by p but mod f p we already know is mod g p dash. So, this is integral over X mod g p dash d mu power 1 by p, Cf a. And therefore, this is equal to the integral of X mod g p dash d mu power 1 minus 1 by p which is equal to 1 minus 1 by p is equal to 1 by p dash and therefore, this is equal to norm g p dash. So, the supremum, which is the norm of the linear transformation, is the supremum of all such quotients that is less than or equal to norm g p dash we have already seen here, and now we are producing an element for which it is actually achieved. So, this implies norm Tg equals norm g p dash.

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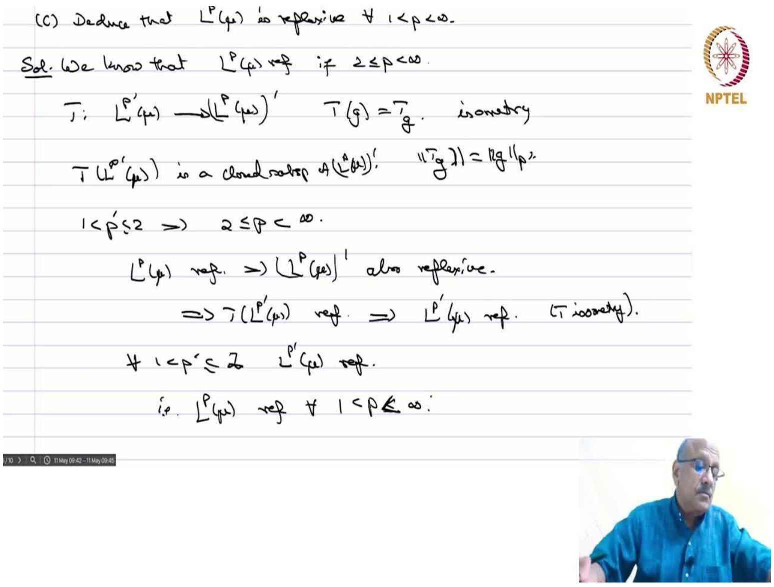

C, deduce that L^{p}(μ) is reflexive for all 1 less than p less than infinity.

So, we know that the solution L^{p}(μ) is reflexive if 2 less than equal to p less than infinity. So, now from L p dash mu to $L^{p'}$(μ). We have a mapping T, T of g equals Tg and this is isometric since norm Tg is equal to norm g p dash. So, T of $L^{p'}$(μ) isometrically sorry is a closed subspace of L p mu dash and if 1 less than p less than equal to 2, this implies 1 less than p dash less than equal 2, this implies 2 less than or equal to p less than infinity.

So, this means that now L p mu reflexive implies L p mu dash also reflexive and this implies T of $L^{p'}$(μ) reflexive implies $L^{p'}$(μ) reflexive because T is an isometric. Therefore, for all 1 less than p dash less than infinity less than or equal to 2, $L^{p'}$(μ) reflexive, so, that is L p mu reflexive for all 1 less than p less than infinity.

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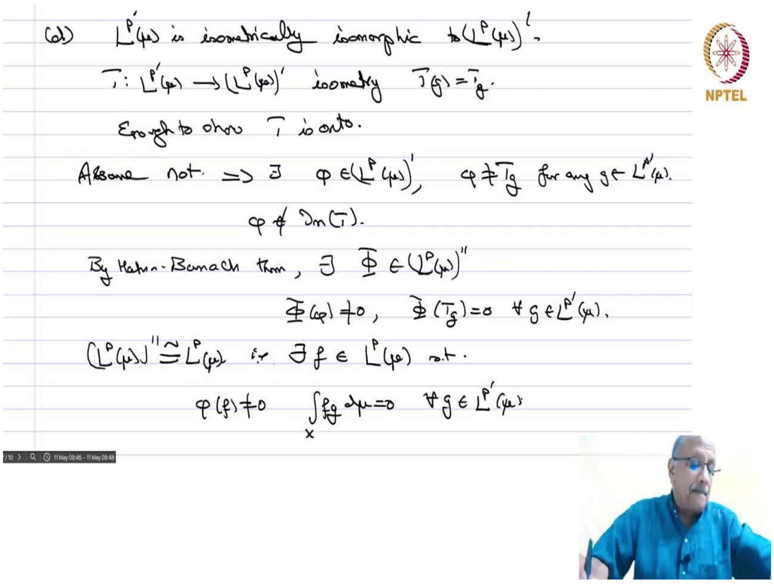

d, LT mu p dash mu is isometrically isomorphic to L p mu dash. So, T from L p dash mu to L p mu dash is an isometric, T of g equal to Tg. So, enough to show T is onto, so they can complete the. So, assume the contrary assume not so, this implies there exists a phi in $(L^p(\mu))'$ and phi is not equal to Tg for any g in L p dash Mu, that is, phi does not belong to the image of T, but the image of t is a closed subspace.

So, by Hahn Banach theorem there exists capital phi in L p mu double dash capital phi of small phi is not 0 and capital phi of Tg equal to 0 for all g in L p dash. So, that is it vanishes on the image of T but does not vanish completely. But L p mu double dash is the same as $L^p(\mu)$. That is, there exists an f in $L^p(\mu)$ such that phi of f is not 0, but integral over X f g d mu equal to 0 for all g in $L^{p'}$.

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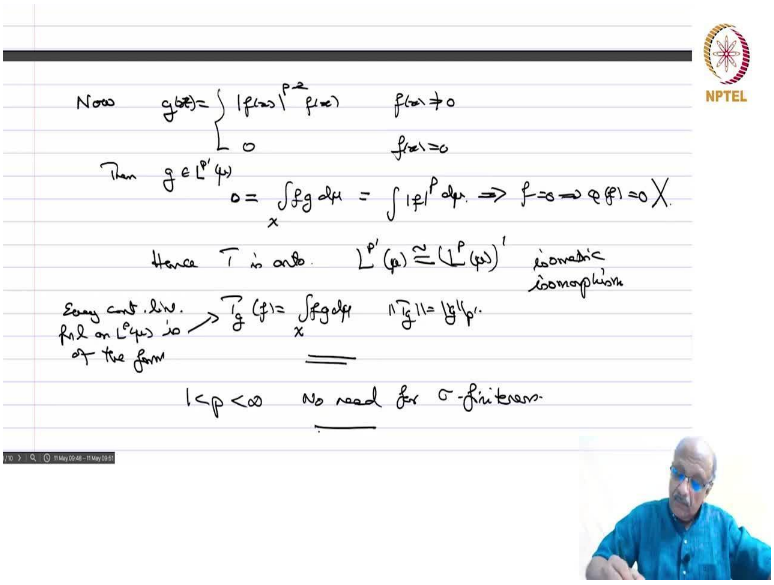

Now again, we play the same trick as we did before: take

$$g(x) = |f(x)|^{p-2} f(x) \ , \ if\ f(x) \neq 0,$$

$$= 0, \quad if \ \ f(x) = 0.$$

Then $g \in L^{p'}(\mu)$ and therefore, $0 = \int\limits_X fg d\mu = \int |f|^p d\mu \Rightarrow f = 0 \Rightarrow \phi(f) = 0$, which is a contradiction. Hence, T is onto, that is, $L^{p'}(\mu) \simeq (L^p(\mu))'$. And you have

$$T_g(f) = \int\limits_X fg \, d\mu \,; \ \ ||T_g|| = ||g||_{p'} .$$

So, this is the theorem which we proved the representation theorem. So, for every continuous linear functional on $L^p(\mu)$ is of the form in the form like this. So, here so, for one less than p less than infinity no need for sigma finite, this beauty of this proof.

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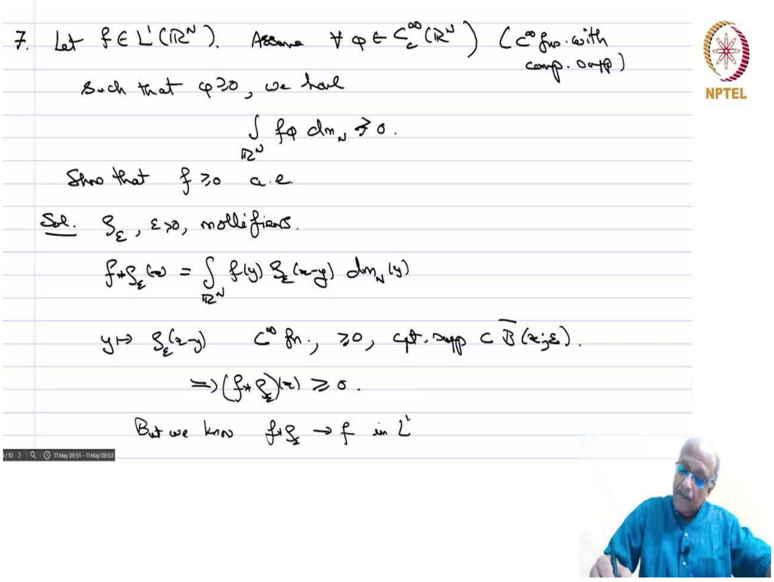

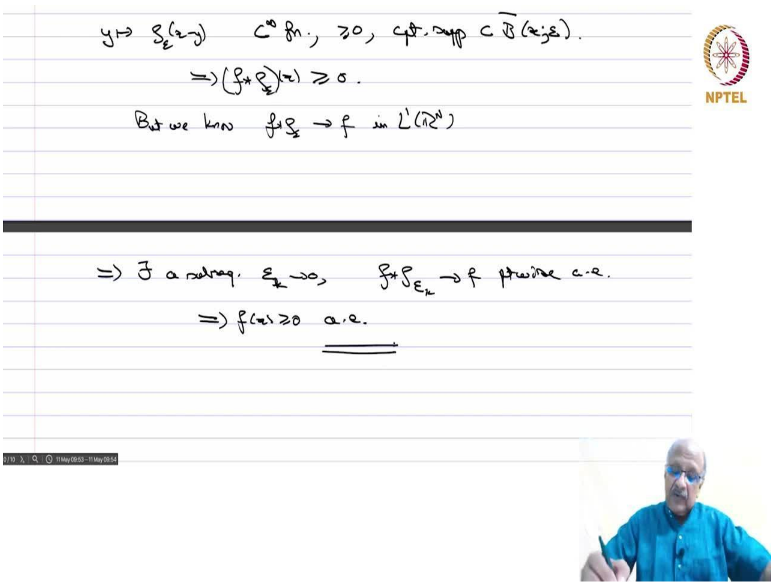

(7) Let $f \in L^1(\mathbb{R}^N)$. Assume for every $\phi \in C_c^{\infty}(\mathbb{R}^N)$, such that ϕ is non-negative, we have

$$\int_{\mathbb{R}^N} f\phi \, dm_N \geq 0.$$

Show that $f \geq 0$ almost everywhere.

Solution: we take ρ_ϵ, $\epsilon > 0$, mollifiers, then

$$f * \rho_\epsilon(x) = \int_{\mathbb{R}^N} f(y)\rho_\epsilon(x - y)dm_N(y).$$

Now, if you take $y \to \rho_\epsilon(x - y)$, this is C^∞ function, it is greater than equal to 0 and compact support contained in $\overline{B}(x; \epsilon)$, and therefore, this has the center at x and epsilon greater or equal to 0. So, this implies that $f * \rho_\epsilon(x) \geq 0$.

But we know $f * \rho_\epsilon \to f \; in \; L^1(\mathbb{R}^N) \Rightarrow \exists$ a subsequence $\epsilon_k \to 0$, then $f * \rho_{\epsilon_k} \to f$ pointwise almost everywhere. And since this is greater than or equal to 0 point wise and it converges to f, so this implies that $f(x) \geq 0$.

So, this proves this theorem. So, it is a very nice application of the convolution idea, a different kind of application.

So, if this integral is all non-negative for non-negative functions then the function itself has to be non-negative almost everywhere. So, we will conclude the exercises with this, so the next very short topic I want to deal with next time is the change of variable formula. We have not seen particular cases of it so I would like to give you the general formula for that which we will next time.