Measure and Integration
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The Institute of Mathematical Sciences
Lecture No-76
12.2 - Exercises

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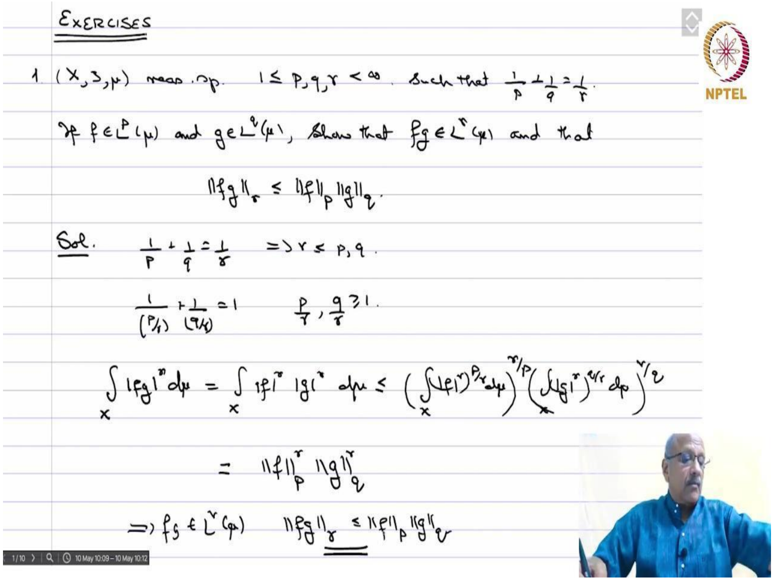

So, now let us do some exercises.

(1) (X, S, μ) measure space, $1 \leq p, q, r < \infty\ s.t.\ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, show that $fg \in L^r(\mu)$ and that $||fg||_r \leq ||f||_p\, ||g||_q$.

sol.- So, you have $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Rightarrow r \leq p, q$ and then you can write this as 1

$$\frac{1}{p/r} + \frac{1}{q/r} = 1,\ \frac{p}{r}, \frac{q}{r} \geq 1.$$

So, then, we can apply a Holder inequality

$$\int_X |fg|^p d\mu = \int_X |f|^r |g|^r d\mu \leq \left(\int (|f|^r)^{\frac{p}{r}} d\mu\right)^{\frac{r}{p}} \left(\int (|g|^r)^{\frac{q}{r}} d\mu\right)^{\frac{r}{q}} = ||f||_p^{\ r}\, ||g||_q^{\ r}.$$

So, this implies that $fg \in L^r(\mu)$ and you have $||fg||_r \leq ||f||_p\, ||g||_q$.

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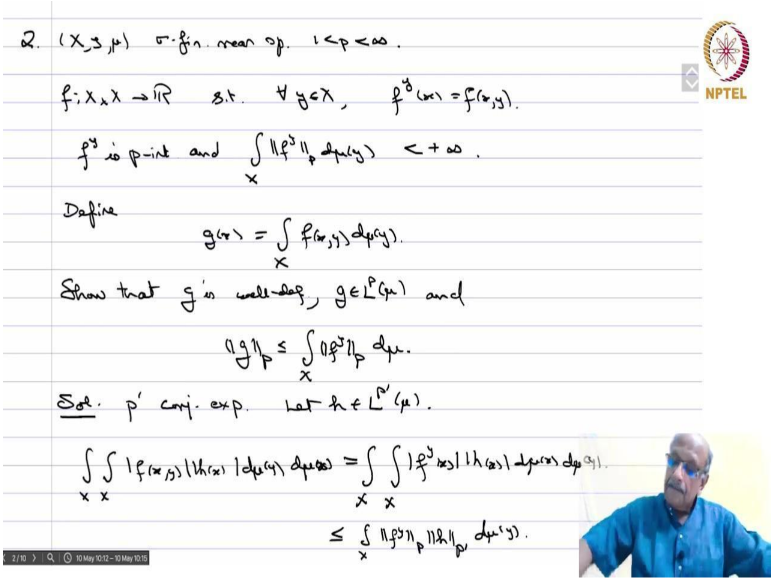

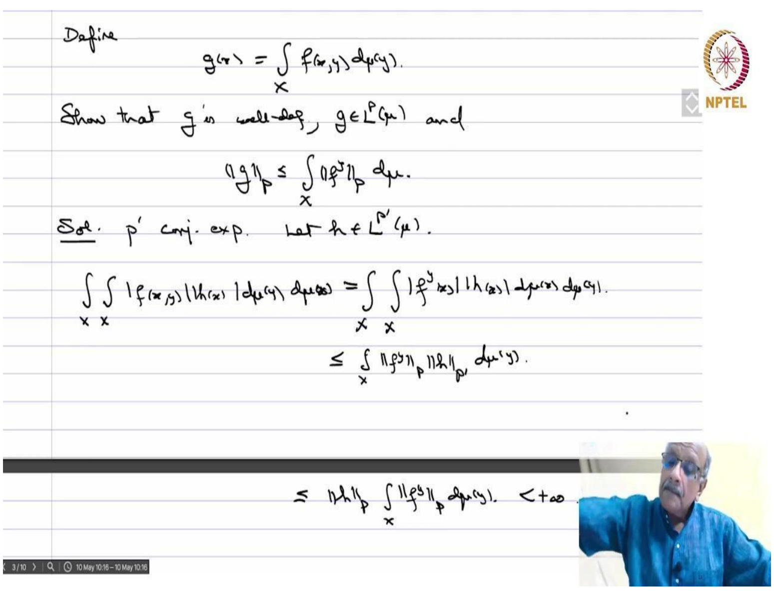

(2) (X, S, μ) $\sigma -$ sigma finite measure space, $1 < p < \infty$, $f: X \times X \to \mathbb{R}$ such that say it is a mapping set for every y in X, we have $f^{y}(x) = f(x, y)$. f^{y} is p-integrable and

$$\int_X ||f^{y}||_p d\mu(y) <+ \infty.$$

Define $g(x) = \int_X f(x, y) d\mu(y)$. Show that g is well defined, $g \in L^{p}(\mu)$ and

$$||g||_p \leq \int_X ||f^{y}||_p d\mu.$$

solution: so, we are going to use the duality method. So, p' conjugate exponent just as we did in the case of convolution. So, let $h \in L^{p'}(\mu)$, so, let us take

$$\int_X \int_X |f(x, y)|\, |g(y)|\, d\mu(y)\, d\mu(x) = \int_X \int_X |f^y(x)|\, |h(x)| d\mu(y)\, d\mu(x)$$

$$\leq \int_X ||f^y||_p\, ||h||_{p'}\, d\mu(y) \leq ||h||_p \int_X ||f^y||_p d\mu(y)$$

$$< +\infty.$$

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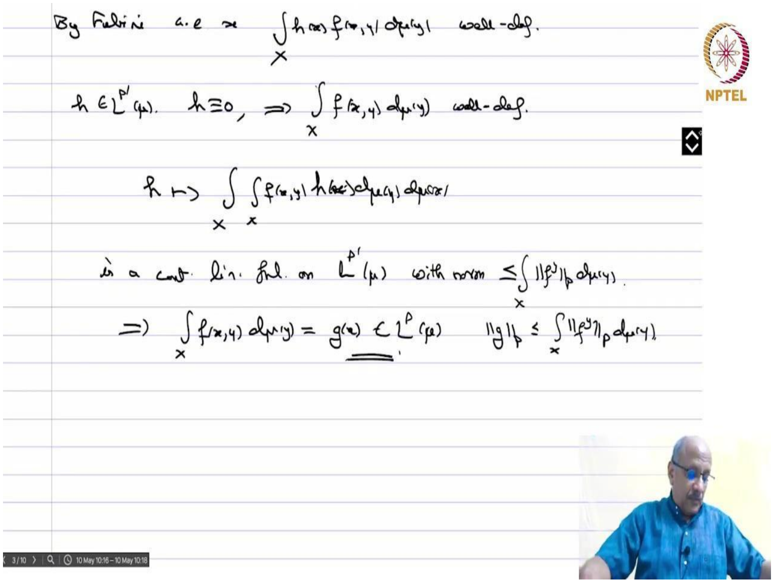

Therefore, by Fubini's theorem for almost every x you have $\int_X h(x) f(x, y) d\mu(y)$ well defined, so, you take $h \in L^{p'}(\mu)$. So, set h is not identically 0 then you have that, so, this implies that integral over x, f of x, y d mu y is well defined. Also h going to integral over x integral x f of x, y d mu y, sorry, the h of x d mu y d mu x is a continuous linear functional on norm on Lp dash mu with norm less than or equal to norm integral over x norm fy p d mu y. So, this implies that integral over x f of x, y d mu y equal to g of x belongs to Lp of mu and g nom gp less than equal to integral over x norm fy p d mu so, that completes the proof of this exercise.

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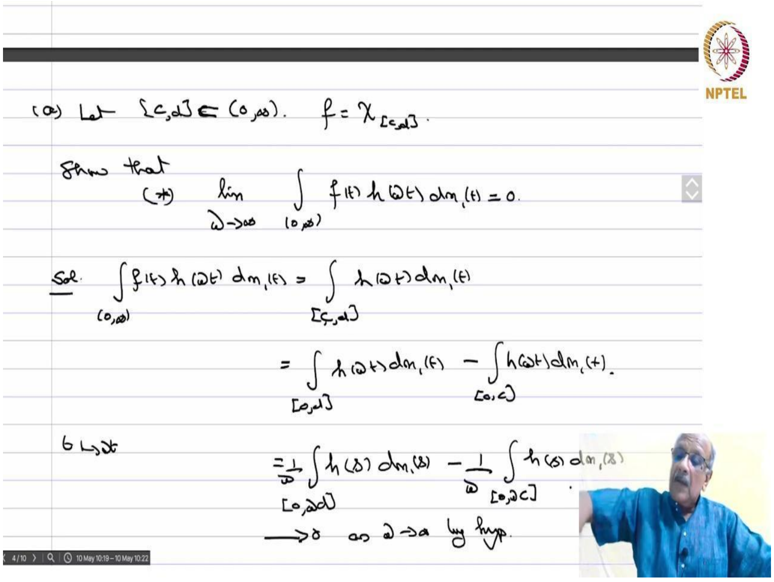

3 This exercise is sometimes called the Riemann Lebesgue lemma. There are several versions of Riemann Lebesgue lemma. So, this is one situation, so h from 0 infinity to R bounded measurable function such that 1 by c integral 0 to c sorry integral 0, c h dm1 limit as c tends to infinity equal to 0. a, let c, d be contained in 0 infinity f equal to chi of c, d. Show that the star limit omega tending to infinity integral 0, infinity f of t h omega t dm1 t equal to 0.

Solution, so that is integral 0 infinity ft h of omega t dm1 t equals integral over c, d of h of omega t dm1 let us see go the integral 0, d of h of omega t dm1 t minus integral 0, c h of omega t dm1. Now, you can change the variable t going to omega t and therefore, this will become integral h so it is a linear thing, determinant is omega power n this diagonal there anyway there is only one dimension so, you have this is 0 to omega t omega d sorry, h of s dm1 s 1 over omega because omega dt is ds, so, you have 1 over omega will come and this will similarly this will be minus 1 over omega integral 0 to omega c of h of s dm1 s and then we know that this goes to 0 as omega tends to infinity by hypothesis so, that is the solution.

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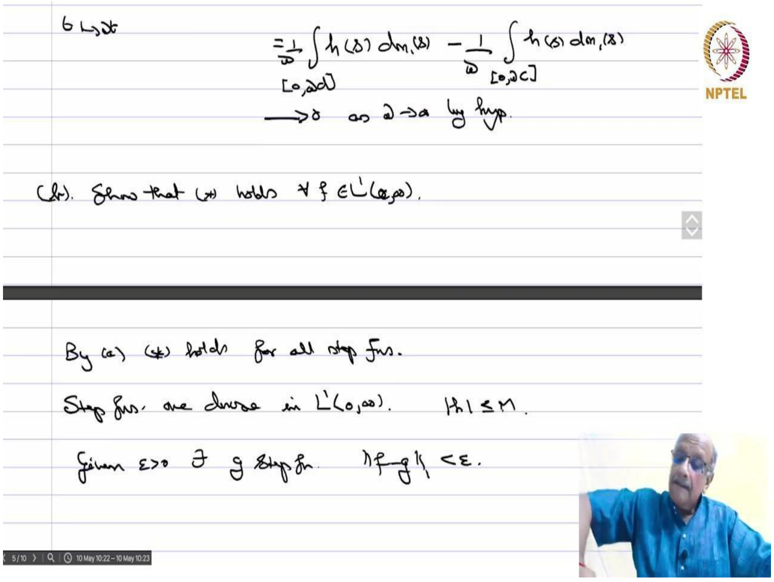

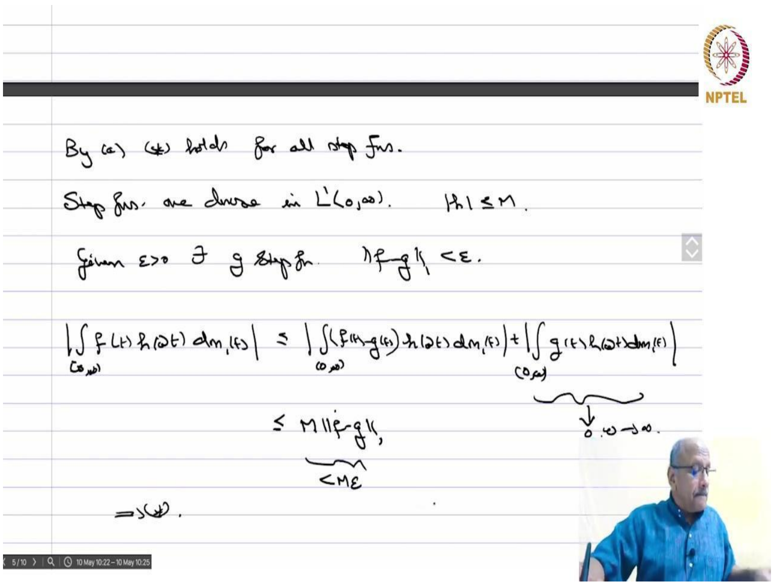

So, b, show that start holds for every f in L1 of 0 infinity, so this is Riemann Lebesgue lemma which you want to show. So, by a star holds for all step functions what is the step function it is a simple function where each element Ei is a form of an interval in 1 dimension that is step function and therefore, now we have also seen step functions are dense in L1 and let us assume that mod h is less than equal to M which is because it is a bounded function. So, given epsilon positive there exists a g in g step function norm of f minus g in L1 less than epsilon.

Therefore, mod integral 0 infinity ft h omega t dm1 t is less than or equal to mod integral 0 infinity, ft minus gt h omega t dm1 t plus integral 0 infinity modulus gt h omega t dm1. Now, we know that this goes to 0 as omega tends to infinity, now, this one is less than or equal to m which is the mod h is just equal to m integral 0 infinity mod f minus g dm1 t and that is just norm f minus g L1 and then this is less than M epsilon. So, you can find g as close as possible so that this can be made as close as you want and once you have g you can find omega (()) (16:13) so this goes as small as possible. So, this implies a star.

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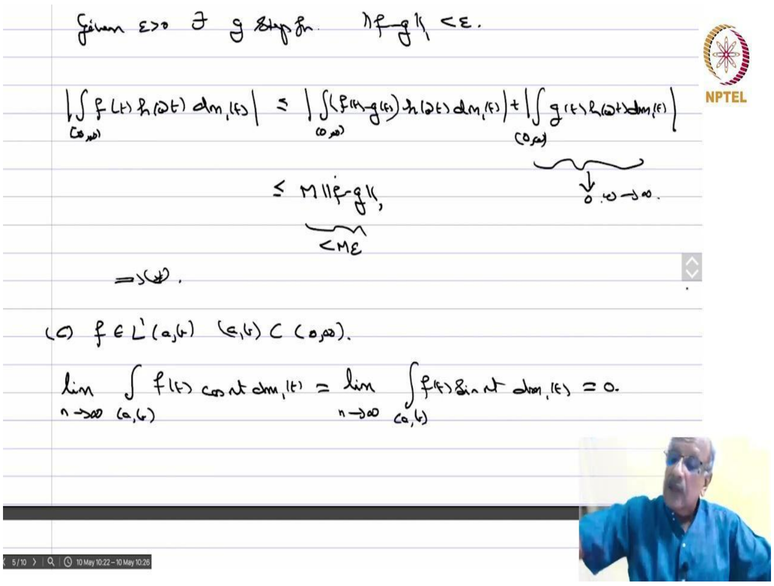

(c) so, $f \in L^1(a, b)$, $(a, b) \subset (0, \infty)$,

$$\lim_{n \to \infty} \int_{(a,b)} f(t) \cos(nt)\, dm_1(t) = \lim_{n \to \infty} \int_{(a,b)} f(t) \sin(nt)\, dm_1(t) = 0.$$

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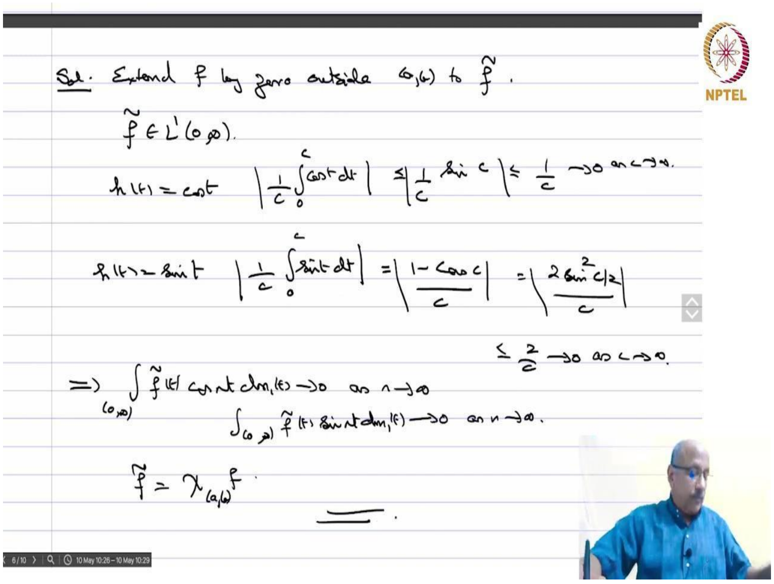

solution: extend f by 0 outside a, b, so, let $\tilde{f}$. Then $\tilde{f} \in L^1(0, \infty)$. Now, if I take

$$h(t) = \cos t \,\left|\frac{1}{c}\int_0^c \cos t\, dt\right| \leq \left|\frac{1}{c}\sin c\right| \leq \frac{1}{c} \rightarrow \ as\ c \rightarrow \infty.$$

Similarly, if you take

$$h(t) = \sin t \,\left|\frac{1}{c}\int_0^c \sin t\, dt\right| = \left|\frac{1-\cos c}{c}\right| = \frac{2\sin^2(\frac{c}{2})}{c}\Big| \leq \frac{2}{c} \rightarrow 0\ as\ c \rightarrow \infty.$$

So, therefore, this implies that integral f tilde t 0, infinity cos nt dm1 t tends to 0 as n tends to infinity we are taking omega equals n, n integral 0, infinity f tilde t sin nt dm1 t also goes to 0 as n tends to infinity which proves the thing because and then f tilde is nothing but chi of a, b times f so, therefore, the result follows.

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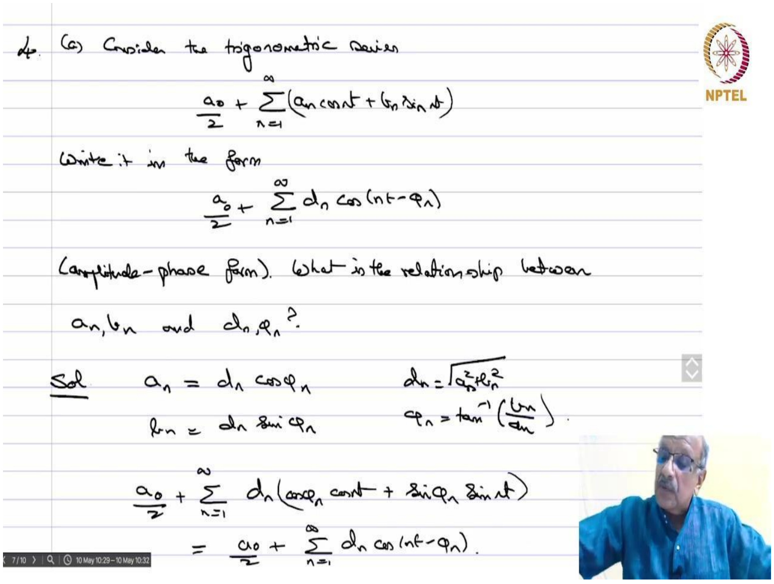

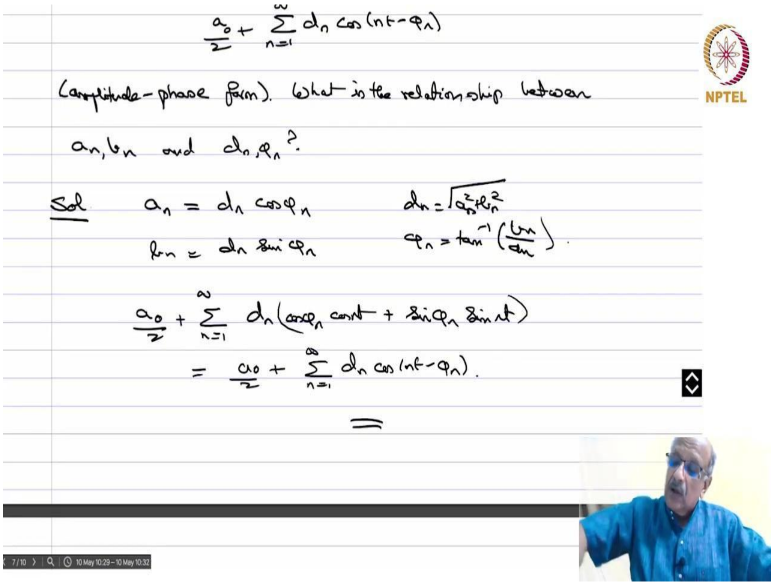

(4) (a) consider the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

So, write it in the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(nt - \phi_n)$

(this is called the amplitude phase form, amplitude is dn, phase is the is phi n,) what is the relationship between a_n, b_n and d_n, ϕ_n?

solution: So, you just write $a_n = d_n \cos(\phi_n),\ b_n = d_n \sin(\phi_n),\ i.e.,\ d_n = \sqrt{a_n^2 + b_n^2}$,

$\phi_n = \tan^{-1}(\frac{b_n}{a_n})$. So, then this is well defined so, this gives you the thing and then if you substitute then you get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} d_n(\cos(\phi_n)\cos\ (nt) + \sin(\phi_n)\sin(nt))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(nt - \phi_n).$$

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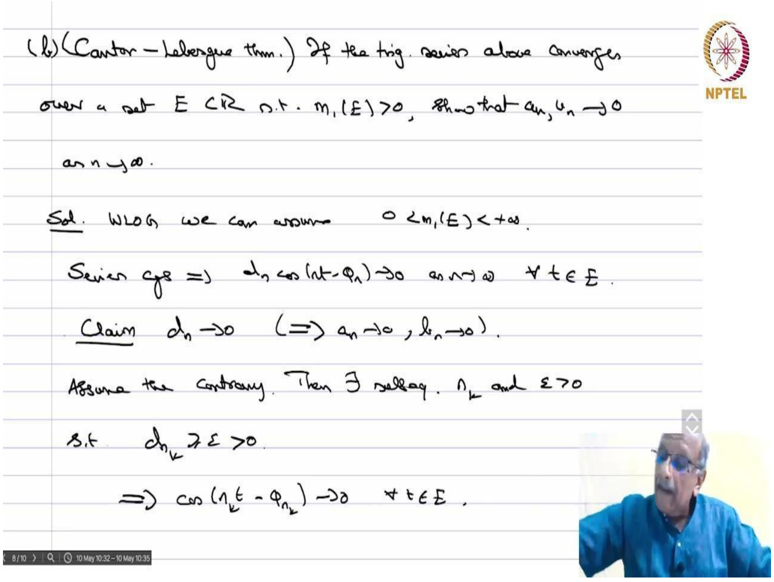

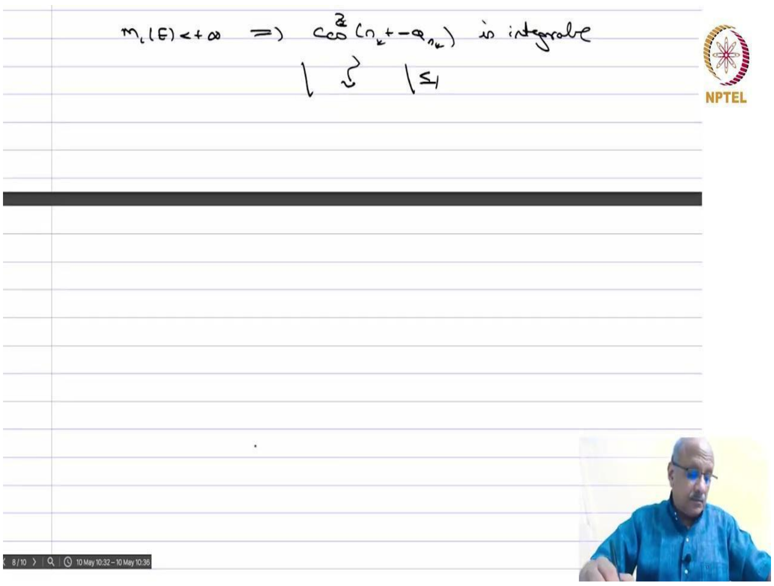

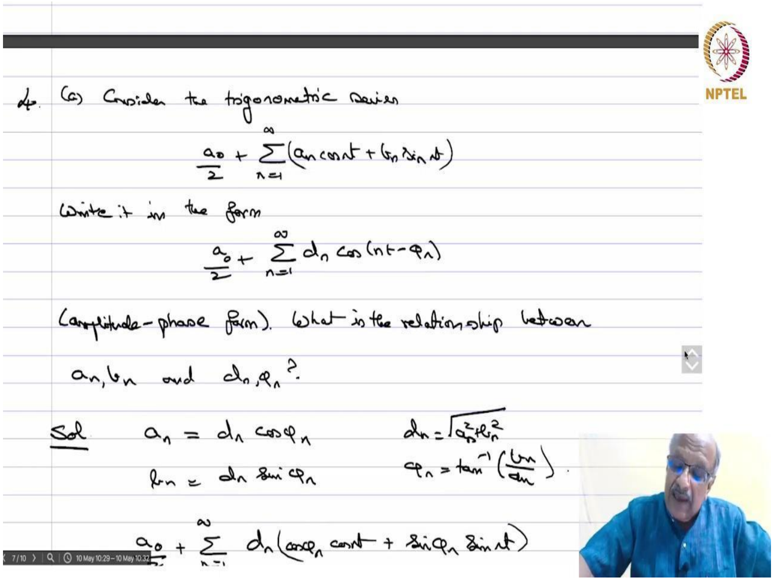

So, now is a very interesting result. So, this is called the Cantor Lebesgue theorem. If the trigonometric series about converges over a set E in R such that the measure of E is positive show that an, bn into 0 as n tends to infinity so, if you look at the trigonometric series, so, you have an cos nt plus bn sin nt, this series converges all you can say an cos nt goes to 0 and bn sin nt goes to 0 that is all that you can say from this but we want to show that an goes to 0 and bn goes to 0 individually.

So, this is different from numerical series where the general term goes to 0 is here it does not fully imply that. So, with solution, without loss of generality, we can assume 0 less than m1 E strictly less than infinity is finite, because you take a smaller set of finite measure, you have a set of positive measures where the convergence takes place. So, you take a finite measure subspace of it, so, there is no loss of generality of that. So, the series converges implies the general term goes to 0. So, dn cos nt minus phi n tends to 0 as n tends to infinity for every t in E.

So, assume, so, claim dn tends to 0, which will of course imply that an tends to 0 bn tends to 0. So, it is enough to prove this claim. So, assume the contrary then, there exists a subsequence nk and epsilon positive such that dnk is greater than equal to epsilon strictly positive and this will then imply the cos nkt minus phi nk goes to 0 for every t in E, so, measure of E is less than plus 1 implies cos square nkt minus phi nk is integrable, so modulus of this is less than equal to 1 and therefore, this is integral.

(Refer Slide Time: 27:16)

By DCT $\Rightarrow \int_E \cos^2(n_k t - \varphi_{n_k})\, dm_1(t) \to 0.$

i.e. $\int_E \frac{1 + \cos 2(n_k t - \varphi_{n_k})}{2}\, dm_1(t) \to 0.$

$$\int_E \cos 2(n_k t - \varphi_{n_k})\, dm_1(t) = \cos 2\varphi_{n_k} \int_{\mathbb{R}} \chi_E \cos 2n_k t\, dm_1(t)$$

$$+ \sin 2\varphi_{n_k} \int_{\mathbb{R}} \chi_E \sin 2n_k t\, dm_1(t).$$

But $\chi_E \in L^1(\mathbb{R})$ $\because$ $m_1(E) < +\infty$.

Prev. result (Pr 3(c)) also applies to $(-\infty, 0)$ and hence to $\mathbb{R}$.

9/10 10 May 10:37 – 10 May 10:40

But $\chi_E \in L^1(\mathbb{R})$ $\because$ $m_1(E) < +\infty$.

Prev. result (Pr 3(c)) also applies to $(-\infty, 0)$ and hence to $\mathbb{R}$.

$\int_{\mathbb{R}} \chi_E \cos 2n_k t\, dm_1(t) \to 0$, $\int_{\mathbb{R}} \chi_E \sin 2n_k t\, dm_1(t) \to 0$

$$\int_E \cos^2(n_k t - \varphi_{n_k})\, dm_1(t) \to \int_E \tfrac{1}{2}\, dm_1(t) = \tfrac{1}{2} m_1(E) > 0.$$

Contradicts (*).

$\Rightarrow d_n \to 0$ $(\Rightarrow a_n, b_n \to 0)$.

10/10 10 May 10:40 – 10 May 10:41

$\Rightarrow \cos(n_k t - \varphi_{n_k}) \to 0 \quad \forall t \in E.$

$m_1(E) < +\infty \Rightarrow \cos^2(n_k t - \varphi_{n_k})$ is integrable

$|\quad \} \quad | \leq 1$

By DCT $\Rightarrow \int_E \cos^2(n_k t - \varphi_{n_k})\, dm_1(t) \to 0.$ (*)

i.e. $\int_E \frac{1 + \cos 2(n_k t - \varphi_{n_k})}{2}\, dm_1(t) \to 0.$

$\int_E \cos 2(n_k t - \varphi_{n_k})\, dm_1(t) = \cos 2\varphi_{n_k} \int \chi_E \cos 2n_k t\, dm_1(t)$

9/10 10 May 10:37 – 10 May 10:40

Also it is less than 1 which is also integrable therefore, by the dominated convergence theorem you have integral over E cos square nkt minus phi nk dm1 t goes to 0 that is integral over E 1 plus cos 2 times nkt minus phi nk by 2 dm1 t goes to 0. Now, integral over E cos 2 times nkt minus phi nk dm1 t, what is it equal to, it is cos 2 times phi nk integral chi over R cos 2 nkt dm1 an integral plus sorry sin 2 phi integral over R chi E sin 2 nkt dm1. But chi belongs to L1 of R since you have that a m1 of E is finite and previous result that is problem 3 c also applies to R minus infinity, 0 and hence to R.

You just have to do everything with the minus sin and you will get the same whatever you did in that exercise so, therefore, you have the integral chi E cos nt dm1 t over R tends to 0 an integral over R chi sin 2 nk dm1 t also goes to 0. So, both these terms go to 0 that means the cosine the integral of the cosine goes to 0 and therefore, we have the integral over E of cos square nkt minus phi nk dm1 t goes to integral over E one half dm1 t which is equal to one half of m1 E is strictly positive.

And but, so, this is a contradiction because we saw that it goes to 0, so, we saw that it goes to 0, so, contradict star and therefore, this implies that dn goes to 0 and that implies that an and bn go to 0. So, this is a very nice application of the Riemann Lebesgue Lemma. And so, we have applied essentially the thing dependent on step functions being dense in L1, so, we will continue the exercises next time.