Measure and Integration
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Lecture No-75
12.1 - Convolutions

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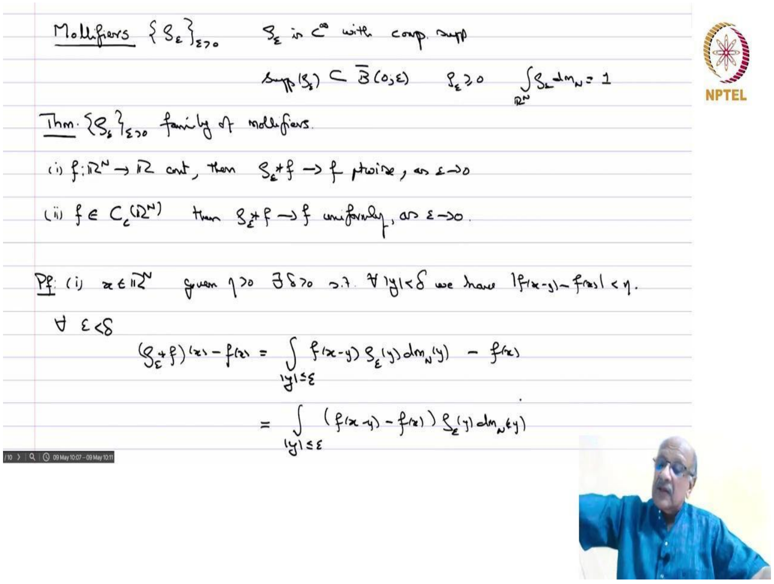

We were looking at the family of mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$. So, ρ_ϵ is C^∞ with compact support, $supp(\rho_\epsilon) \subset \overline{B}(0, \epsilon)$ and then rho epsilon is greater than or equal to 0 and integral of rho epsilon dmN over RN which is also the integral over the ball radius epsilon that is equal to 1. So, this is the family of mollifiers and we will see how they are very useful together with the tool of convolution.

Theorem: so, $\{\rho_\epsilon\}_{\epsilon>0}$ family of mollifiers, so,

(i) $f: \mathbb{R}^N \to \mathbb{R}$ continuous, then $\rho_\epsilon * f \to f$ pointwise as $\epsilon \to 0$.

(ii) $f \in C_c(\mathbb{R}^N)$, then $\rho_\epsilon * f \to f$ uniformly as $\epsilon \to 0$.

Proof: so, recall that rho x on the C^∞ function compact support f is a continuous function. So, the convolution is well defined and in fact it is a C^∞ function. So, you are approximating point wise a given function continuous function by C^∞ function and here we are approximating it uniformly when f is continuous with compact support. So, one, x in $\mathbb{R}^N$ then given eta positive there exists a delta positive such that for all mod y less than delta y is the vector in $\mathbb{R}^N$. So, mod is the Euclidean distance we have mod f of x minus y minus fx is less than eta.

So, choose, so, for all epsilon less than delta we have rho epsilon star f of x minus f of x equals integral mod y less than or equal to epsilon it is enough to f of x minus y rho epsilon y dmN y minus f of x. Now, this can be witness integral mod y less than or equal to epsilon f of x minus y minus f of x times ρ_ϵ y dmN y because the integral rho epsilon is 1 and fx is a constant as far as this integral is concerned and therefore, it just pulls out. So, this can be like this.

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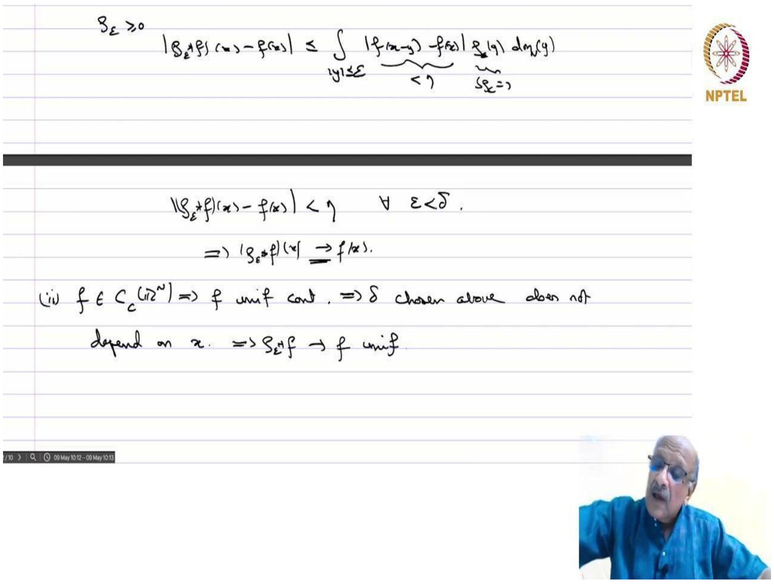

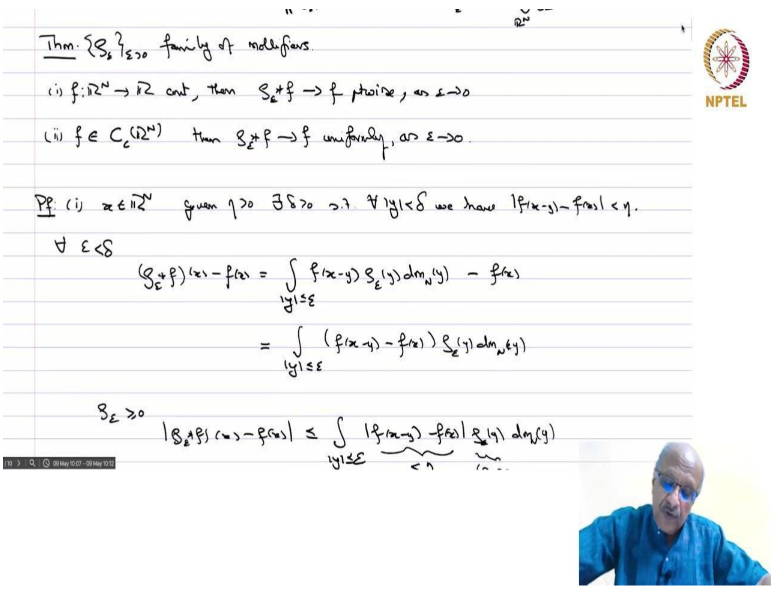

Therefore, you have and also rho epsilon is greater than or equal to 0. Therefore, mod rho epsilon star f x minus f of x is less than equal to integral mod y less than equal to epsilon mod of f of x minus y minus f of x into ρ_ϵ y dmN y. Now, this is less than eta and this integral rho epsilon is equal to 1 and therefore, you have mod rho epsilon star fx minus f of x is less than or equal to eta or less than eta in fact, does not matter, for every epsilon less than delta and that so, this implies that rho epsilon star fx converges to f of x. 2, f in $C_c(\mathbb{R}^N)$ implies f is uniformly continuous implies delta chosen above does not depend on x, this implies that the $\rho_\epsilon * f$ converges to f uniformly.

So, there is just, so, in particular, this is a little more particular the general statement would be if fn f is uniformly continuous than $\rho_\epsilon * f$ goes to f uniformly that is what we had.

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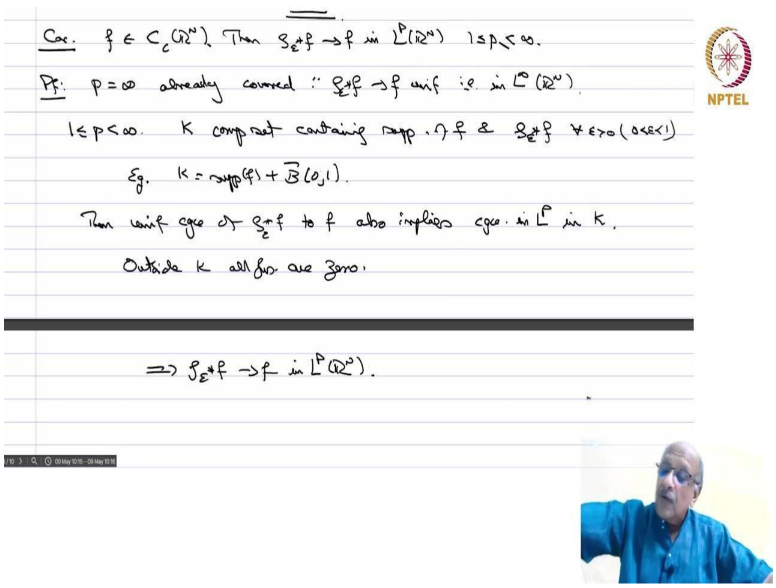

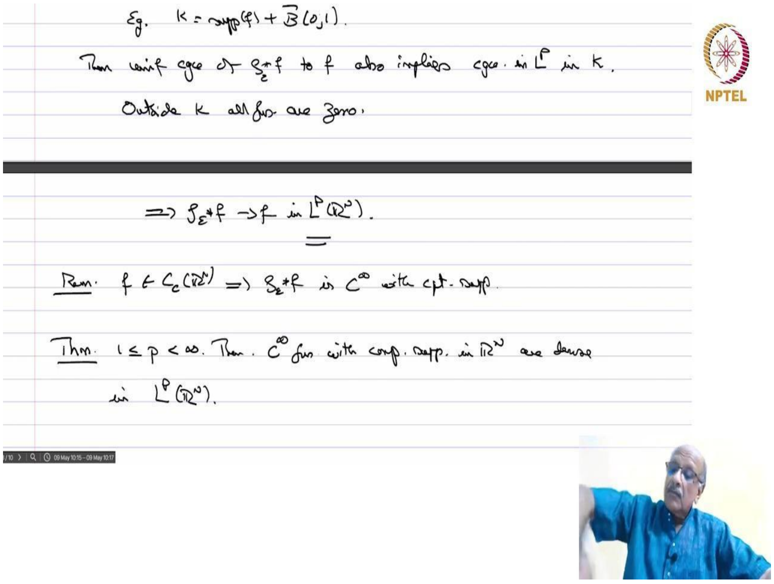

Corollary: $f \in C_c(\mathbb{R}^N)$, then $\rho_\epsilon * f$ converges to f in LP of RM 1 less than equal to p less than infinity. Proof in fact, let us say, so, p equals infinity already covered since rho epsilon star f converges to f uniformly that is in L infinity, uniform converges L infinity convergence are the same, so, 1 less than equal to p less than infinity. So, K compact set containing support of f and $\rho_\epsilon * f$ for all epsilon positive let us say 0 less than epsilon less than 1. So, for instance you can take K equal to support of f plus the close ball center (()) (8:16) and radius 1, so, example.

So, then uniform convergence of $\rho_\epsilon * f$ to f also implies convergence in $L^p(K)$. Outside K all functions are zero so, therefore, you have $\rho_\epsilon * f$ converges to f in $L^p(\mathbb{R}^N)$. K is a finite measure since it is compact and therefore, uniform convergence implies LP convergence. So, remark $f \in C_c(\mathbb{R}^N)$ implies $\rho_\epsilon * f$ is C infinity with compact support. So, that leads us to the next theorem 1 less than equal to p less than infinity then C infinity functions with compact support in $\mathbb{R}^N$ are dense in $L^p(\mathbb{R}^N)$.

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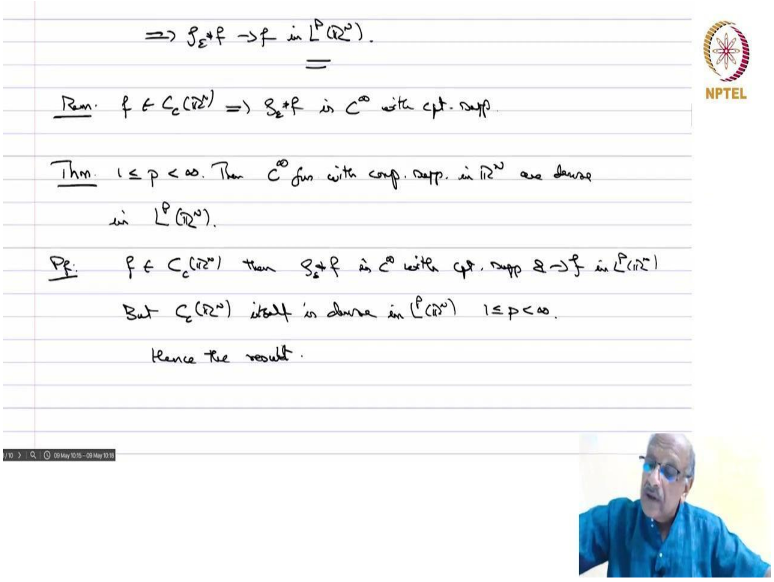

Proof: so, if $f \in C_c(\mathbb{R}^N)$ then rho epsilon star f is C infinity with compact support and converges to f in LP RN but $C_c(\mathbb{R}^N)$ itself is dense in $L^p(\mathbb{R}^N)$ 1 less than equal to p strictly less than infinity. So, this implies hence the result, then we see infinity functions with compact support with an approximate in LP norm the continuous functions with compact support and they approximate in LP norm functions in LP. So, that completes that.

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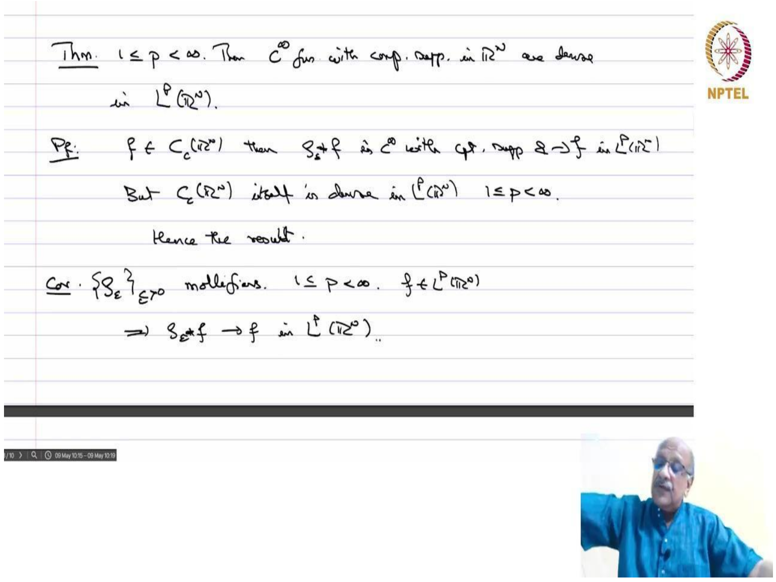

So, corollary rho epsilon positive mollifiers 1 less than equal to p less than infinity then f in of $L^p(\mathbb{R}^N)$ implies rho epsilon star f converges to f in $L^p(\mathbb{R}^N)$. We already saw this continuous function with compact support.

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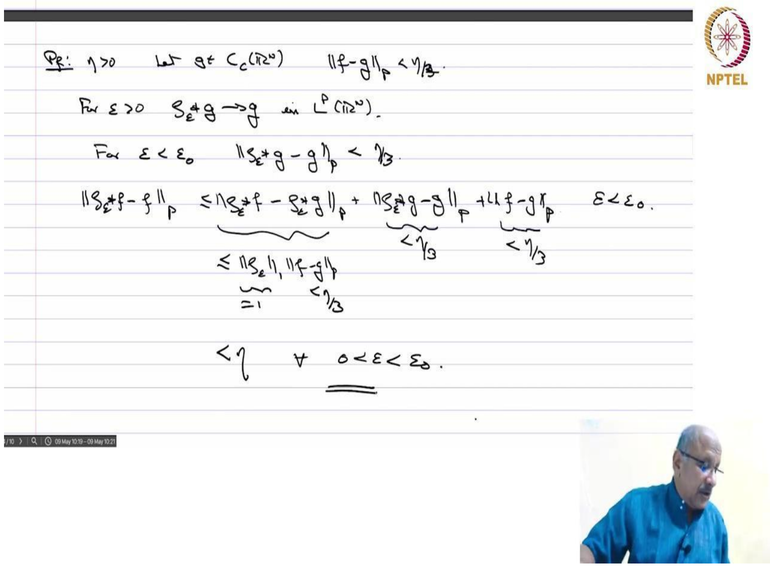

Now, we just have to use the density so, eta be positive and let g in $C_c(\mathbb{R}^N)$ norm f minus g in L^p less than eta by 2. Now, for epsilon less than or equal to positive rho epsilon star f star g converges to g in $L^p(\mathbb{R}^N)$. Therefore, for epsilon less than some epsilon naught norm of rho epsilon star g minus g in L^p is less than say, so, let us take eta by 3. Now, norm rho epsilon star f minus f in L^p is less than equal to rho epsilon star f minus rho epsilon star g in L^p plus norm rho epsilon star g minus g L^p plus norm of f minus g in L^p. Now, this is then eta by 3 this also less than eta by 3 where for epsilon less than epsilon naught.

Now, this one by Young's inequality is less than or equal to norm rho epsilon 1 norm f minus g in L^p . This is equal to 1 and this is less than eta by 3 and therefore, the whole thing is less than eta for all 0 less than epsilon less than epsilon naught and therefore, that proves the corollary.

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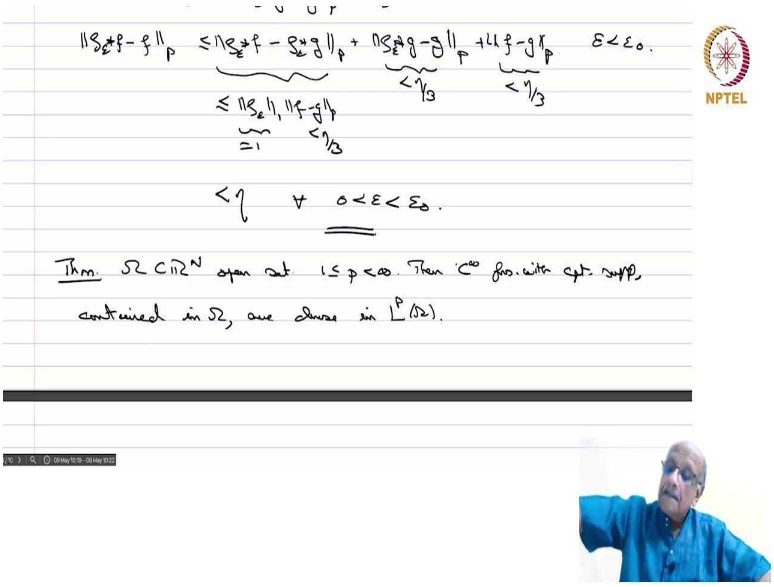

So, we prove that C infinity functions with compact support $\mathbb{R}^N$ in LP or dense in LP of RN. Now we want to show it for any open set omega in $\mathbb{R}^N$, so, omega contained in $\mathbb{R}^N$ open set 1 less than equal to p less than infinity then C infinity functions with compact support contained in omega are dense in LP of omega.

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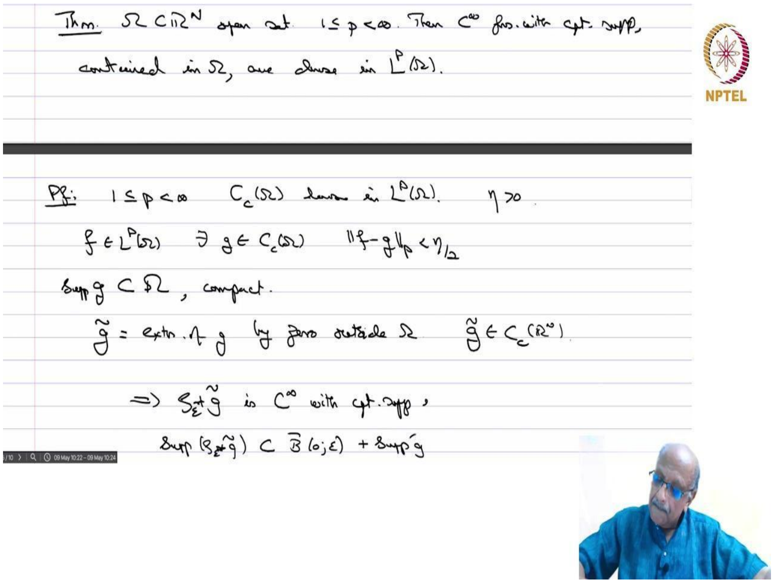

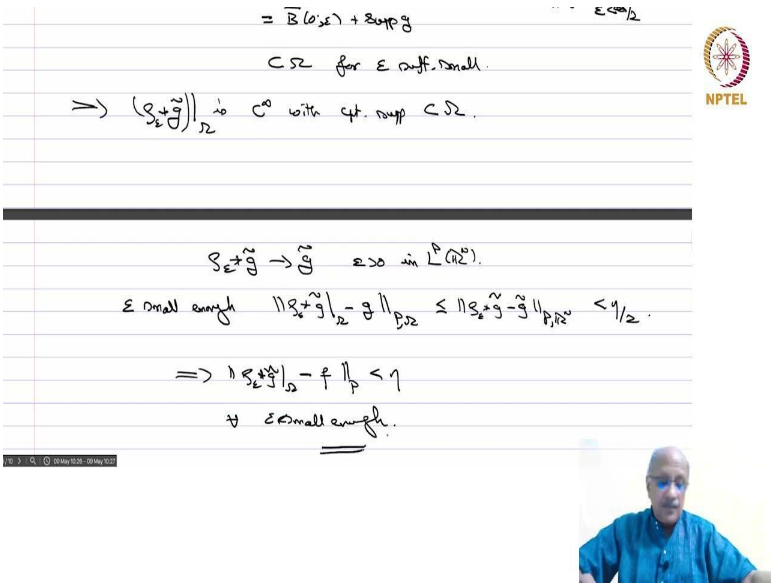

Proof, so, 1 less than equal to p less than infinity then CC of omega continues functions with compact support is dense in $L^p(\Omega)$, so, given f in $L^p(\Omega)$ there exists g in $C_c(\Omega)$ norm f minus g p is less than say eta by 2, so, eta greater than 0 is greater. Now, G is support of g is contained in omega and compact let g tilde equal to extension of g by zero outside domain then g tilde belongs to $C_c(\mathbb{R}^N)$ then rho epsilon star g tilde is C infinity with compact support and support of rho epsilon star g tilde is contained in B closure 0 epsilon that is a ball close ball center zero radius epsilon plus support of g tilde but that is equal to B zero epsilon plus support of g because g tilde is nothing but extension of g by zero and this is contained in omega for epsilon sufficiently small, what do we mean by this.

So, you have omega is here and then you have compact set K which is a support of g which is compact now, so, this is a compact set closed the boundary for omega is also close therefore, you take d to be the distance, shortest distance between the set and this way they are 2 disjoint compact and hence closed sets therefore, you know that the distance is positive. So, you just have to take epsilon less than d or d by 2 whatever you like d is enough let us say d by 2 to be absolutely safe and then you have this set will be contained in omega itself.

So, rho epsilon star g tilde restricted to omega is C infinity with compact support contained in omega and rho epsilon star g tilde converges to g tilde as epsilon goes to zero in $L^p(\mathbb{R}^N)$ that we already know and therefore, you have for epsilon small enough norm rho epsilon star g tilde restricted omega minus g in L^p, I will put omega this is less than or equal to norm of rho epsilon star g tilde minus d tilde in $L^p(\mathbb{R}^N)$ and this can be made less than eta by 2 and therefore, you have norm of rho epsilon star g tilde restricted omega minus f is therefore in L^p is less than eta for all epsilon less than epsilon naught epsilon small enough and that completes the proof.

So with this we complete this chapter, we will do some exercises next time.