Measure and Integration
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Lecture No-74
11.6 - Convolutions

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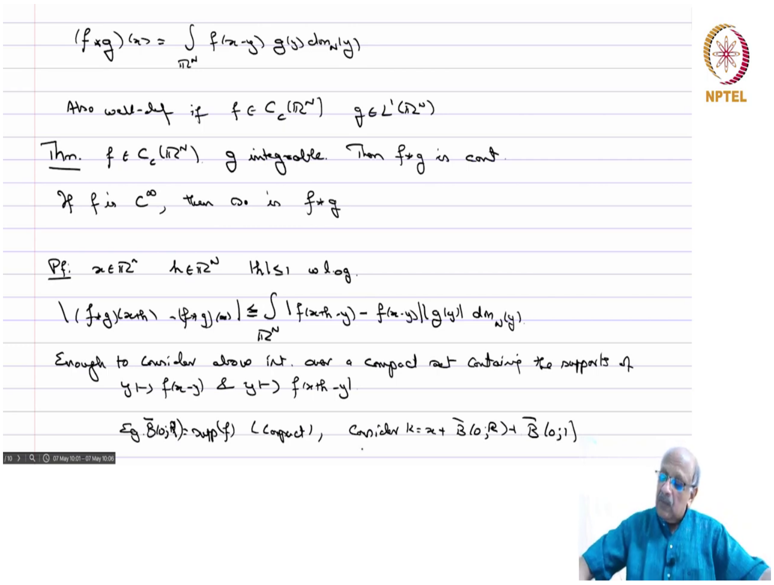

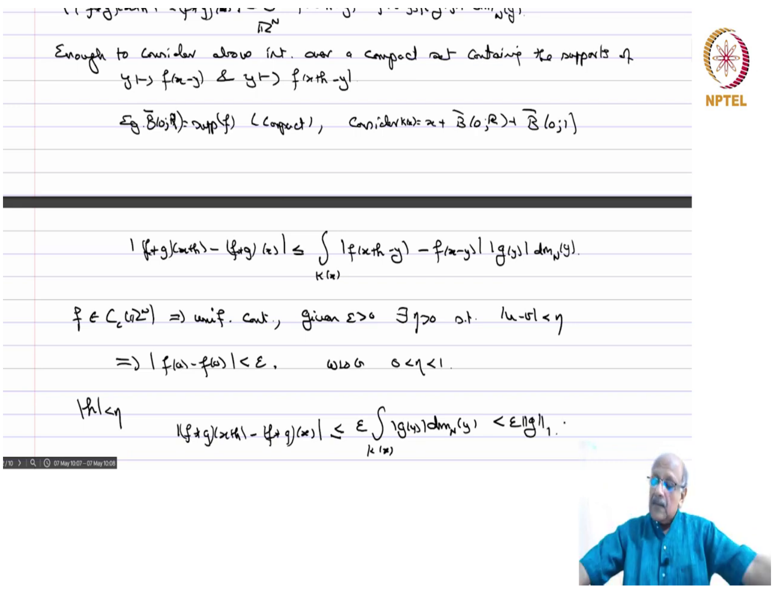

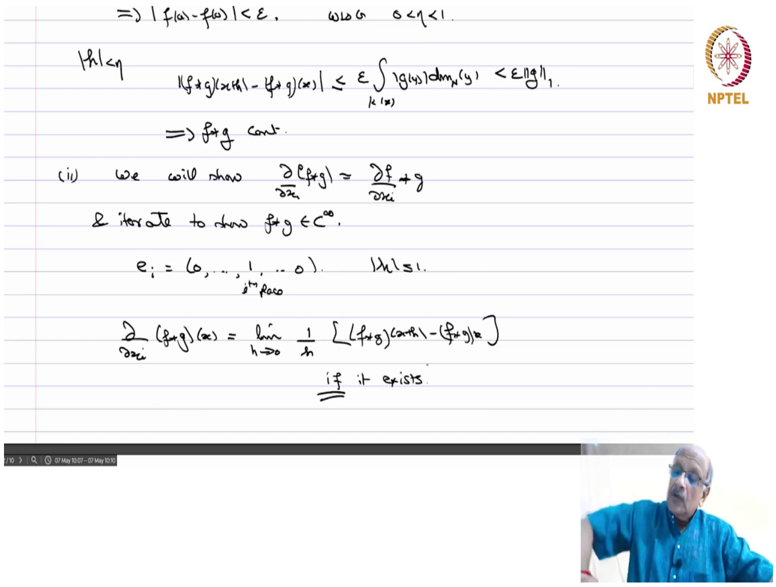

We were looking at convolutions. So, if you have two functions f star g, f and g, then

$$f * g(x) = \int_{\mathbb{R}^N} f(x - y)g(y)dm_N(y).$$

And we saw that you can define this if f and g are both integrable or f is and LP and g is, fs in L1, g is in LP and or if f and g are continuous with at least one of them of compact support. So, these were various things, you can also define.

So, also well-defined if $f \in C_c(\mathbb{R}^N)$, and $g \in L^1(\mathbb{R}^N)$. And therefore, you have that this is an integral part. So, we now have the really beautiful property of the convolution that it has a smoothing effect, you take a continuous function with compact support and convolve it with any L1 function, it can be any rough function, then the convolution is continuous.

Theorem: If $f \in C_c(\mathbb{R}^N)$, *and* g integrable, then $f * g$ is continuous. If f is C^∞ then so is $f * g$.

proof: So, you take $x \in \mathbb{R}^N$ fixed and then you take $h \in \mathbb{R}^N$ and let us assume $|h| \le 1$ without loss of generality. So,

$$|f * g(x + h) - f * g(x)| = \int_{\mathbb{R}^N} |f(x + h - y) - f(x - y)|\, g(y)dm_N(y)$$

Now, enough to take, enough to consider the above integral over a compact set containing the supports of the functions y going to f of x minus y and y going to f of x plus h minus 1. So, example, if k equals support of f which is a compact set, consider x plus k plus B 0 1 or rather if B 01 closure that is the closed ball central origin radius 1 is a support of f then you can consider k equals x plus B 0 R and because h is less than 1 you can take B 0 1.

So, this is a sum of two compact sets is compact and is translated by x which is also compact and outside this set it is clear that the integrand is 0, and therefore, it is enough to consider this integral over k. Now, let us call that k of x. So, you have mod f star g x plus h minus f star g of x is less than equal to integral over k of x of mod f of x plus h minus y minus f of x minus y mod g y d m n y.

Now, f has compact support, therefore, f is uniformly continuous, $f \in C_c(\mathbb{R}^N)$ implies uniform continuity, therefore, given an $\epsilon > 0$, $\exists\, \eta > 0$ such that

$$|u - v| < \eta \Rightarrow |f(u) - f(v)| < \epsilon.$$

So, if you take $|h| < \eta$, we can choose without loss of generality again $0 < \eta < 1$.

So, in that case you have

$$|f * g(x + h) - f * g(x)| \leq \epsilon \int_{K(x)} |g(y)| dm_N(y) < \epsilon ||g||_1,$$

and therefore, this shows that $f * g$ is continuous. So, now we want to show the second part, namely the differentiability.

What will we show? We will show d by dxi of f star g is the same as df by dxi star g then this is continuous with compact support and this is integrable, and therefore, this is still continuous. That means, the first derivative of f star g is continuous. Now, you can repeat this for any day. We do one by one and iterate this procedure and so iterate to show $f * g$ is C infinite.

So, we take e i to be the standard basis 0 1 in the i-th place and 0 elsewhere. And then you take again mod h less than equal to 1. So, d by dxi of f star g at x equal to limit h tending to 0 of 1 over the h integral $f * g$ at x plus h minus f star g at x if it exists. So, the limit must not be obvious that it is going to exist. We have to show that also.

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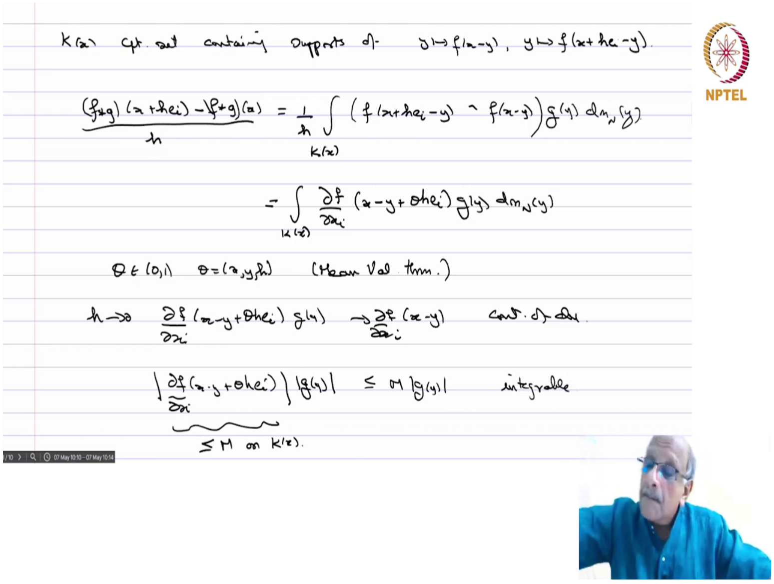

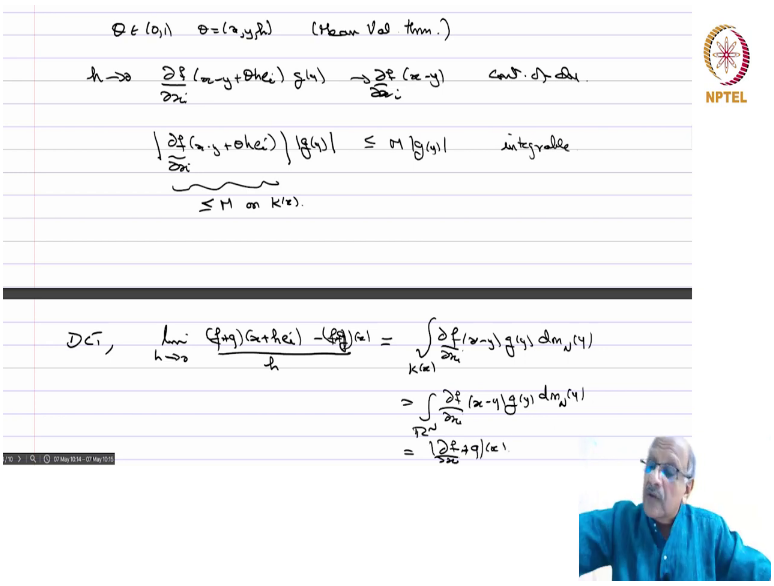

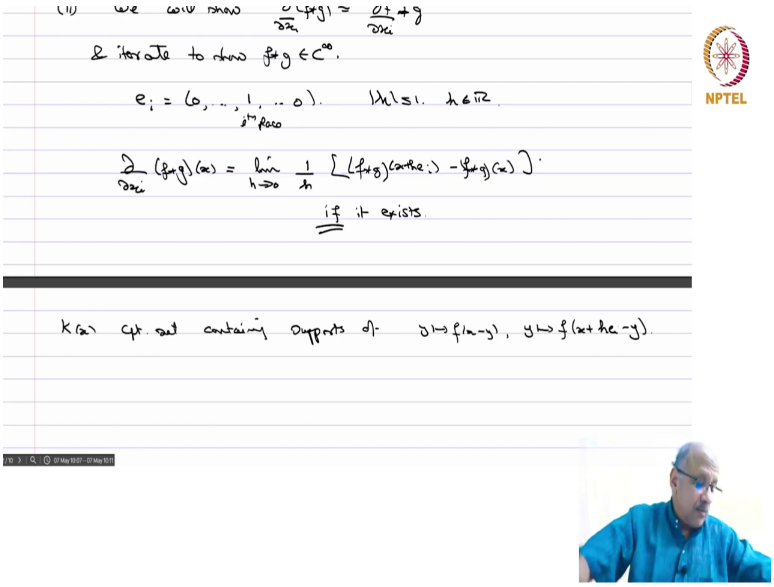

So, you again take the K(x)compact set containing supports of $y \rightarrow f(x - y)$ and $y \rightarrow f(x + he_i - y)$. Now, h is the real number, so, let me not confuse you, so h less than equal to 1 h in r. So, we were giving an increment in the i-th direction that is why we brought in (())(10:09).

So, let me correct this right away, this is h e i minus f star g of x. So, f star g of x plus h e i minus f star g of x divided by h is equal to 1 by h integral again it is enough for me to take the integral over k x, f of x plus h e i minus y minus f of x minus y times g y d m n y. That is equal to integral over k x.

Now, you have a differentiable function and therefore, you can apply the mean value theorem this will be nothing but df by dxi at x plus x minus y plus theta times h e i g y d m n y, where theta belongs to 0, 1, theta depends on x, y and h, this is the mean value theorem. The value of the derivative at any intermediary points and then into h will come and that will cancel with the 1 by h here and that is why we only have this.

Now, as h goes to 0 df by dxi x minus y plus theta h e i g y this will go to df by dxi of x minus y because by continuity of derivative. And also, mod df by dxi x minus y plus theta h e i mod g y. Now, this is a continuous function and over the compact set, this is less than equal to the M on k x, it is bounded and continuous function is bounded over a compact set, anyway, it is a continuous function with compact support. So, it is bounded everywhere in fact, so, this is simply M times mod g y and this is integral.

Therefore, by the dominated convergence theorem, you have limit h tending to 0 f star g of x plus h e i minus f star g of x by h is equal to integral over k x of f of x df by dxi and x minus y g y d m n y and that is equal to integral over Rn df by dxi x minus y g y dx d m n y, it is equal to df by dxi star g evaluated at x and that is continuous, and therefore, now we can iterate this procedure and obtain the result.

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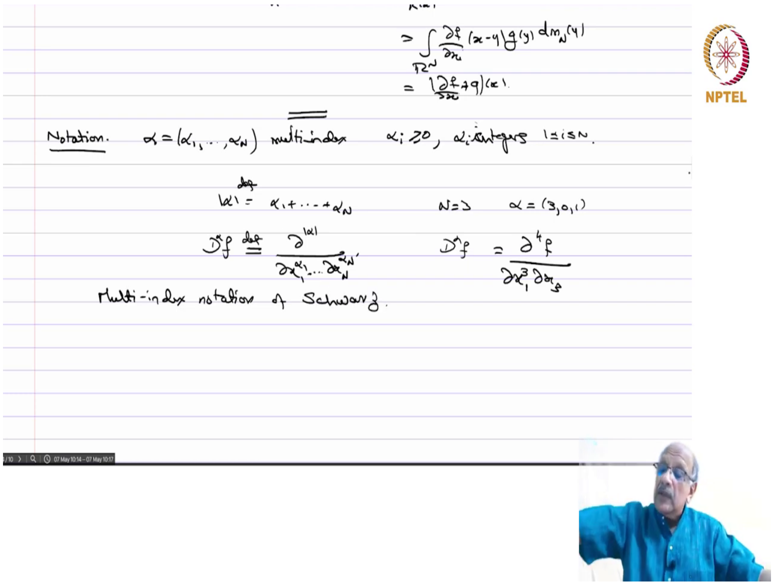

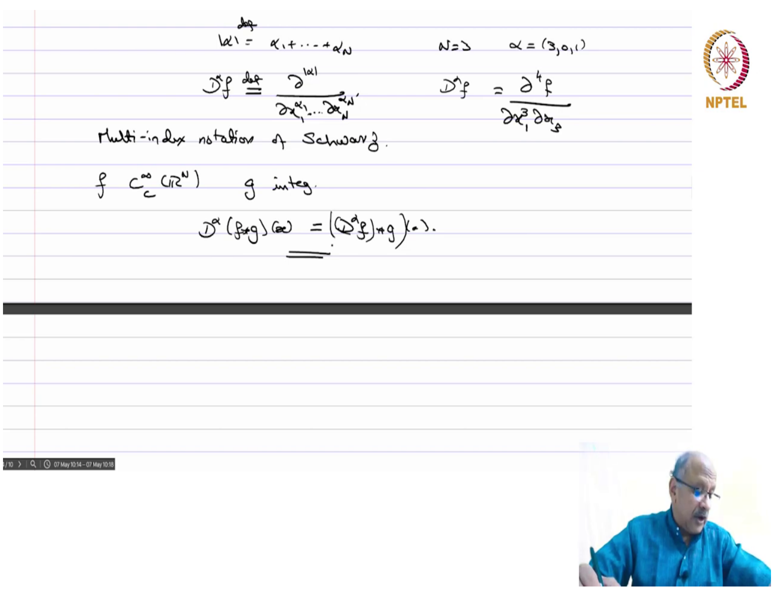

Notation: $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ where $\alpha_i \geq 0$, α_i integers. Then we set

$$|\alpha| = |\alpha_1| + |\alpha_2| + ... = |\alpha_N|.$$

And then $D^{\alpha} f = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}_1 \partial x^{\alpha_2}_2 .... \partial x^{\alpha_N}_N}$.

So, this is how you write this very compact notation, this is due to Laurent Schwartz, so multi-index notation of Schwartz for dealing with partial derivatives in Rn. So, then we have if $f \in C^{\infty}_c(\mathbb{R}^N)$, g integrable then $D^{\alpha}(f * g)(x) = ((D^{\alpha} f) * g)(x)$. So, this is how the theorem is proved. So, now this we can use, so, what we do to get C^{∞} function all you have to do is take an integrable function and take the convolution with the C infinity function with compact support.

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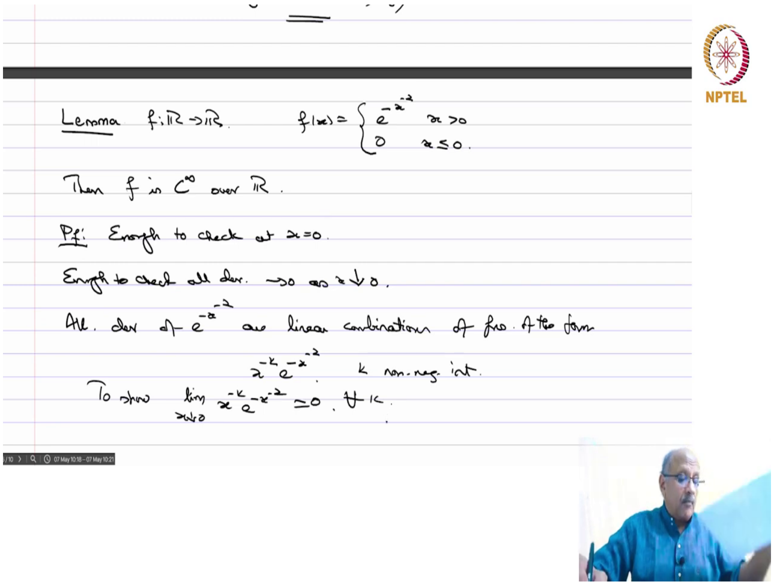

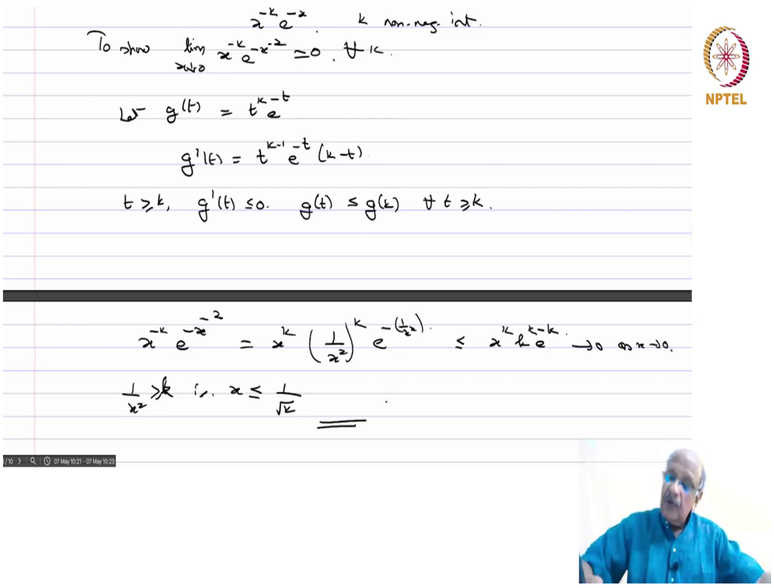

Lemma: $f: \mathbb{R} \rightarrow \mathbb{R}$. So, you take

$$f(x) = e^{-x^{2}}, \ x > 0,$$

$$= 0, \ x \leq 0.$$

Then f is C^{∞} over $\mathbb{R}$.

proof: enough to check at $x = 0$, because if x is strictly positive we have a nice analytic function here, and therefore, it is certainly seeming at e, if x is strictly less than 0 it is 0, everything is 0, derivatives are also 0. So, all you have to show is that the derivatives are continuous across the origin which is the only point where you may have some difficulty.

So, we want to show that, if we check all derivatives tend to 0 as x tends to 0, as x decreases to 0 because on the left you do not have anything to check, so as x decreases to 0, we have to check this. But all derivatives of e power minus x power minus 2 are linear combinations of functions of the form x power minus k e power minus x power minus 2.

Because when you differentiate this function f of x any number of times it will just be a linear combination of terms of this form, so, where k is a non-negative integer. So, we have to show, so, to show $\lim_{x\downarrow 0} x^{-k}e^{-x^2} = 0, \forall\ k$. So, we consider let $g(t) = t^k e^{-t}$. Then

$g'(t) = t^{k-1}e^{-t}(k - t)$. So, if $t \geq k$, $g'(t) \leq 0$. And therefore, $g(t) \leq g(k)$, $\forall\, t \geq k$.

So, you take $x^{-k}e^{-x^2} = x^k(\frac{1}{x^2})^k e^{-\frac{1}{x^2}} \leq x^k h^{1-k} e \to 0\ as\ x \to 0$.

And therefore, that completes the proof that this from all derivatives go to 0.

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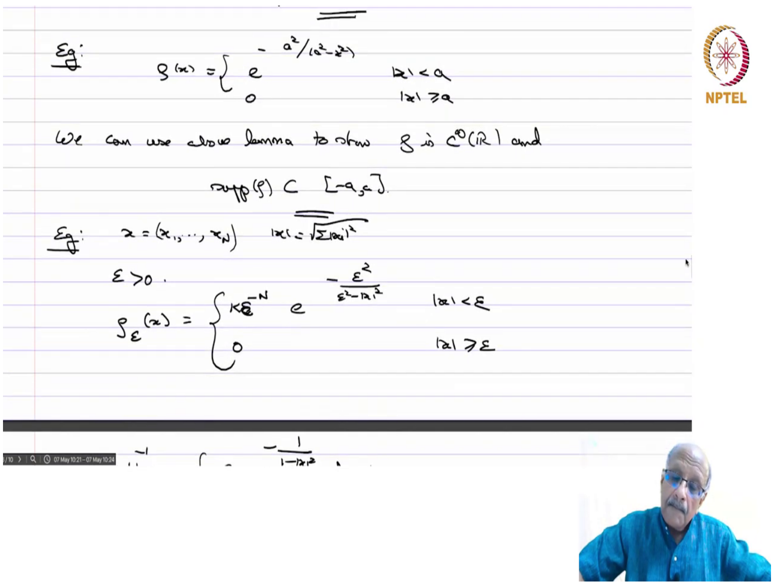

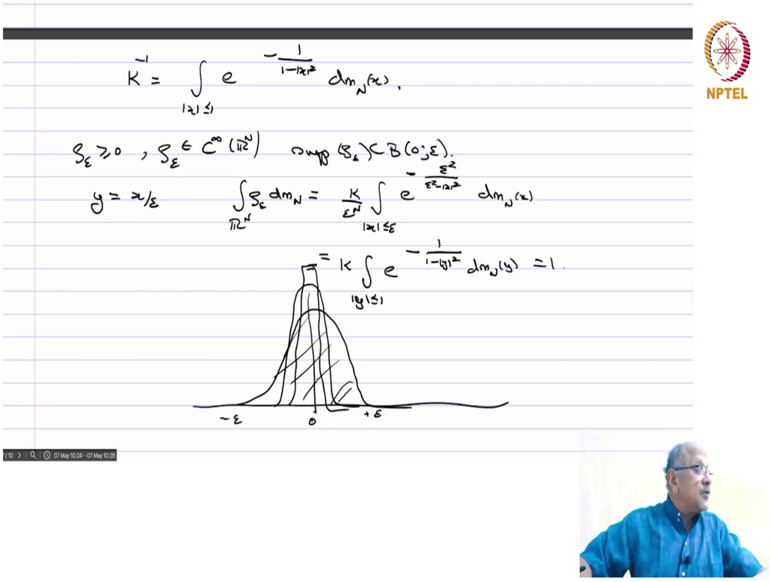

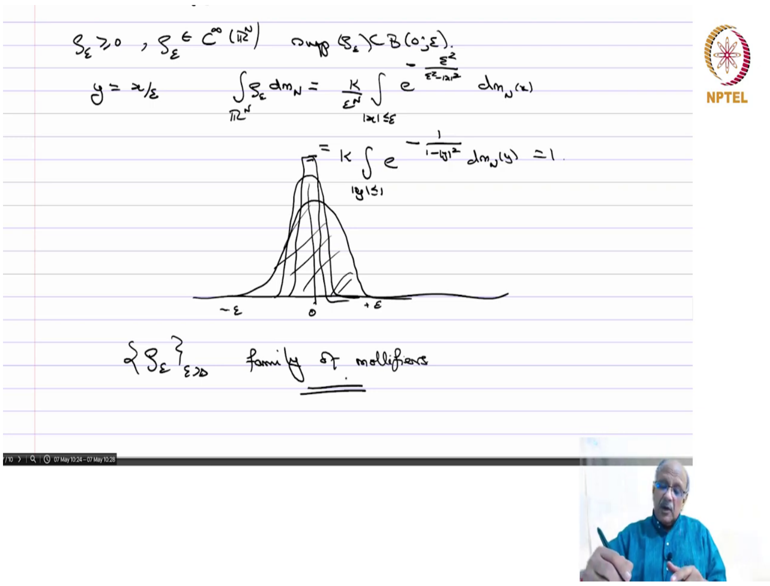

(1) So, now, we can use this to construct C infinity functions with compact support. You define

$$\rho(x) = e^{-\frac{a^2}{a^2-x^2}}, \quad if\ |x| < a,$$

$$= 0, \quad if\ |x| \geq a.$$

So, we can use the above lemma to show rho is C infinity of R and support the ρ, of course, is contained in minus a and a. Now, we can copy this and then do it in higher dimensions.

(2) So, if you take $x = (x_1, x_2, ..., x_N)$, $|x| = \sqrt{\sum |x_i|^2}$.

and then you have that if let $\epsilon > 0$. So, you define

$$\rho_{\epsilon}(x) = k\epsilon^{-N} e^{-\frac{\epsilon^2}{\epsilon^2 - |x|^2}}, \ if\ |x| < \epsilon,$$

$$= 0, \ \ if\ |x| \geq \epsilon.$$

where $k^{-1} = \int_{|x| \leq 1} e^{-\frac{1}{1-|x|^2}} dm_N(x)$.

Then of course, ρ_{ϵ} is non-negative, $\rho_{\epsilon} \in C^{\infty}(\mathbb{R}^N)$ and support of ρ_{ϵ} is contained in the ball center origin and radius epsilon. So, this gives you a C^{∞} function with compact support. Now, if you chain the variable y equals x by epsilon this linear change of variable then you know the determinant of this is 1 by epsilon power n.

So, you get that $\int_{\mathbb{R}^N} \rho_{\epsilon} dm_N(x) = \frac{k}{\epsilon^N} \int_{|x| \leq \epsilon} e^{-\frac{\epsilon^2}{\epsilon^2 - |x|^2}} dm_N(x) = k \int_{|y| \leq 1} e^{-\frac{1}{1-|y|^2}} dm_N(y) = 1.$

So, these functions are the familiar bell-shaped functions which you have probably seen in probability. So, you have 0 here and this is the minus epsilon and this is plus epsilon and then the area under this curve is equal to 1.

And so, if epsilon becomes smaller this function will become steeper and steeper like this, so that the area under the curve is always equal to 1, but its support is smaller and smaller. So, at the origin it will rise very rapidly and itself. Now, these are very useful functions and this family of functions is called the mollifiers. So, rho epsilon, epsilon greater than 0 are called family of mollifiers. So, next time we will see how we can use this together with the idea of, so these C infinity functions with compact support. In addition, they have the property that the integral is equal to 1. Now, we will use these functions in a nice way to prove some density theorems in LP space.