Measure and Integration
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Lecture No-73
11.5 - Convolutions

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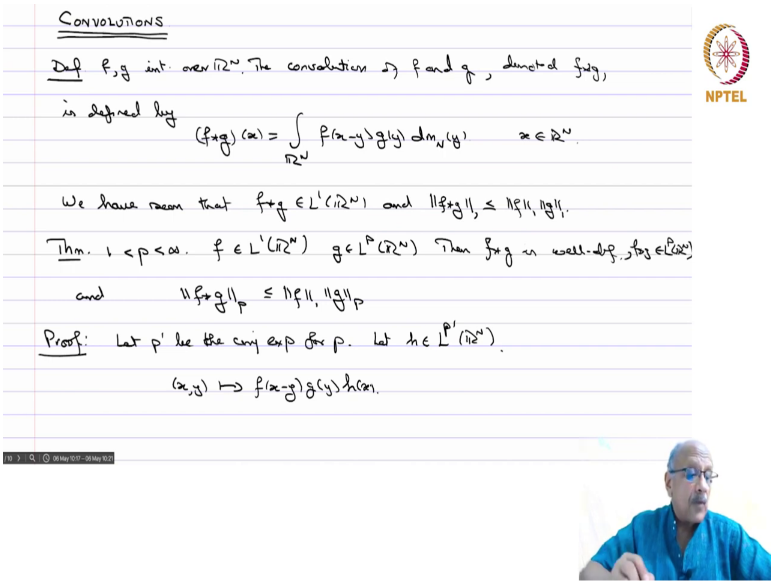

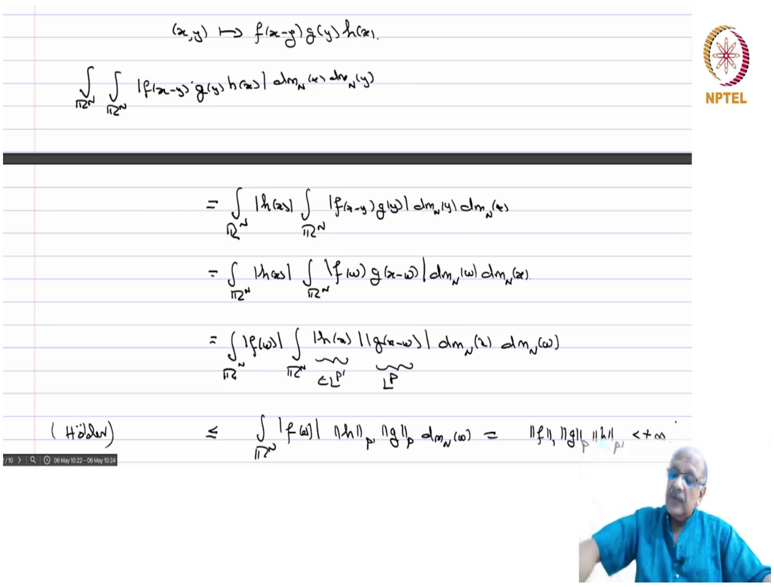

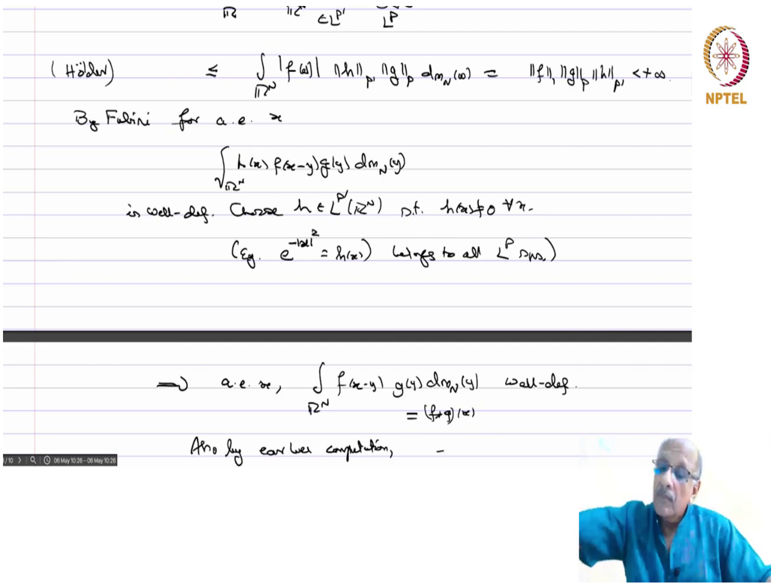

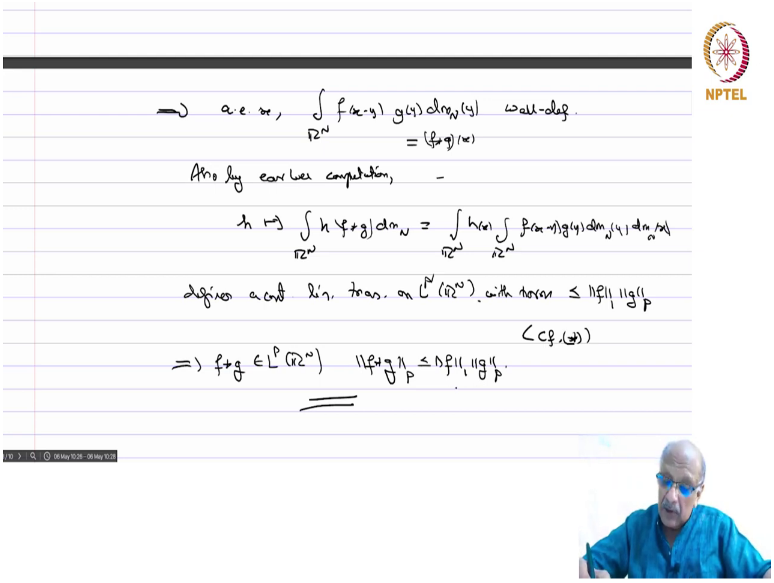

We will all study Convolutions. This is a very important tool in analysis and especially in the study of partial differential equations and also in probability theory. So, we have already seen this in the context of Fubini's theorem, and so, we have the following, I recall the following:

Definition. So, f and g integrable over $\mathbb{R}^N$. The convolution of f and g, denoted by $f * g$ is defined by $f * g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy\ ,\ for\ x \in \mathbb{R}^N$.

We have seen that this is well defined, in fact, we started with Borel measurable functions, and she used Fubini's theorem to show it is well defined and then every measurable function can be written almost everywhere equal to Borel measurable function and where integration goes it makes no difference.

And therefore, we showed that this is also true for lebesgue measurable functions. Further, we also saw we have seen that $f * g \in L^1(\mathbb{R}^N)$ and $||f * g||_1 \leq ||f||_1 \, ||g||_1$, this is a particular case of what is called Young's inequality, which we will now extend. So, we will now try to extend the definition of the convolution product to other functions, not just L1 functions.

Theorem: $1 < p < \infty$, $f \in L^1(\mathbb{R}^N)$, $g \in L^p(\mathbb{R}^N)$. Then $f * g$ is well defined and $f * g \in L^p(\mathbb{R}^N)$, $||f * g||_p \leq ||f||_1 \, ||g||_p$.

proof: we are going to use the duality method. So, let p' be the conjugate exponent for p. So, let $L^p(\mathbb{R}^N)$. Now, you consider the map $(x, y) \rightarrow f(x - y)g(y)h(x)$. And we are going to use Fubini's theorem, translation invariance of Lebesgue measure and Hölder's inequality, all of these results we are going to use. So,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x - y)g(y)h(x)| dm_N(x) dm_N(y) = \int_{\mathbb{R}^N} |h(x)| \int_{\mathbb{R}^N} |f(x - y)g(y)| dm_N(y) dm_N(x)$$

$$= \int_{\mathbb{R}^N} |h(x)| \int_{\mathbb{R}^N} |f(w)g(x - w)| dm_N(w) dm_N(x)$$

$$= \int_{\mathbb{R}^N} |f(w)| \int_{\mathbb{R}^N} |h(x)g(x - w)| dm_N(w) dm_N(x)$$

$$\leq \int_{\mathbb{R}^N} |f(w)| ||h||_{p'} \, ||g||_p \, dm_N(w) = ||f||_1 \, ||h||_{p'} \, ||g||_p \ <+ \infty.$$

So, by Fubini for almost every x you have that $\int_{\mathbb{R}^N} f(x - y)g(y)h(x)dm_N(y)$ is well-defined. Now, chose $h \in L^{p'}(\mathbb{R}^N)$ such that $h(x) \neq 0$ for all x. So, an example of such a function e power minus mod x square equals h x. We have seen how to evaluate this integral and this in fact belongs to all Lp spaces. So, this is a function which you have. So, if you substitute that here, this is Rn. So, that will imply then you can cancel out the h, so, that implies that for almost every x, $\int_{\mathbb{R}^N} f(x - y)g(y)dm_N(y) = f * g(x)$ is well-defined.

What is this? This is equal to integral over Rn h of x integral Rn f of x minus y g y d m n y d m n x defines a continuous linear transformation on Lp dash Rn. And with norm less than equal to norm f 1 norm g p that comes from star, c f star.

So, this implies that $f * g \in L^p(\mathbb{R}^N)$ and $||f * g||_p \leq ||f||_1 \, ||g||_p$.

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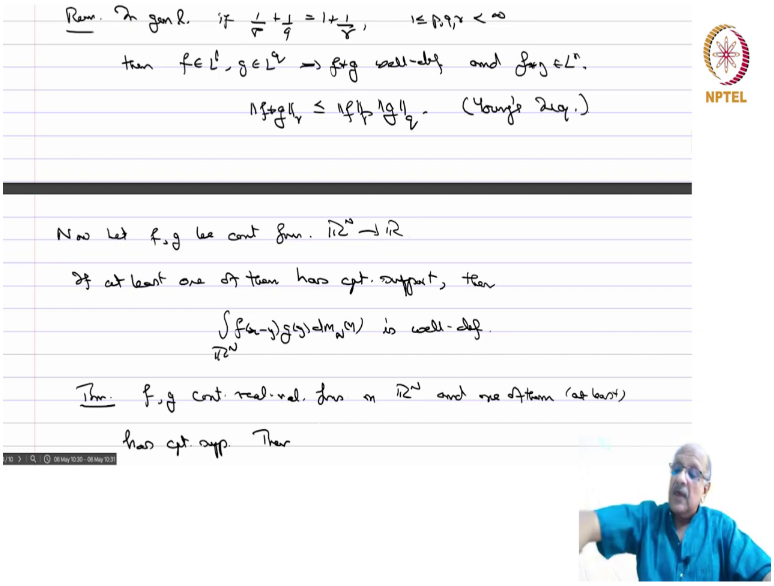

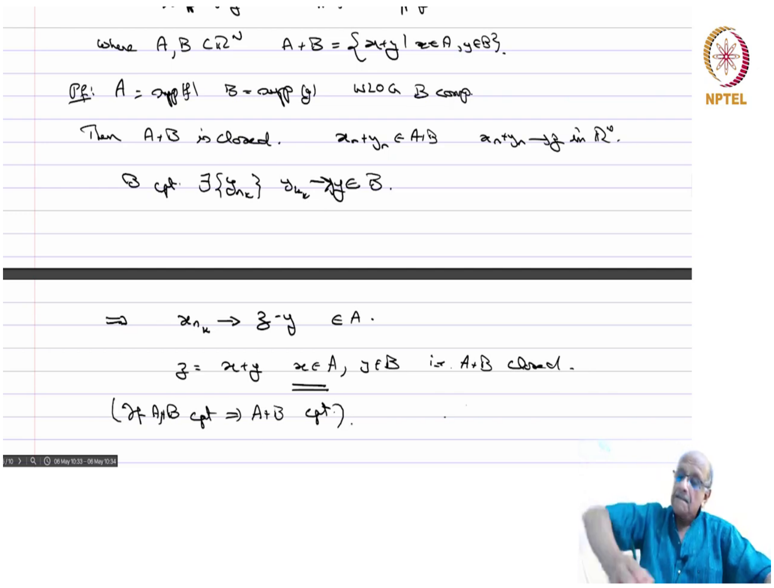

Remark: In general if $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $1 \leq p, q, r < \infty$, then $f \in L^1, g \in L^q$

$$\Rightarrow f * g \text{ is well defined and } f * g \in L^r \text{ and } ||f * g||_r \leq ||f||_p \, ||g||_q .$$

and this is called Young's Inequality. So, what we have proved so far is a particular case of Young's Inequality. So, now, these are the convolutions Lp functions.

Now, let f g be continuous functions Rn to R if at least one of them has compact support then

$\int_{\mathbb{R}^N} f(x - y)g(y)h(x)dm_N(y)$ is well-defined because you have to integrate for instance if g has compact support then you just integrate over the support of g which is a compact set then the f and g are continuous, therefore, they are bounded compact sets of finite measure. So, this integral is finite, and therefore, it is well-defined.

Theorem: f, g continuous real-valued functions on $\mathbb{R}^N$ and one of them at least has compact support. Then, we have seen that $f * g$ is well-defined, $supp(f * g) \subset supp(f) + supp(g)$.

proof: so, A equals support of f, B equals support of g, one of them is compact, so without loss of generality, we take B compact. Because when you define f of x minus y because by commutativity of the integral it is f of x minus y g y or g of x minus y f y it does not matter.

So, you can take any one of them to be of compact support. Then A plus B is closed. Normally, when you have closed sets the sum is not closed. Whereas if you have one closed and one compact then the sum is closed. So, let x n plus y n belong to A plus B and x n plus y n converges to z in Rn. Now, B is compact. So, there exists y n k subsequence, y n k converges to y and compact sets are closed, therefore, y will also be in B.

Then this will mean that x n k will converge to z minus y and this has to be in the A. So, that means that z equals x plus y when x is in the A and y is in B that is A plus B is closed. So, if A plus B is both compact, then you can easily see that A plus B is also compact, because you have sequences that have convergent subsequences, and therefore, if f g, so, this is just a remark. So, A plus B is closed.

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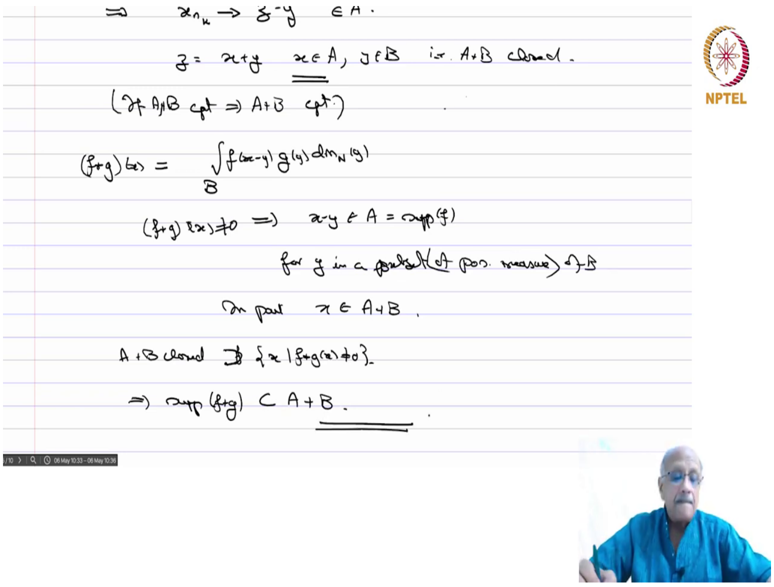

So, now, if you look $f * g(x) = \int_B f(x - y)g(y)dm_N(y)$. And therefore, if $f * g(x) \neq 0$, this implies that $x - y \in A$, which is a support of f for y in a subset of positive measure of B.

Only then, so only if this is nonzero for some set of positive measure subset to B will this is the nonzero, with the integral will be nonzero. In particular, you have $x \in A + B$. Now, $A + B$ is closed and contains the set of all x such that f star g x is not zero. And therefore, this implies that $supp(f * g) \subset A + B$.

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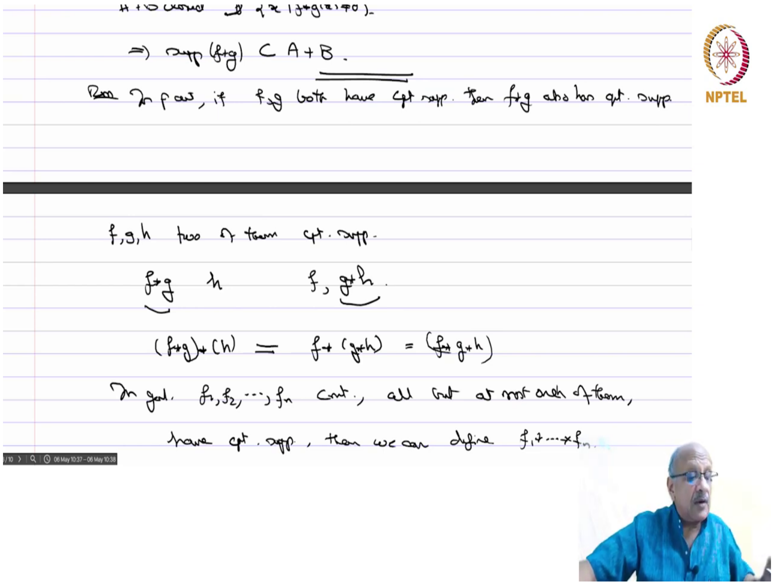

So, in particular, remark, if f, g both have compact support then $f * g$ also has compact support. So, now, if you have f, g, h two of them compact support then $f * g$ and you take h and f and $g * h$. Now, if f and g have compact support then $f * g$ has compact support, if g and h have compact support then $g * h$ have compact support.

So, in either case whether you take $(f * g) * h$ for the f and g star h both of them at least one of them has compact support, and consequently, you have the f star g star h is well-defined and similarly f star g star h is well-defined. And because of associativity these two are in fact equal and you can write this as f star g star h without worrying about putting, that means you are defining it 2 by 2. In general, f_1, f_2,....., f_n continuous at least n minus 1 of them.

So, let us not write like that. All but at most one of them have complex support, then we can define $f_1 * f_2 * \ldots * f_n$. The same way as I did for three functions you can define it 2 by 2 for various functions. So, we will continue with the properties of convolutions. So, we will show that its utility is namely it has a smoothing effect, you can convolve, suppose I take an integrable function and a continuous function with compact support then the convolution is continuous like that. So, we will see such properties next time.