Measure and Integration
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No-72
11.4 - Duality

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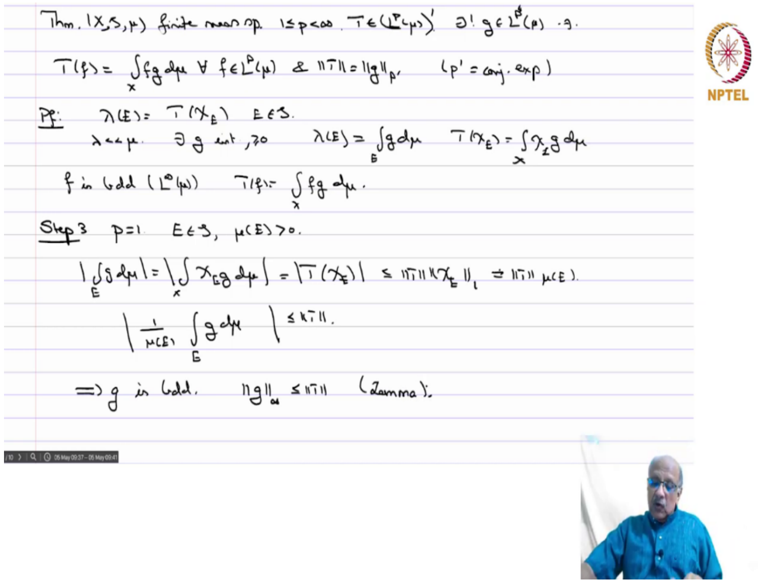

We are now in the process of the proof of the theorem.

Theorem: (X, S, μ) finite measures space, $1 \leq p < \infty$, and $T \in (L^p(\mu))'$, and then there exists a unique $g \in L^{p'}(\mu)$, such that

$$T(f) = \int_X fgd\mu \, \forall f \in L^p(\mu) \ and \ ||T|| = ||g||_{p'}.$$

proof: so we defined $\lambda(E) = T(\chi_E), \ E \in S$. And this defined assigned measure and lambda was absolutely continuous with respect to mu and therefore there exists a g, which is and this also finite measure, so g is integrable non negative, and such that

$$\lambda(E) = \int_E gd\mu, \ T(\chi_E) = \int_X \chi_E gd\mu.$$

And then by linearity this extends to simple functions, and finally we prove that f is bounded, that is L infinity mu, then $T(f) = \int_X fgd\mu$.

So now, we want to show that this is true in fact for all f in Lp m not just bounded functions, bounded functions are Lp mu because we are in a finite measure space but otherwise, we will not show this.

Step 3: assume that p =1, let E in S, mu E positive, then

$$|\int_E g d\mu| = |\int_X \chi_E g d\mu| = |T(\chi_E)| \leq ||T||\, ||\chi_E||_1 = ||T||\, \mu(E).$$

$$\Rightarrow |\frac{1}{\mu(E)}\int_E g d\mu| \leq ||T||.$$

So this implies that g is bounded, and $||g||_\infty \leq ||T||$.

This is by the lemma which we proved already. The average over all sets of positive measures is bounded by the same constant, then g is also bounded by the same constant, so this is.

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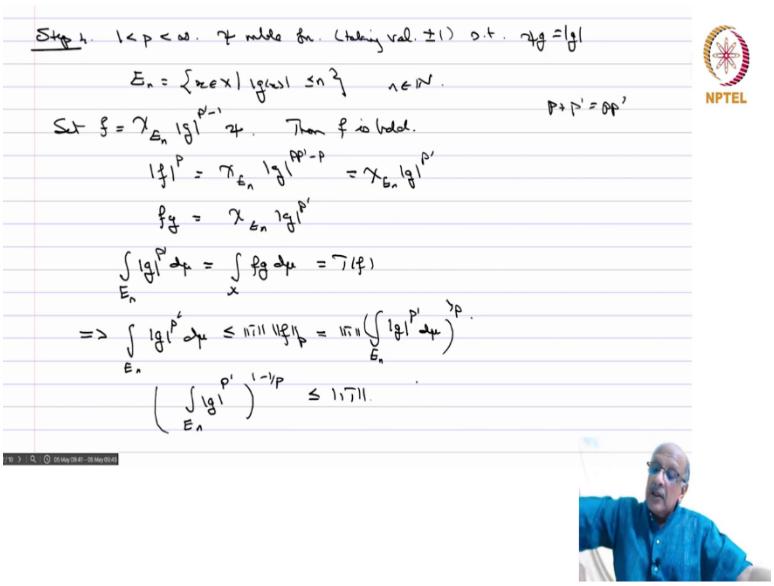

Step 4: So now, $1 < p < \infty$, ψ psi measurable function, taking values plus or minus 1, such that $\psi g = |g|$, when g is positive you put plus 1, so you get psi g equals much.

Now, you set $E_n = \{x \in X : |g(x)| \leq n\}$, $n \in \mathbb{N}$.

Now, you set $f = \chi_{E_n} |g|^{p'-1}\psi$. Then, f is bounded. And also,

$$|f|^{p} = \chi_{E_n}|g|^{pp'-1} = \chi_{E_n}|g|^{p'}.$$

And $fg = \chi_{E_n}|g|^{p'}$. So, we get

$$\int_{E_n}|g|^{p'}d\mu = \int_{X} fg\,d\mu = T(f) \Rightarrow \int_{E_n}|g|^{p'}d\mu \le ||T||\,||f||_{p} = ||T||(\int_{E_n}|g|^{p'}d\mu)^{\frac{1}{p}}.$$

$$\Rightarrow (\int_{E_n}|g|^{p'}d\mu)^{1-\frac{1}{p}} \le ||T||.$$

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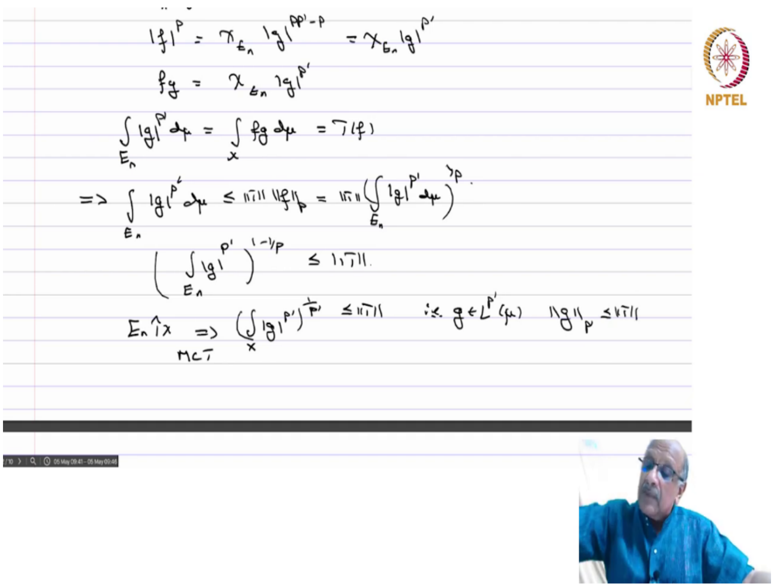

Now, En increases to x, so this implies that integral over x by the monotone convergence theorem, this implies that mod g p dash, for 1 minus 1 by p is just 1 by p dash is less than equal to t, that is g belongs to Lp dash mu, and norm g p dash is less than equal to norm, so this is the same result which we had, g belongs to the conjugate exponent, and the corresponding norm is less than equal to norm t, so that is what we have proved after this.

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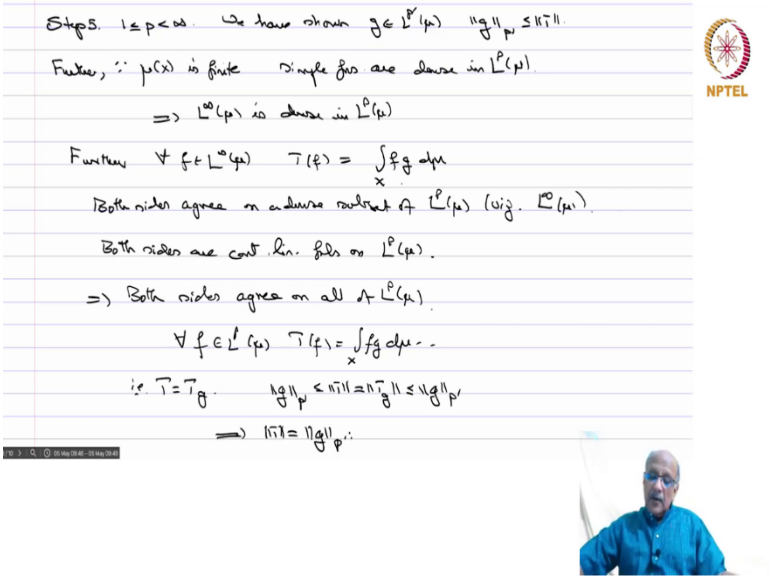

Step 5: $1 \leq p < \infty$. Now, we have shown g belongs to Lp dash mu and norm g in Lp dash is less than equal to norm T. Further, since mu X is finite, simple functions are dense in Lp mu, why? Because simple functions which vanish outside the set of finite measure are dense in Lp mu is what we have already seen in a lemma, but now since anyway the set has finite measure, all simple functions are dense, so this implies sort L infinity mu is dense in Lp mu.

Because L infinity mu is a bigger set than the simple functions, and therefore L infinity mu is density mu. Further for every $f \in L^{\infty}(\mu)$, $T(f) = \int_X fg d\mu$, both sides agree on a dense subset of $L^p(\mu)$, namely L infinity mu. Both sides are continuous linear functionals on $L^p(\mu)$, implies both sides agree on all of $L^p(\mu)$.

So, for every $f \in L^p(\mu)$, you have $T(f) = \int_X fg d\mu$, $i.e.$, $T = T_g$ and you know that

$$||g||_{p'} \leq ||T|| = ||T_g|| \leq ||g||_{p'} \Rightarrow ||T|| = ||g||_{p'}.$$

That completes the proof of the theorem. So, we have shown everything which we wanted to show, and therefore you have in the case of finite measure spaces, every continuous linear functional occurs from the space of the conjugate exponent.

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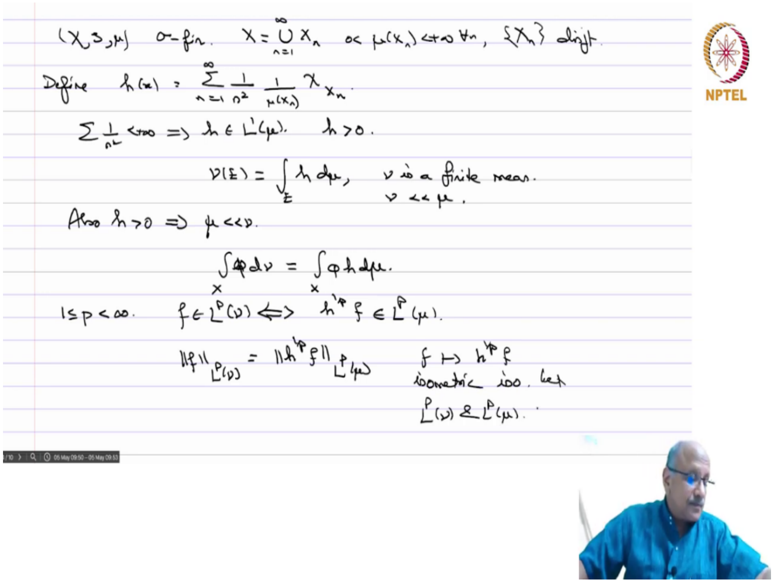

So now, let us take (X, S, μ) σ −sigma finite, then $X = \cup_{n=1}^{\infty} X_n$, $0 < \mu(X_n) < +\infty$, $\{X_n\}$ disjoint. So, you can take them all or positive measure also. So, now you define

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{\mu(X_n)} \chi_{X_n}.$$

h of x equals sigma, n equals 1 to infinity, 1 over n square, 1 over mu of x n that is all will define because it is into chi of xn.

Then, sigma 1 by n square is finite and therefore this implies h belongs to L 1 of mu, and h is also strictly positive. So, if you define $\nu(E) = \int_E h d\mu$, then you have ν is a finite measure, also you have that ν is absolutely continuous with respect to mu, also h strictly positive, so this means that $\mu << \nu$. So now, $\int_X \phi d\mu = \int_X \phi h d\mu$. So therefore, let 1 less than equal to p less than infinity, then $f \in L^p(\nu) \Leftrightarrow h^{\frac{1}{p}} \in L^p(\mu)$, and $||f||_{L^p(\nu)} = ||h^{\frac{1}{p}}||_{L^p(\mu)}$.

So this gives you an isometric isomorphism between, so $f \to h^{\frac{1}{p}} f$ is an isometric isomorphism, between $L^p(\nu)$ and $L^p(\mu)$.

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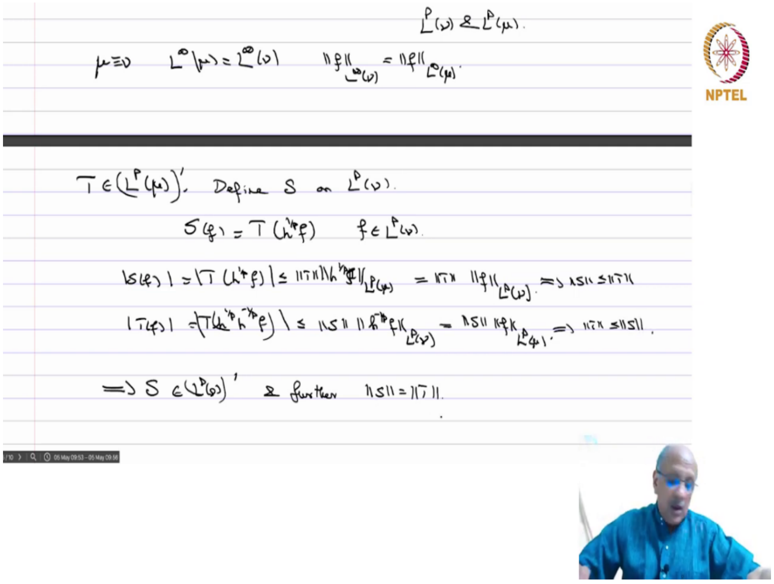

But mu is equivalent to nu, therefore L infinity mu equals L infinity of nu, and norm f infinity L infinity nu is the same as norm f. L infinity of mu, there is no change because your sets of measures 0, are one and the same. So now, let T belong to Lp mu prime, so define S on Lp nu as S of f equals T of h power 1 by pf, this is well defined because this in Lp, and this is for f in Lp nu.

So, mod sf equal to T of h power 1 by pf, that is less than equal to norm T into norm h power 1 by pf, Lp mu which equal to norm t, norm f Lp mu, okay. And T of f is equal to T of h power 1 by pM h power minus 1 by pf which is less than equal to norm S, T of h power 1 by p into norm of h power minus 1 by pf in Lp nu which is equal to norm S, norm f, Lp nu.

So, this implies that S is a belongs to Lp nu, prime and further norm of S is the same as norm (())(18:48), because this tells you that norm T is less than equal to norm S, norm S is less equal to norm t, and therefore you have, so this implies that norm is less equal to norm T, and this implies that norm T less than equal to norm s, and therefore norm S is equal to norm T.

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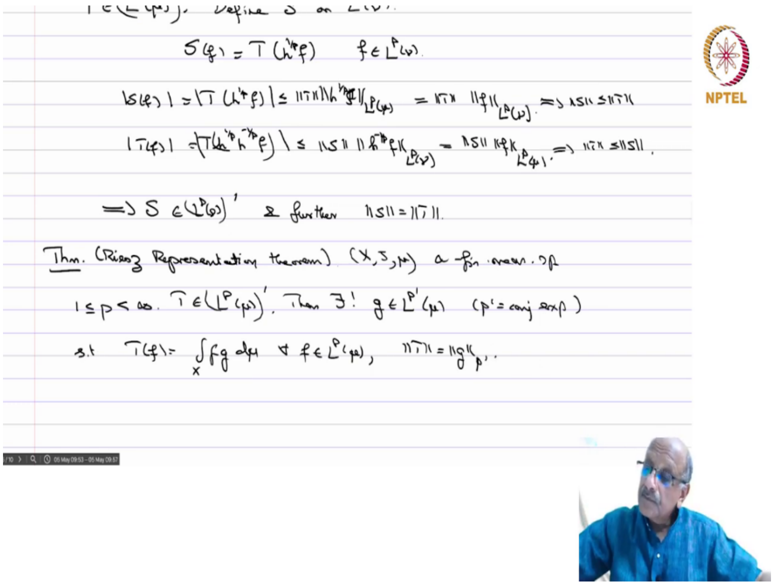

So, with these notations, we now prove the following theorem, so theorem, so this is called the Riesz representation theorem.

Theorem: so (X, S, μ) $\sigma -$ sigma finite measure space, $1 \leq p < \infty$, $T \in (L^p(\mu))'$. Then there exists a unique $L^{p'}(\mu)$, such that a of $T(f) = \int_X fgd\mu$ $\forall f \in L^p(\mu)$, $||T|| = ||g||_{p'}$.

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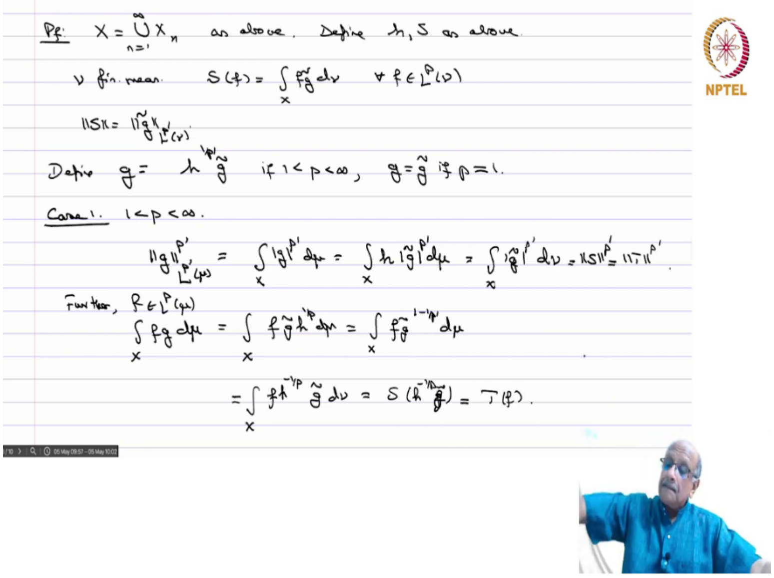

proof: So you write $X = \cup_{n=1}^{\infty} X_n$, then h defines h, S, as above. So, nu is a finite measure, and therefore $S(f) = \int_X fgd\mu \, \forall f \in L^p(\mu)$ and you have that $||S|| = ||\tilde{g}||_{L^{p'}(\mu)}$.

Now, define $g = h^{\frac{1}{p}}\tilde{g}\ if\ 1 < p < \infty,\ g = \tilde{g}\ if\ p = 1.$

case 1: $1 < p < \infty$: so

$$||g||^{p'}_{L^{p'}(\mu)} = \int_X |g|^{p'} d\mu = \int_X h\ \tilde{g} d\mu = \int_X |\tilde{g}|^{p'} d\mu = ||S||_{p'} = ||T||_{p'}.,$$

Further, $\int_X fg d\mu = \int_X f h^{\frac{1}{p}} \tilde{g} d\mu = \int_X f \tilde{g}^{1-\frac{1}{p'}} d\mu = \int_X f h^{-\frac{1}{p}} \tilde{g} d\mu = S(h^{-\frac{1}{p}} f) = T(f).$

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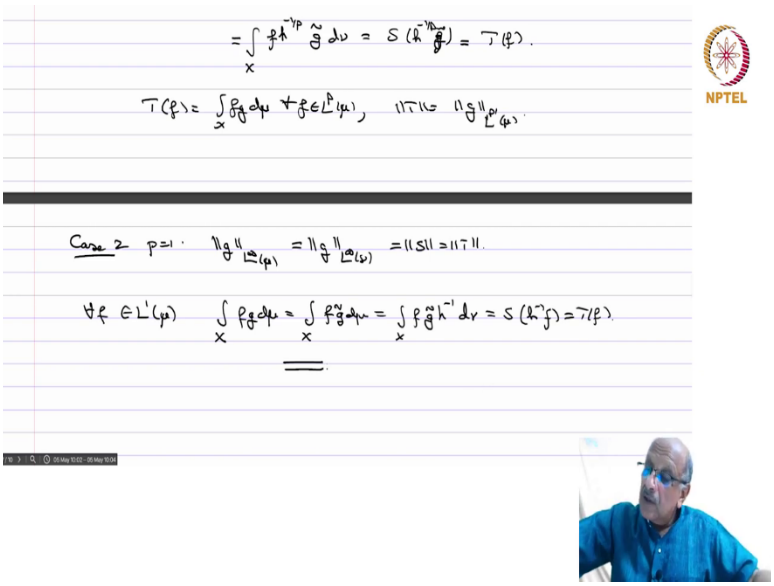

So, you have Tf equals integral over x, fg d mu, for all f in Lp mu, and norm g norm T is equal to norm g, beta Lp dash, so this is proves the case for p, now we only have to do the

case 2: p = 1, so this should be easier. So, then you have norm g, L infinity mu is the same as norm g, L infinity nu which is norm S, which is equal to norm T.

And for all f, in L 1 of mu, you have integral over x, fg d mu which is integral over x, f h power 1 by p g tilde d mu, g is g tilde, and then d mu, sorry, which is equal to integral over x, fg tilde h power minus 1 d nu which is S of h minus 1 f which is T of f, and that completes the proof for the case p equals 1, and so we have completely proved the Riesz representation theorem in the case of sigma finite measures.

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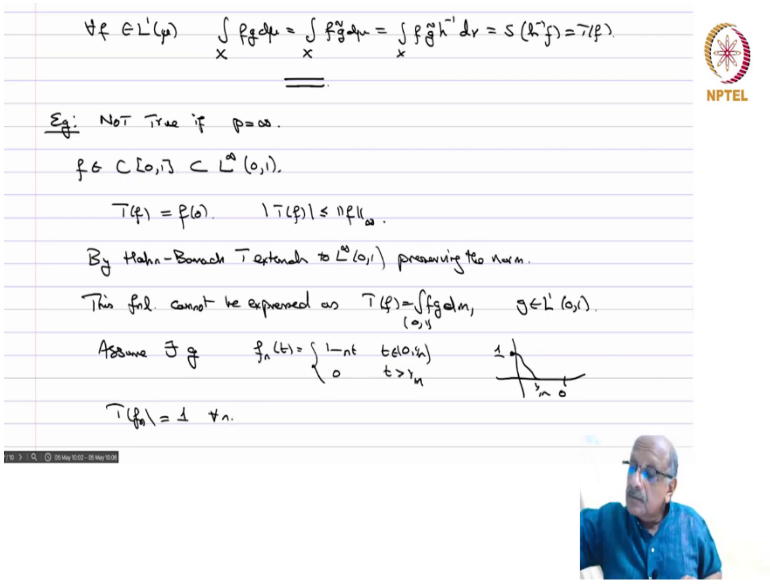

Now, not true if p equals infinity, that means there do exist continuous linear functionals on L infinity, which do not come from an L1 function, in the way we have done, because the conjugate exponent of L in infinity is 1.

Eg. So, let us take $f \in C[0, 1] \subset L^{\infty}[0, 1]$, and you take $T(f) = f(0)$, this is well defined, and $|T(f)| \le ||f||_{\infty}$ and therefore this is a continuous linear function.

And therefore, by Hahn–Banach theorem, T extends to $L^{\infty}[0, 1]$, preserving the norm, that part is really not very important. This functional cannot be expressed as

$$T(f) = \int_{(0,1)} fg\,d\mu, \ \ g \in L^{1}(0, 1).$$

Assume the contrary assume, there exists g satisfying this thing. So, now you take fn of T equals 1 minus n, T for T belonging to 01 by n, and 0 for T greater than 1 by n. So, this is the function fn of T, this is 1 here, it comes to 0 at the point 1 by n, and then goes on at this place 0, so then, T of f n equal to 1 for all n.

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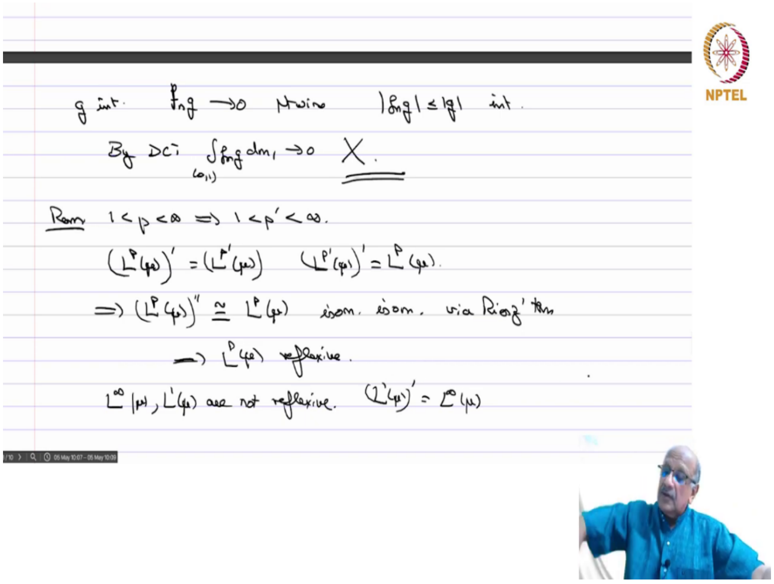

But g integrable and therefore, $f_n g \rightarrow 0$ and $|f_n g| \leq |g|$, because f n is less than equal to 1, and integrable. Therefore, by dominated convergence theorem, you have

$$\int_{(0,1)} f_n g dm_1 \rightarrow 0,$$

and that gives you a contradiction, so that there are continuous linear functions L infinity which do not come from L1 functions.

Remark: 1 less than p, less than infinity, so this implies that 1 less than p dash, less than infinity, so $L^p(\mu)' = L^{p'}(\mu)$, and $L^{p'}(\mu) = L^p(\nu)$, and the it is always the same, the continuous linear function is given by the integral of a product of 2 functions, and therefore this implies that $(L^p(\mu)')'$ is isometrically isomorphic to $L^p(\mu)$ isometric isomorphism via this theorem, and this implies that Lp mu is reflective. L infinity mu, L1 mu are not reflexive, L1 mu infinity equals L infinity mu, but not vice versa, so L1 mu, sorry, dash is converse is not.

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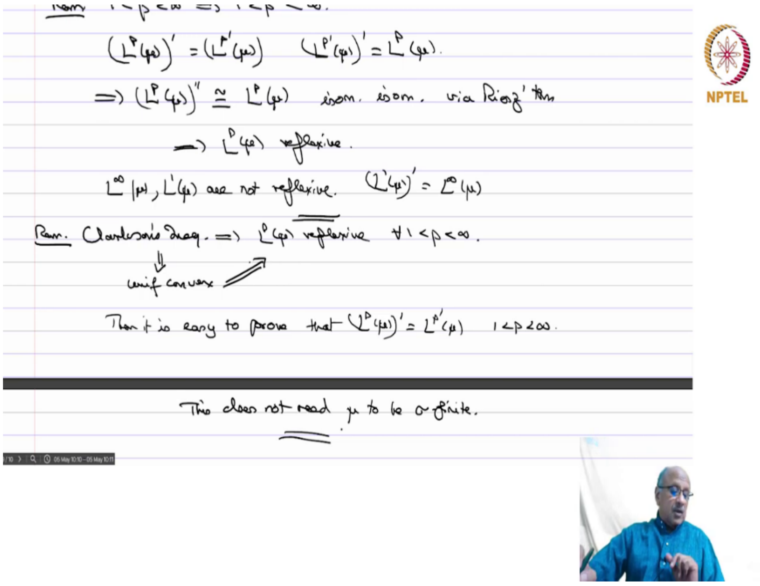

Now, we can prove by Clarkson's inequality $\Rightarrow$ $L^p(\mu)$ reflexive for all $1 < p < \infty$. This Clarkson's inequality implies $L^p(\mu)$ is uniformly convex, which implies that $L^p(\mu)$ is reflexive for all this. Then, it is easy to prove, we will see that in the exercises, this is Clarkson's inequality. Then, it is easy to prove that $L^p(\mu)' = L^{p'}(\mu)$ for $1 < p < \infty$.

So, and this does not need μ to be sigma finite. So, if you are, if you only have worried about 1 less than p, less than infinity, then the Riesz representation theorem is true, even without the sigma finiteness property, but when you want it for 1 also, then the sigma finiteness, that is where it is really important, the rest of the form, but the proof we have given here using measure theoretic arguments is a package deal, you get all of it together, and so 1 and p you get it all at the same time, and therefore we have presented that here.

But you can purely by functional analytic methods, using the notion of uniform convexity and reflexivity, we can show that for p between 1 and infinity strictly, even if mu is not sigma finite, then you have this representation theorem. So, this completes this topic of duality, next time we will take up convolution.