Measure and Integration
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Lecture No-71
11.3- Duality

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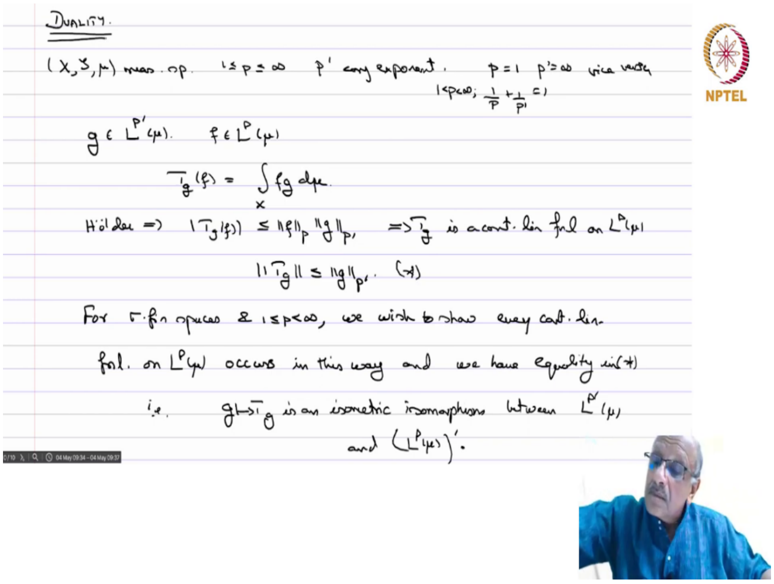

We will now discuss an important topic in Lp spaces namely that of duality. So, the Lp spaces are all Banach spaces. Therefore, whenever you have a Banach space, it is interesting to know what is the dual space, the dual space is the space of all continuous linear functionals on the Banach space, that itself forms on the Banach space. So, we would like to often compute what is that dual space, and the study of the dual space often gives us a lot of information about the original space itself. And therefore, it is important to know what is the dual space, and that is what we are going to do now for the Lp spaces.

So, we have (X, S, μ), measure space, and you have $1 \leq p \leq \infty$. And then, p' is the conjugate exponent, that means if p=1, then $p' = \infty$, and vice versa, and $\frac{1}{p} + \frac{1}{p'} = 1$.

So, let us take $g \in L^{p'}(\mu)$ *and* $f \in L^{p}(\mu)$, then we define

$$T_g(f) = \int_X fg d\mu.$$

So Holder implies $|T_g(f)| \leq ||f||_p ||g||_{p'} \Rightarrow T_g$ is a continuous linear functional on Lp mu, and $||T_g|| \leq ||g||_{p'}$. ------- (*)

So, our aim is to show for sigma finite spaces. So, for sigma finite spaces, and $1 \le p < \infty$, we wish to show every continuous linear functional on Lp mu occurs in this way, and we have equality in (*). Stars, so not only so in other words this g going to Tg, that is g going to Tg is an isometric, isomorphism, its 1, 1 on 2 continuous map, and therefore it is isomorphism by the open mapping theorem, and it is an isometric isomorphism, because the norm is preserved so between Lp dash mu, and Lp mu dash which is the dual space of the.

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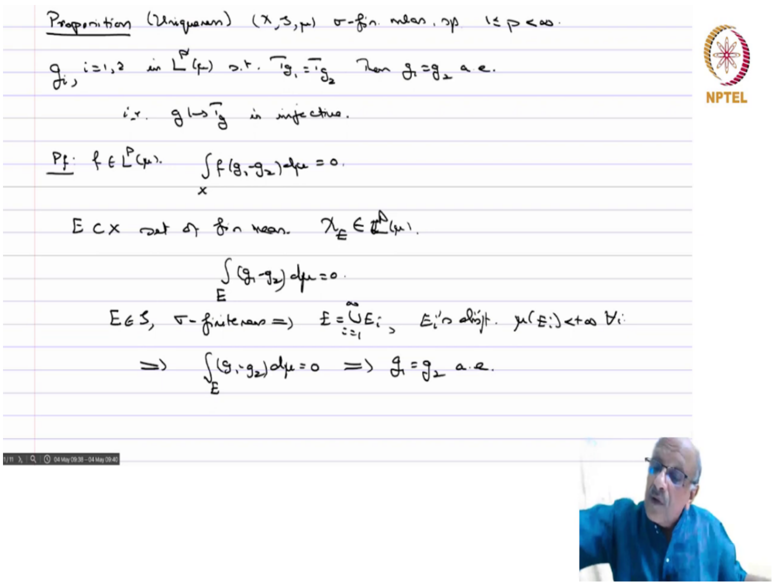

Proposition. (X, S, μ) σ-finite measurable measure space, $1 \le p < \infty$, g_i, $i = 1, 2,\ in\ L^{p'}(\mu)$, such that $T_{g_1} = T_{g_2}$, then $g_1 = g_2$ almost everywhere. So, in other words, the map $G: \rightarrow T_g$ is injecting.

proof: $f \in L^p(\mu)$, then $\int_X f(g_1 - g_2) d\mu = 0$. So, E contained in X set a finite measure, because its finite measure then $\chi_E \in L^p(\mu)$, therefore $\int_E (g_1 - g_2) d\mu = 0$. Now, if E is S because of sigma finiteness, $E = \cup_{i=1}^{\infty} E_i$, $\mu(E_i)$ finite for all i. Then, this implies the

$$\int_E f(g_1 - g_2) d\mu = 0 \Rightarrow g_1 = g_2\ a.e.$$

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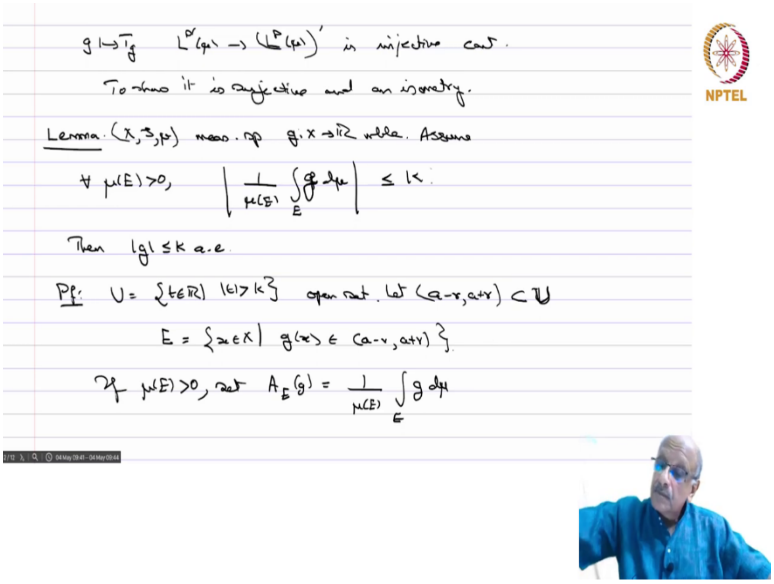

So, this proves the uniqueness, so now you have that g going to Tg, from Lp dash mu to Lp mu dash, the dual space is injective and continuous, to show it is subjective and isometric. So, we will first move it to the finite measure space, then we will look at it in a general case, sigma finite. Before that, anyway we need a very interesting lemma. This is a nice lemma.

Lemma: (X, S, μ) measure space $g: X \rightarrow \mathbb{R}$ measurable, assume for every $\mu(E) > 0$, you have that $|\frac{1}{\mu(E)}\int_E fgd\mu| \leq k$. Then $|g| \leq k\ a.e.$

proof: $U = \{t \in \mathbb{R}:\ |t| > k\}$ open set.Let $(a - r, a + r) \subset U$. Define

$$E = \{x \in X:\ g(x) \in (a - r, a + r)\}.$$

Now, if $\mu(E)$ is positive, then you have $A_E(g) = \frac{1}{\mu(E)}\int_E gd\mu$.

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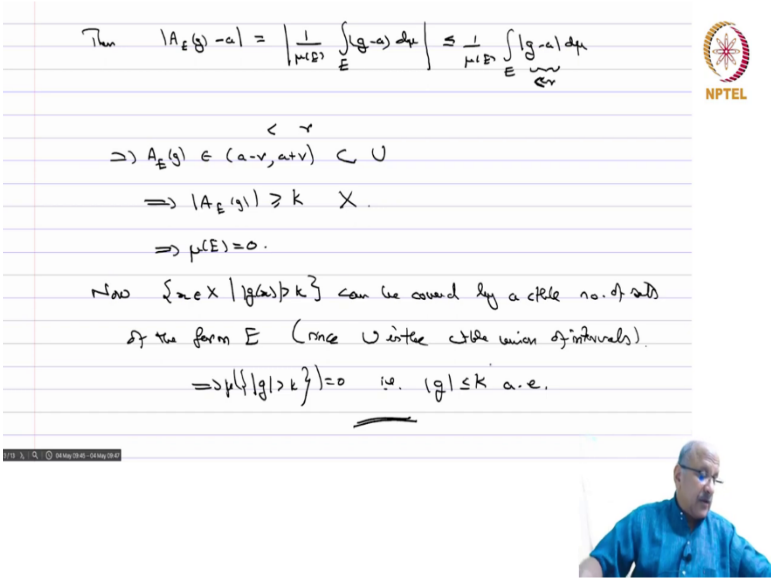

Then, mod Ae of g minus a, equals 1 by mu E of integral over E, g minus a d mu, a is a constant so if I integrate over E, I will just get mu of E times a, divided by mu of E is give you a, again, so this I can write like this. But g minus a, because we are in E, g minus a is less than or equal to r, so this is less equal to 1 by mu E, integral over E, mod g minus a, d mu, but this is less than or equal to less than r, and therefore, you have then this is equal to sorry less than or equal let us strictly less than r.

So, this means that AE of g also belongs to a minus r, a plus r. And, that is less than contained in U. And this implies that mod of AE of g is greater than equal to k, which is a contradiction. Because you have told that for every set of positive measure. The A average is less than or equal to k. So, this implies that mu E equal to 0.

Now, set of all x in X, so say gx is greater than k can be covered by a countable number of sets of the form E, since U is the countable union of intervals. So, each set of this form is of measure 0, and therefore this implies that mod g greater than k, the measure of this equal to 0, that is g mod g is less than equal to k, almost everywhere. So, that proves this.

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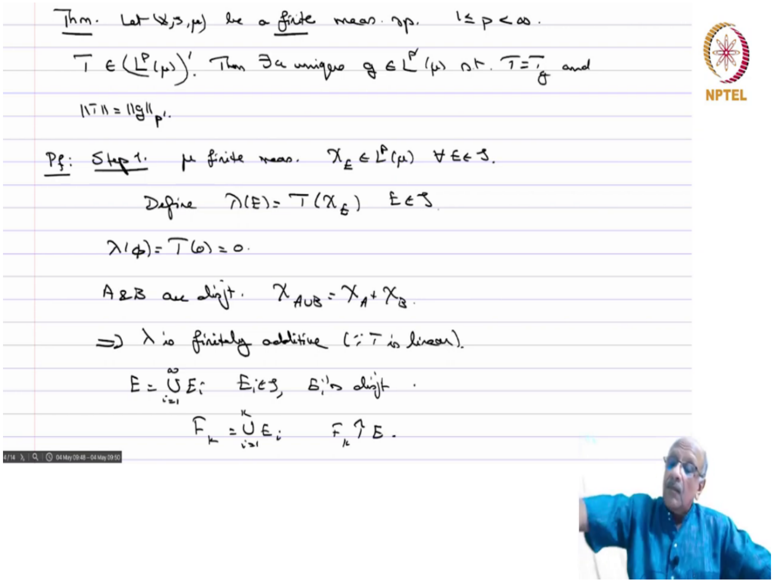

Theorem: Let (X, S, μ) be a finite measure space, $1 \leq p < \infty$, $T \in (L^p(\mu))'$. Then there exists a unique $g \in L^{p'}(\mu)$, such that $T = T_g$, and $||T|| = ||g||_{p'}$.

proof: step 1: now, μ is finite measure, therefore chi of E belongs to Lp of mu for all E and S, and therefore define lambda of E, equals t of chi f. Then of course the lambda of the empty set is chi_f for the empty set is the identically 0 function, so t of 0 is equal to 0. Now, if A and B are disjoint, when chi of A union B is chi of A, plus chi of B. So, this implies that lambda is finitely additive, since p is linear. Now, let E be the union Ei, Eies, Ei is disjoint, then we do the usual thing f k equals union i equals 1 to k of E, Ei, then fk increases to b.

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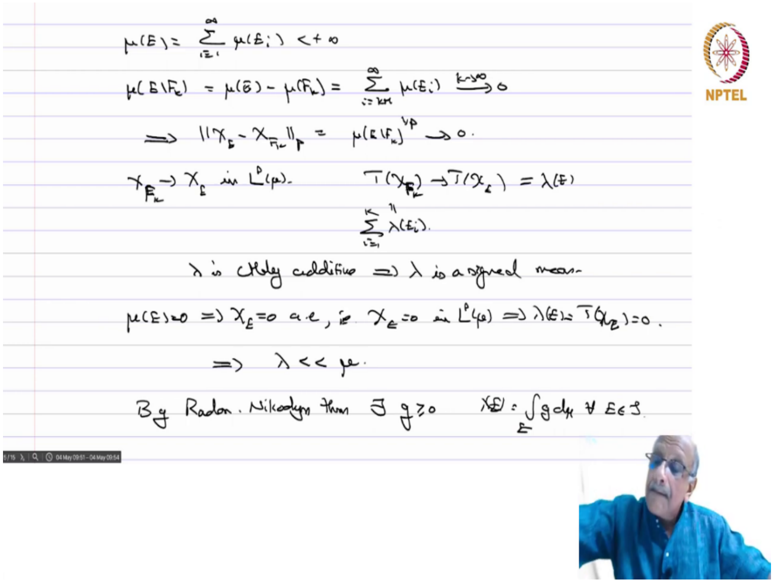

Then, mu of E minus fk is equal to, since you are in the finite measure space, you are allowed to subtract, let me write this, so mu E equals sigma, i equals 1 to infinity mu of Ei and that is finite, and mu of E minus fk is mu E minus mu of fk, and that is equal to sigma i equals k plus 1 to infinity, mu of Ei. And, this tends to 0, as k tends to infinity, because you have this the tail of a convergent series, and therefore this has to go to 0.

Then, therefore this implies that norm of chi E minus chi of fk in Lp, this is nothing but mu of E minus fk because that is where it is, (())(18:34) everywhere, else it is 0, and power 1 by 3, and therefore this goes to 0. So, chi E converge, chi is chi of fk converges to chi of E in Lp mu, therefore t of chi of E, chi of fk converges to t of chi of U, which is equal to lambda E, but this is a finite disjoint in the n, so this equal to i equals 1, 2, k lambda of chi of Ei lambda of Ei, and therefore you have lambda is countably additive, implies lambda is a signed measure.

Now, if mu E equal to 0, this implies that chi E equal to 0, almost everywhere, that is chi E equal to 0 in Lp mu, and therefore this implies lambda of E which is t of mu E, and t of chi E equal to 0. This implies that lambda is absolutely continuous with respect to mu, because no, so by the Radon–Nikodym theorem, there is a g, which is greater than equal to 0, such that lambda E equal to integral gd mu over E for all E in s.

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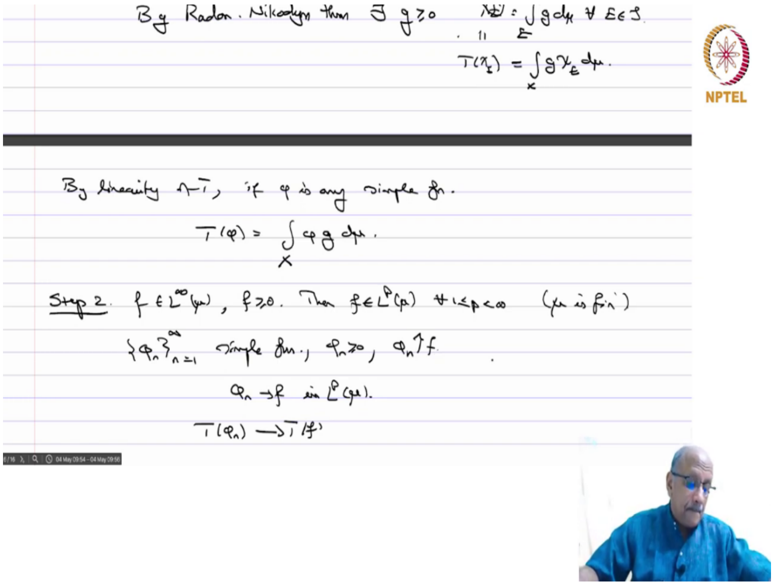

So, by linearity of T, if ϕ is any simple function, we have

$$T(\phi) = \int_X \phi g d\mu.$$

step 2: f in l infinity mu, f non negative, then of course f is in Lp mu for all 1 less than equal to p, less than infinity, this is because mu is finite.

So, phi n, n equals 1 t infinity, simple functions, phi n non negative, phi n increasing to f. Then, we also saw phi n converges to f in Lp mu, we have already seen this earlier, phi n converges to f in Lp mu to a simple application of the dominated convergence theorem. Therefore, T of phi n converges to T of f.

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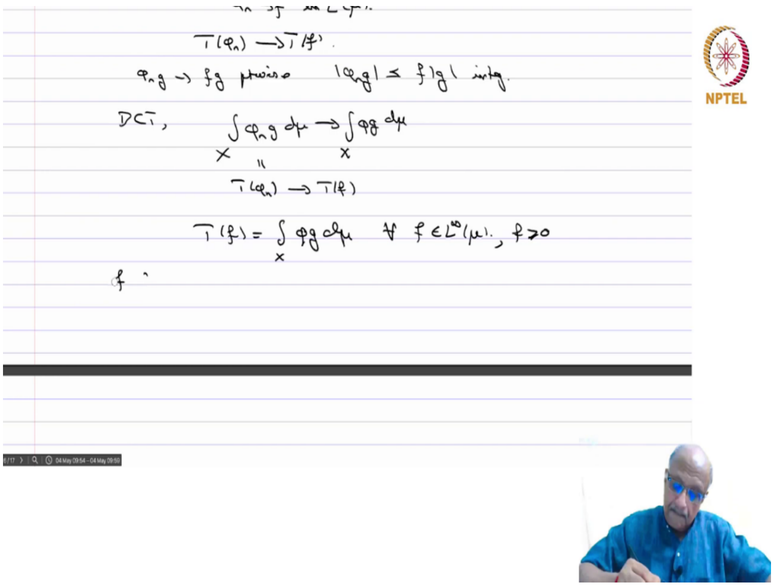

On the other hand, $\phi_n g \rightarrow fg$ pointwise $|\phi_n g| \leq f|g|$. Therefore by dominated convergence theorem, we have $\int_X \phi_n g d\mu \rightarrow \int_X \phi g d\mu$. And therefore, and this is nothing but t of phi n, which converges to t of f, therefore you have $T(f) = \int_X \phi g d\mu, \forall f \in L^{\infty}(\mu), f \geq 0.$

t of f equal to integral over x, phi g, d mu, for every f in l infinity.

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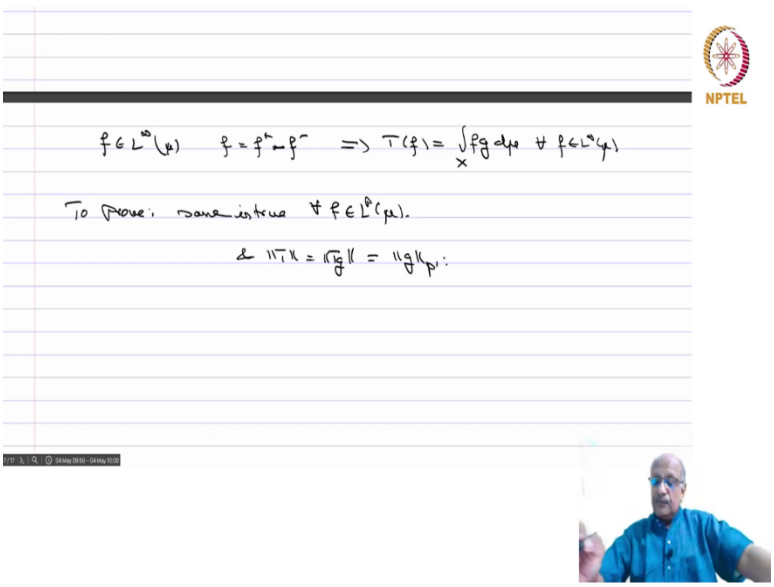

So, if $f \in L^{\infty}(\mu)$, $f = f^{+} - f^{-} \Rightarrow T(f) = \int_X fgd\mu \ \forall f \in L^{\infty}(\mu)$. So, now to prove, the same is true for every $f \in L^{p}(\mu)$, and $||T|| = ||T_g|| = ||g||_{p'}$.

So, this is what we need to prove, which we will do next time.