

**Measure and Integration**  
**Professor S. Kesavan**  
**Department of Mathematics**  
**The Institute of Mathematical Sciences**  
**Lecture No-70**  
**11.2 - Applications**

So, we proved an important theorem namely that if one is less than p less than infinity and omega is an open set in  $\mathbb{R}^N$  with Lebesgue measure then continuous functions with compact support in Omega dense in  $L^p$  of omega and as a consequence, we proved that  $L^p$  of omega is separable if p is less than infinity and it is not separable if p is equal to infinity. Now, in the last thing when I mentioned it, the countable dense set is  $\bigcup_{m,n} p_m$  and tilde union over m and n, not just  $p_m$  and I say wrongly what I have corrected it in the lecture materials. So, now, we will see some more applications of this particular result.

(Refer Slide Time: 01:10)

Theorem (Lusin's thm.)  
 $E \subset \mathbb{R}^N$  a measurable set of finite measure.  $f: E \rightarrow \mathbb{R}$  measurable.  $\epsilon > 0$ . Then  $\exists \phi \in C_c(\mathbb{R}^N)$   
s.t.  $m_N(\{x \in E \mid \phi(x) \neq f(x)\}) < \epsilon$  & if  $f$  is bounded,  $\|\phi\|_\infty \leq \|f\|_\infty$ .

Pf: Step 1:  $n \in \mathbb{N}$   
Let  $E_n = \{x \in E \mid |f(x)| \leq n\}$ .

$E_n \uparrow E$   $E$  has fin. meas.  $m_N(E_n) \uparrow m_N(E) \Rightarrow \exists n$  s.t.  $m_N(E \setminus E_n) < \epsilon/3$ .

Define  $\tilde{f}: \mathbb{R}^N \rightarrow \mathbb{R}$  by  $\tilde{f}(x) = \begin{cases} f(x) & x \in E_n \\ 0 & x \in \mathbb{R}^N \setminus E_n \end{cases}$ .

$\tilde{f}$  bdd,  $E_n$  has fin. meas.  $\Rightarrow \tilde{f}$  is integrable on  $\mathbb{R}^N$ . i.e.  $\tilde{f} \in L^1(\mathbb{R}^N)$ .

$\{\phi_n\}$  in  $C_c(\mathbb{R}^N)$   $\phi_n \rightarrow \tilde{f}$  in  $L^1(\mathbb{R}^N)$   $\exists$  subseq.  $\{\phi_{n_k}\}$   $\phi_{n_k} \rightarrow \tilde{f}$  a.e.



So, now, we are going to prove a fairly important theorem.

**Theorem:** (Lusin's theorem). So,  $E \subset \mathbb{R}^N$  a measurable set of finite measure,  $f: E \rightarrow \mathbb{R}$  measurable function, epsilon greater than 0, then there exists  $\phi \in C_c(\mathbb{R}^N)$  such that

$$m_N(\{x \in E: \phi(x) \neq f(x)\}) < \epsilon \text{ and if } f \text{ is bounded, } \|\phi\|_\infty \leq \|f\|_\infty.$$

proof. **Step 1:** so, n in  $\mathbb{N}$  positive integer, so, we define  $E_n = \{x \in E: |f(x)| \leq n\}$ . Then  $E_n$  is measurable and  $E_n \uparrow E$ , That is clear.

Now,  $E$  has finite measure so,  $m_N(E_n) \uparrow m_N(E)$  implies there exists an  $m$  such that  $m_N(E_n \setminus E) < \frac{\epsilon}{3}$ . Define  $\tilde{f}: \mathbb{R}^N \rightarrow \mathbb{R}$  by

$$\begin{aligned} \tilde{f}(x) &= f(x) \text{ if } x \in E_m \\ &= 0 \text{ if } x \in \mathbb{R}^N \setminus E_m. \end{aligned}$$

Now,  $\tilde{f}$  is bounded because it is less than equal to  $m$  because otherwise it is 0 so, it is bounded and  $E_m$  has finite measure since it where the  $\tilde{f}$  is not 0. So, this implies  $\tilde{f}$  is integrable. That is  $\tilde{f} \in L^1(\mathbb{R}^N)$ . So, then that exists  $\phi_N$  in  $C_c$  of  $\mathbb{R}^N$  such that  $\phi_n \rightarrow f$  in  $L^1(\mathbb{R}^N)$ , and there exists a subsequence  $\phi_{n_k}, \phi_{n_k} \rightarrow f$  pointwise almost everywhere all these things we know.

(Refer Slide Time: 5:37)

The slide contains handwritten mathematical notes in five steps:

- Step 2:**  $E_m$  has fin. meas.  $\phi_n \rightarrow f$  pointwise  $\Rightarrow \exists F \subset E_m, m_N(E_m \setminus F) < \frac{\epsilon}{3}$ .  $\{\phi_{n_k}\}$  conv. unif. on  $F$ . (Egoroff).
- $F$  has fin. meas. then  $\exists K$  compact,  $K \subset F, m_N(F \setminus K) < \frac{\epsilon}{3}$ .
- $K \subset F \subset E_m \subset E \Rightarrow m_N(E \setminus K) < \epsilon$ .
- Step 3:**  $K \subset F, \phi_{n_k} \rightarrow \tilde{f}$  unif. on  $K \Rightarrow \tilde{f}|_K$  is cont. But  $K \subset E_m$ .  
i.e.  $\tilde{f} = f$  on  $E_m \Rightarrow f|_K$  is cont.
- Step 4:** Tietze extn. thm.  $\exists g: \mathbb{R}^N \rightarrow \mathbb{R}$  cont. s.t.  $g = f$  on  $K$ .  
 $\|g\|_\infty \leq \|\tilde{f}\|_\infty \leq m$ .
- Step 5:** Urysohn's lemma  $\Rightarrow \exists \psi \in C_c(\mathbb{R}^N), 0 \leq \psi \leq 1, \psi \equiv 1$  on  $K$ .  
Set  $\varphi = g\psi \Rightarrow \varphi = g$  on  $K \Rightarrow \varphi = f$  on  $K$ .

**Step 2:**  $E_m$  has finite measure and  $\phi_{n_k} \rightarrow \tilde{f}$  point wise implies there exists  $F \subset E_m$ ,  $m_N(E_m \setminus F) < \frac{\epsilon}{3}$  and  $\{\phi_{n_k}\}$  converges uniformly on  $F$ . So, this theorem we have said if you have pointwise convergence on a set of finite measure, then you can find a subset where the convergence is uniform, that is it is almost uniformly convergent. So, then by Egoroff's of this theorem you can do it, now  $f$  also has finite measure then there exists a  $K$  compact,  $K$  contained in  $F, m_N(E \setminus K) < \epsilon$ .

**Step 3.** So,  $K \subset F$ ,  $\phi_{n_k} \rightarrow \tilde{f}$ ,  $\Rightarrow \tilde{f}|_K$  is continuous but,  $K \subset E_m$ . And on  $E_m$   $\tilde{f}$  is the same as  $f$  and that is, so  $\tilde{f}|_E$  equals  $f$  on  $E_m$  implies that  $f$  restricted to  $K$  is continuous.

**Step 4.** So, by Tietze extension theorem, there exists a  $g: \mathbb{R}^N \rightarrow \mathbb{R}$  continuous such that  $g = f$  on  $K$  and  $\|g\|_\infty \leq \|f\|_{\infty, K} \leq m$ .

**Step 5.** so, using Tietze extension theorem, we can also use Urysohn's lemma now, there exists  $\phi \in C_c(\mathbb{R}^N)$ ,  $0 \leq \phi \leq 1$ ,  $\phi \equiv 1$  on  $K$ . Then set  $\psi = g\phi \Rightarrow \psi = g$  on  $K \Rightarrow \psi = f$  on  $K$ .

(Refer Slide Time: 10:52)

Step 5. Urysohn's lemma  $\Rightarrow \exists \psi \in C_c(\mathbb{R}^N)$ ,  $0 \leq \psi \leq 1$ ,  $\psi \equiv 1$  on  $K$ .  
 Set  $\psi = g\phi \Rightarrow \psi = g$  on  $K \Rightarrow \psi = f$  on  $K$ .  
 $\|g\|_\infty \leq \|f\|_\infty \leq m$  ( $L \leq \|f\|_\infty$ ) (if  $f$  is bounded).

---

$m_N(\psi \neq f) \leq m_N(E \setminus K) < \epsilon$ .

---

Prop.  $1 \leq p < \infty$   $f \in L^p(\mathbb{R}^N)$ . For  $h \in \mathbb{R}^N$  define  
 $(\tau_h f)(x) = f(x-h)$   $x \in \mathbb{R}^N$ .  
 Then  $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$ .



And therefore,  $\|\phi\|_\infty \leq \|g\|_\infty \leq m$ ,  $m_N(\phi \neq f) \leq m_N(E \setminus K) < \epsilon$ .

So, this is Lusin's theorem.

**Proposition.** So,  $1 \leq p < \infty$ ,  $f \in L^p(\mathbb{R}^N)$ . For  $h \in \mathbb{R}^N$ , define

$$(\tau_h f)(x) = f(x - h), x \in \mathbb{R}^N.$$

Then  $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$ .

(Refer Slide Time: 13:16)

$(\tau_h f)(x) = f(x-h) \quad x \in \mathbb{R}^N.$   
 Then  $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0.$   
Pr. By translation inv. of Lebesgue meas.,  $\tau_h f \in L^p(\mathbb{R}^N)$ ,  $\|\tau_h f\|_p = \|f\|_p.$   
 $\epsilon > 0.$  choose  $\phi \in C_c(\mathbb{R}^N)$   $\|f - \phi\|_p < \epsilon/3.$   
 $\Rightarrow \|\tau_h f - \tau_h \phi\|_p = \|f - \phi\|_p < \epsilon/3.$   
 $\phi$  has compact support  $\Rightarrow \phi$  unif. cont. Let  $\text{supp } \phi \subset [-a, a]^M.$   
 $\Rightarrow \exists \delta > 0$  (we can choose  $0 < \delta < 1$ ) s.t.  $|h| < \delta \Rightarrow$   
 $| \phi(x-h) - \phi(x) | < \frac{\epsilon}{3} [2(a+1)]^{-N/p} \quad \forall x \in \mathbb{R}^N$



*proof:* so, by translation invariance of Lebesgue measure,  $\tau_h f \in L^p(\mathbb{R}^N)$ ,  $\|\tau_h f\|_p = \|f\|_p.$

Now, let  $\epsilon > 0.$  So, choose  $\phi \in C_c(\mathbb{R}^N)$ , such that  $\|f - \phi\|_p < \frac{\epsilon}{3}.$

$$\Rightarrow \|\tau_h f - \tau_h \phi\|_p = \|f - \phi\|_p < \frac{\epsilon}{3}.$$

So,  $\phi$  has compact support implies  $\phi$  is uniformly continuous. Now, let  $\text{supp}(\phi) \subset [-a, a]^M$  and some boxes, big boxes you can put there. Then there exists a delta greater than 0 we can choose 0 to be less than delta less than 1 because the smaller you go the answer the uniform continuity (15:10) such that  $|h| < \delta,$



$$\Rightarrow |\phi(x-h) - \phi(x)| < \frac{\epsilon}{3} [2(a+1)]^{-N/p}, \quad \forall x \in \mathbb{R}^N.$$

(Refer Slide Time: 15:50)

Then  $|h| < \delta \Rightarrow$

$$\int_{\mathbb{R}^N} |\tau_h(\phi) - \phi|^p dm_N = \int_{\{|x-h| < \delta\}} |\phi(x-h) - \phi(x)|^p dm_N < \left(\frac{\epsilon}{3}\right)^p$$

$$\Rightarrow \|\tau_h \phi - \phi\|_p < \frac{\epsilon}{3}.$$

$$\|f - \tau_h f\|_p \leq \|f - \phi\|_p + \|\phi - \tau_h \phi\|_p + \|\tau_h \phi - \tau_h f\|_p < \epsilon.$$



Then  $|h| < \delta \Rightarrow$

$$\int_{\mathbb{R}^N} |\tau_h(\phi) - \phi|^p dm_N = \int |\phi(x-h) - \phi(x)|^p dm_N < \left(\frac{\epsilon}{3}\right)^p \Rightarrow \|\tau_h \phi - \phi\|_p < \frac{\epsilon}{3}.$$

And now,  $\|f - \tau_h f\|_p \leq \|f - \phi\|_p + \|\phi - \tau_h \phi\|_p + \|\tau_h \phi - \tau_h f\|_p < \epsilon.$

That proves the theorem.

So, with this so, the next we will take up the equation of duality, we have a norm linear space, we have Banach spaces, we would like to know what is the dual of this Banach space  $L^p$  of  $\omega$  so, we will take that up next.