Measure and Integration
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Lecture No-70
11.2 - Applications

So, we proved an important theorem namely that if one is less than p less than infinity and omega is an open set in RN with Lebesgue measure then continuous functions with compact support in Omega dense in l P of omega and as a consequence, we proved that l P of omega is separable if P is less than infinity and it is not separable if p is equal to infinity. Now, in the last thing when I mentioned it, the countable dense set is pm and tilde union over m and n, not just pm and I say wrongly what I have corrected it in the lecture materials. So, now, we will see some more applications of this particular result.

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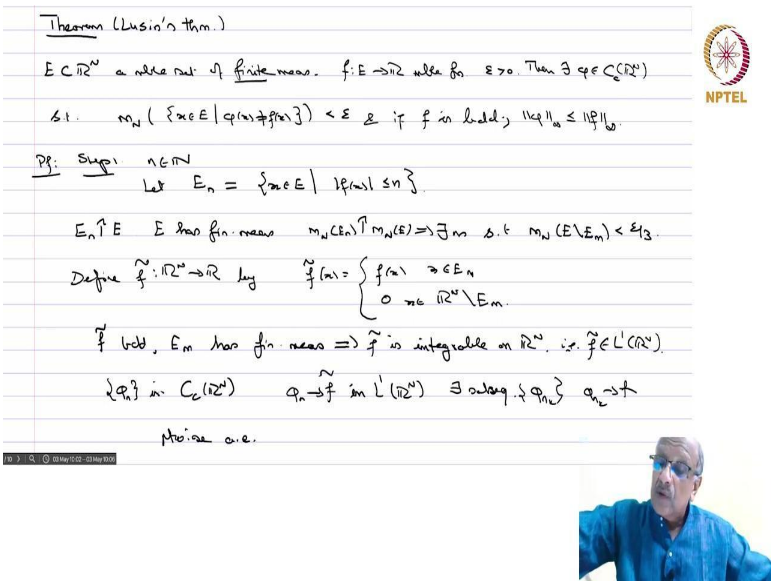

So, now, we are going to prove a fairly important theorem.

Theorem: (Lusin's theorem). So, $E \subset \mathbb{R}^N$ a measurable set of finite measure, $f: E \to \mathbb{R}$ measurable function, epsilon greater than 0, then there exists $\phi \in C_c(\mathbb{R}^N)$ such that

$m_N(\{x \in E: \phi(x) \neq f(x)\}) < \epsilon$ and if f is bounded, $||\phi||_\infty \leq ||f||_\infty$.

proof. **Step 1:** so, n in N positive integer, so, we define $E_n = \{x \in E: |f(x)| \leq n\}$. Then E_n is measurable and $E_n \uparrow E$, That is clear.

Now, E has finite measure so, $m_N(E_n) \uparrow m_N(E)$ implies there exists an m such that $m_N(E_n \backslash E) < \frac{\epsilon}{3}$. Define $\tilde{f}: \mathbb{R}^N \to \mathbb{R}$ by

$$\tilde{f}(x) = f(x) \quad if\ x \in E_m$$

$$= 0 \quad if \quad x \in \mathbb{R}^N \backslash E_m.$$

Now, $\tilde{f}$ is bounded because it is less than equal to m because otherwise it is 0 so, it is bounded and E_m has finite measure since it where the $\tilde{f}$ is not 0. So, this implies f tilde is integrable. That is $\tilde{f} \in L^1(\mathbb{R}^N)$. So, then that exists phi N in Cc of RN such that $\phi_n \to f\ in\ L^1(\mathbb{R}^N)$, and there exists a subsequence $\phi_{n_k}, \phi_{n_k} \to f$ pointwise almost everywhere all these things we know.

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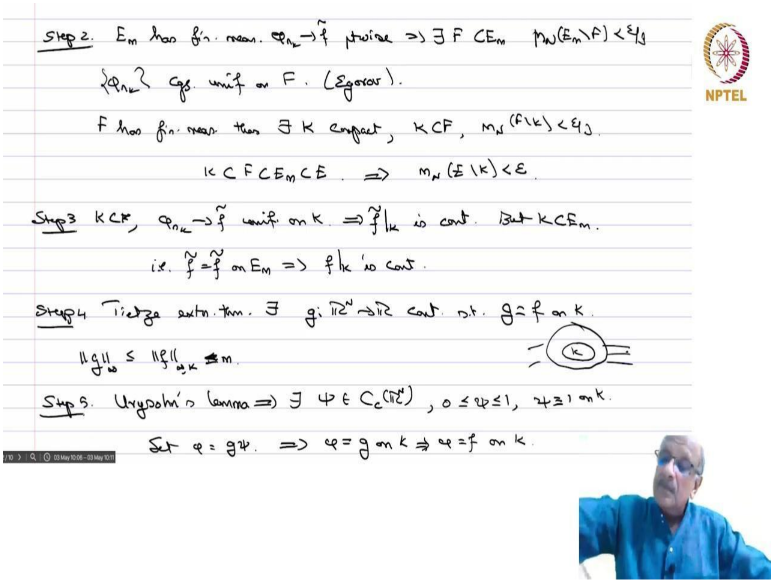

Step 2: E m has finite measure and $\phi_{n_k} \to \tilde{f}$ point wise implies there exists $F \subset E_m$, $m_N(E_m \backslash F) < \frac{\epsilon}{3}$ and $\{\phi_{n_k}\}$ converges uniformly on F. So, this theorem we have said if you have pointwise convergence on a set of finite measure, then you can find a subset where the convergence is uniform, that is it is almost uniformly convergent. So, then by Egoroff's of this theorem you can do it, now f also has finite measure then there exists a K compact ,K contained in F $m_N(E \backslash K) < \epsilon$.

Step 3. So, $K \subset F, \phi_{n_k} \to \tilde{f}, \ \Rightarrow \tilde{f}|_K$ is continuous but, $K \subset E_m$. And on Em f tilde is the same as f and that is, so f tilde E equals F on Em implies that f restricted to k is continuous.

Step 4. So, by Tietze extension theorem, there exists a $g\colon \mathbb{R}^N \to \mathbb{R}$ continuous such that $g = f$ on K and $||g||_\infty \le ||f||_{\infty,K} \le m$.

Step 5. so, using Tietze extension theorem, we can also use Urysohn's lemma now, there exists $\phi \in C_c(\mathbb{R}^N),\ 0 \le \phi \le 1,\ \phi \equiv 1\ on\ K.$ Then set $\phi = g\psi \Rightarrow \phi = g\ on\ K \Rightarrow \phi = f\ on\ K.$

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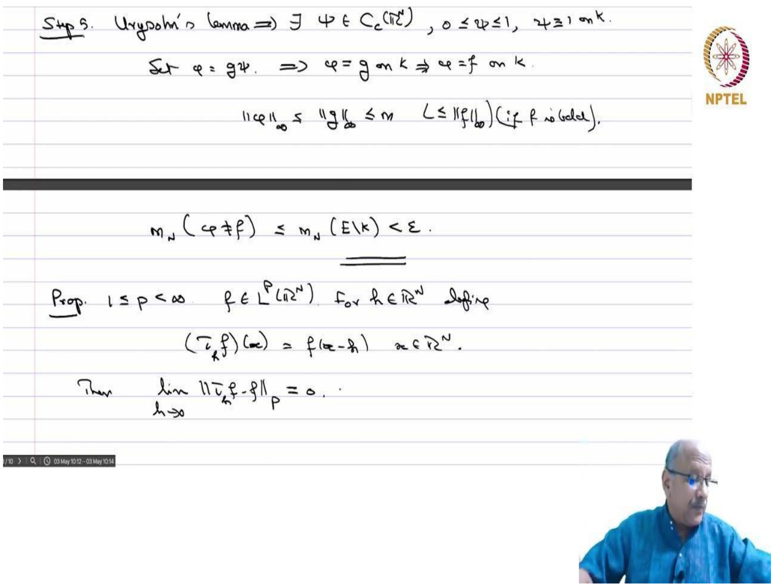

And therefore, $||\phi||_\infty \le ||g||_\infty \le m,\ m_N(\phi \ne f) \le m_N(E\backslash K) < \epsilon.$

So, this is Lusin's theorem.

Proposition. So, $1 \le p, \infty,\ f \in L^p(\mathbb{R}^N)$. For $h \in \mathbb{R}^N$, define

$$(\tau_h f)(x) = f(x - h),\ x \in \mathbb{R}^N.$$

Then $\lim_{h \to 0} ||\tau_h f - f||_p = 0.$

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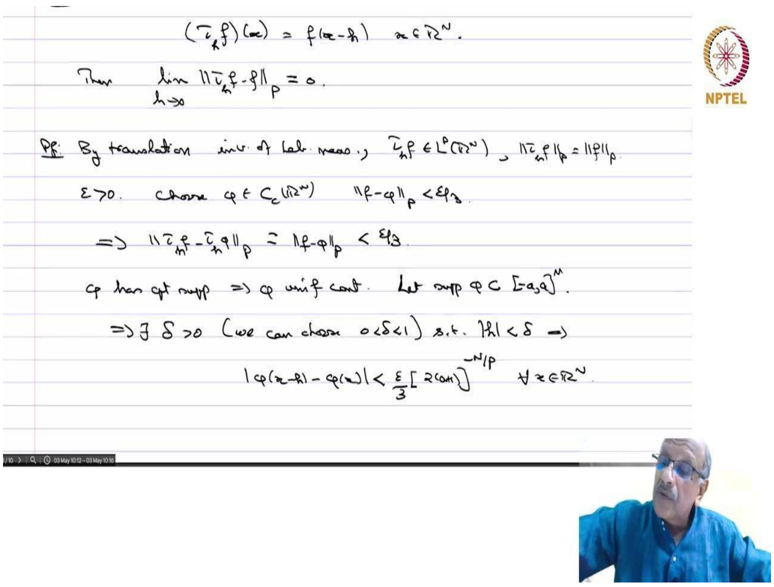

proof: so, by translation invariance of Lebesgue measure, $\tau_h f \in L^p(\mathbb{R}^N)$, $||\tau_h f||_p = ||f||_p$. Now, let $\epsilon > 0$. So, choose $\phi \in C_c(\mathbb{R}^N)$, such that $||f - \phi||_p < \frac{\epsilon}{3}$.

$$\Rightarrow ||\tau_h f - \tau_h \phi \, ||_p = ||f - \phi||_p < \frac{\epsilon}{3}.$$

So, ϕ has compact support implies ϕ is uniformly continuous. Now, let $supp(\phi) \subset [-a, a]^M$ and some boxes, big boxes you can put there. Then there exists a delta greater than 0 we can choose 0 to be less than delta less than 1 because the smaller you go the answer the uniform continuity (())(15:10) such that $|h| < \delta$,

$$\Rightarrow |\phi(x - h) - \phi(x)| < \frac{\epsilon}{3}[2(a + 1)]^{-\frac{N}{p}}, \forall\, x \in \mathbb{R}^N.$$

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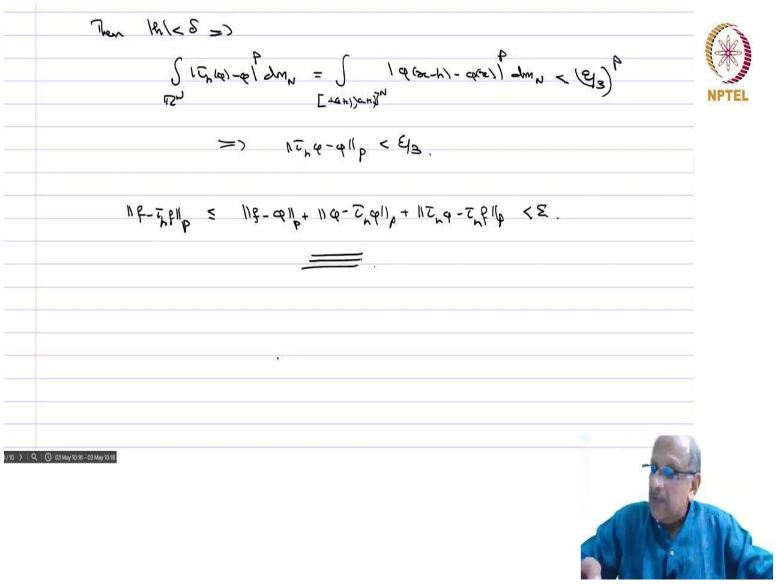

Then $|h| < \delta \Rightarrow$

$$\int_{\mathbb{R}^N} |\tau_h(\phi) - \phi|^p dm_N = \int |\phi(x - h) - \phi(x)|^p dm_N < (\frac{\epsilon}{3})^p \Rightarrow ||\tau_h \phi - \phi||_p < \frac{\epsilon}{3}.$$

And now, $||f - \tau_h f||_p \leq ||f - \phi||_p + ||\phi - \tau_h \phi||_p + ||\tau_h \phi - \tau_h f||_p < \epsilon.$

That proves the theorem.

So, with this so, the next we will take up the equation of duality, we have a norm linear space, we have Banach spaces, we would like to know what is the dual of this Banach space l P of omega so, we will take that up next.