

Measure and Integration
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Lecture-7
2.2 - Exercises

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EXERCISES

1. Let $X (\neq \phi)$ be a set, \mathcal{R} a ring of subsets of X . Let

$$S = \{F \subset X \mid F \in \mathcal{R} \text{ or } F^c \in \mathcal{R}\}.$$

Show that S is the smallest algebra containing \mathcal{R} .

Solution: S' is any algebra containing \mathcal{R} , then it contains all complements of members of \mathcal{R} as well. Thus $S \subset S'$. Enough to show S is an algebra.

$\phi \in \mathcal{R} \Rightarrow X \in S$ Enough to show S is closed under finite unions and complementation.

Closed under complementation is obviously def. of S

$E, F \in S$. To show $E \cup F \in S$.

Exercises:

So, let us now do some exercises. So, the first one.

(1) Let $X (\neq \phi)$ be a set, \mathcal{R} ring of subsets of X let

$$S = \{F \subset X, F \in \mathcal{R} \text{ or } F^c \in \mathcal{R}\}.$$

Show that S is the smallest algebra containing \mathcal{R} .

solution: If S' is any algebra containing \mathcal{R} , then it contains all complements of members of \mathcal{R} as well. Thus, $S \subset S'$. So, enough to show S is an algebra.

So, $\phi \in \mathcal{R} \Rightarrow X \in S$. Now, enough to show we have already seen this S is closed under finite unions and complementation. So, closed under complementation is obvious by construction because if any member of S either it is a member of \mathcal{R} in which case its complement is also in S . If it is not in \mathcal{R} that means its complement is in \mathcal{R} and therefore, it is also in S that member also the complement is also in S . So, this is obvious by definition. So,

now we have to show that it is closed under union. So, let us take $E, F \in S$. So, to show $E \cup F \in S$.

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Handwritten notes on a slide showing the proof that the collection of sets S is closed under union. The notes include:

- $E, F \in S$, to show $E \cup F \in S$.
- (i) $E, F \in R \Rightarrow E \cup F \in R \Rightarrow E \cup F \in S$.
- (ii) $E^c, F^c \in R \Rightarrow E^c \cap F^c \in R$ i.e. $(E \cup F)^c \in R \Rightarrow E \cup F \in S$.
- (iii) $E \in R, F^c \in R \Rightarrow F^c \setminus E \in R$, i.e. $F^c \cap E^c \in R$, i.e. $(E \cup F)^c \in R \Rightarrow E \cup F \in S$.

2. Let $X (\neq \emptyset)$. Let $E \subset X$. Let $S = \{F \subset X \mid E \subset F\}$. Compute $R(S)$.

Solution: Clearly $X \in S \Rightarrow R(S)$ is an algebra
 \Rightarrow closed under complementation.
 $\Rightarrow R(S) \supset \{F \subset X \mid E \subset F\} \cup \{F \subset X \mid F \subset E^c\}$.

So, let us look at different cases. So, first case.

(i) if $E, F \in R \Rightarrow E \cup F \in R \Rightarrow E \cup F \in S$.

(ii) if $E^c, F^c \in R \Rightarrow E^c \cap F^c \in R$, i.e., $(E \cup F)^c \in R \Rightarrow E \cup F \in S$.

(iii) $E, F^c \in R \Rightarrow F^c \setminus E \in R$, i.e., $F^c \cap E^c \in R$, i.e., $(E \cup F)^c \in R \Rightarrow E \cup F \in S$.

E belongs to R and let us say F complement belongs to this is the third possibility.

Therefore, it is an algebra and it is in fact the smallest algebra containing R .

(2) Let $X (\neq \emptyset)$ and let $E \subset X$ be given. Let $\hat{E} = \{F \subset X : E \subset F\}$. Compute $R(\hat{E})$.

solution: Clearly, $X \in \hat{E} \Rightarrow R(\hat{E})$ is an algebra \Rightarrow closed under the complementation.

$$\Rightarrow \{F \subset X : E \subset F\} \cup \{F \subset X : F \subset E^c\} \subset R(\hat{E}) = S.$$

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⇒ closed under complementation.

⇒ $\mathcal{R}(E) \supset \{F \subset X \mid E \subset F\} \cup \{F \subset X \mid F \subset E^c\} = \mathcal{S}$

Enough to show \mathcal{S} is a ring. \mathcal{S} closed under complementation.

Enough to show $F, G \in \mathcal{S} \Rightarrow F \cup G \in \mathcal{S}$.

(i) If F or G contains E , then $F \cup G \supset E \Rightarrow F \cup G \in \mathcal{S}$.

(ii) $F, G \subset E^c \Rightarrow E \subset F^c, E \subset G^c \Rightarrow E \subset F^c \cap G^c = (F \cup G)^c \Rightarrow F \cup G \subset E^c \Rightarrow F \cup G \in \mathcal{S}$.

3. X ($\neq \emptyset$), \mathcal{R} a ring of X , μ a meas. on \mathcal{R} . $E, F \in \mathcal{R}$

Show that $\mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F)$.



So, enough to show it is algebra whatever they want. So, it is closed under complementation and therefore, enough to show, $F, G \in \mathcal{S} \Rightarrow F \cup G \in \mathcal{S}$.

So, again we have to take the 3 cases.

(i) if F or G contains E , then $E \subset F \cup G \Rightarrow F \cup G \in \mathcal{S} \subset \mathcal{S}$.

(ii)

$$F, G \subset E^c \Rightarrow E \subset F^c, E \subset G^c \Rightarrow E \subset F^c \cap G^c = (F \cup G)^c \Rightarrow F \cup G \subset E^c \Rightarrow F \cup G \in \mathcal{S}.$$

So, this becomes an algebra.

(3) X non-empty set, \mathcal{R} ring on X , μ - a measure on \mathcal{R} , $E, F \in \mathcal{R}$. Show that

$$\mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F). \text{ ----- (*)}$$

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Sketch that $\mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F)$. (*)

Solution. $E \cup F = (E \setminus F) \cup F$ disjoint

$(E \setminus F) \cup (E \cap F) = E$ disjoint

If $\mu(E)$ or $\mu(F) = +\infty$, then $\mu(E \cup F) = +\infty$ and so (*) obvious.

$\mu(E), \mu(F) < +\infty \Rightarrow \mu(E \setminus F) < +\infty$.

Additivity $\mu(E \cup F) = \mu(E \setminus F) + \mu(F)$

$\mu(E \setminus F) + \mu(E \cap F) = \mu(E)$

Add & cancel $\mu(E \setminus F) < +\infty$.



solution: So $E \cup F = (E \setminus F) \cup F$ and $(E \setminus F) \cup (E \cap F) = E$.

So, this can be written and this is disjoint.

So, if $\mu(E)$ or $\mu(F) = +\infty$, then $\mu(E \cup F) = +\infty$ and so (*) is obvious.

So, we can assume that both $\mu(E)$ and $\mu(F) < +\infty \Rightarrow \mu(E \cup F) < +\infty$. So, now by additivity $\mu(E \cup F) = \mu(E \setminus F) + \mu(F)$ and $\mu(E \setminus F) + \mu(E \cap F) = \mu(E)$.

So, add and cancel, $\mu(E \setminus F) < +\infty$. Remember so I am allowed to cancel and everything is finite in this equation and that is the solution.

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4. $X (\neq \emptyset)$, \mathcal{R} ring on X , $\mu \geq 0$, finite, additive set fn. on \mathcal{R} which is cont. from below at every $E \in \mathcal{R}$, i.e. if $E_i \in \mathcal{R}$, $E_i \uparrow E$, then $\mu(E) = \lim_{n \rightarrow \infty} \mu(E_n)$. Show that μ is a measure.

Solution $\mu(E) < +\infty \forall E$. (finitely) additive. $\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset)$
 $\Rightarrow \mu(\emptyset) = 0$ ($\because \mu(E) < +\infty$).

$\{E_n\}_{n=1}^{\infty}$ disjoint seq. in \mathcal{R} , $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$.
 $F_n = \bigcup_{i=1}^n E_i \rightarrow F_n \in \mathcal{R}$. $F_n \uparrow E$.
 $\mu(E) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.



(4) X non-empty set, \mathcal{R} ring on X , and $\mu \geq 0$, finite, additive set function on \mathcal{R} , which is continuous from below at every $E \in \mathcal{R}$. i.e., if $E_i \in \mathcal{R}$, $E_i \uparrow E$, then $\mu(E) = \lim_{n \rightarrow \infty} \mu(E_n)$.

Show that μ is a measure.

Solution: $\mu < +\infty \forall E$, and finitely additive. Therefore,
 $\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset)$.

$$\Rightarrow \mu(\emptyset) = 0. \text{ (as } \mu(E) < +\infty \text{).}$$

So, we only have to show countable additivity. So $\{E_n\}_{n=1}^{\infty}$ disjoint sequence in \mathcal{R} and

$$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{R}. \text{ So, you take } F_n = \bigcup_{i=1}^n E_i \rightarrow E \in \mathcal{R}, F_n \uparrow E.$$

$$\text{So, } \mu(E) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

So that proves the results.

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$\{E_n\}_{n=1}^{\infty}$ seq. in \mathcal{E} , $Z = \bigcup_{i=1}^{\infty} E_i \in \mathcal{E}$.
 $F_n = \bigcup_{i=1}^n E_i \rightarrow F_n \in \mathcal{E}$. $F_n \uparrow Z$.
 $\mu(Z) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

5. $X (\neq \emptyset)$ \mathcal{S} σ -ring on X . μ meas on \mathcal{S} . $\{E_n\}_{n=1}^{\infty}$ in \mathcal{S} .
 $\liminf_{n \rightarrow \infty} E_n \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i$.

Show that $\mu(\liminf_{n \rightarrow \infty} E_n) \leq \liminf_{n \rightarrow \infty} \mu(E_n)$.



$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i$.

Show that $\mu(\liminf_{n \rightarrow \infty} E_n) \leq \liminf_{n \rightarrow \infty} \mu(E_n)$.

Solution. Let $F_n = \bigcap_{i=n}^{\infty} E_i$. $E = \liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} F_n$.
 $F_n \uparrow E$. $\mu(E) = \lim_{n \rightarrow \infty} \mu(F_n) = \sup_n \mu(F_n)$.
 $\mu(F_n) \leq \mu(E_i) \forall i \geq n \Rightarrow \mu(F_n) \leq \inf_{i \geq n} \mu(E_i)$
 $\mu(E) \leq \sup_n \inf_{i \geq n} \mu(E_i) = \liminf_{n \rightarrow \infty} \mu(E_n)$.



(5) X non-empty set, \mathcal{S} a σ -ring on X , μ measure on \mathcal{S} , $\{E_n\}_{n=1}^{\infty} \in \mathcal{S}$. Now, you define

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i. \text{ Then show that } \mu(\liminf_{n \rightarrow \infty} E_n) \leq \liminf_{n \rightarrow \infty} \mu(E_n).$$

solution: So, let $F_n = \bigcap_{i=n}^{\infty} E_i$, $E = \liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} F_n$. So, $F_n \uparrow E$, $\mu(F_n) \uparrow$ and

therefore, $\mu(E) = \lim_{n \rightarrow \infty} \mu(F_n) = \sup_n \mu(F_n)$. Now,

$$\mu(F_n) \leq \mu(E_i) \forall i \geq n \Rightarrow \mu(F_n) \leq \inf_{i \geq n} \mu(E_i).$$

Therefore, $\mu(E) \leq \sup_n \inf_{i \geq n} \mu(E_i) = \liminf_{n \rightarrow \infty} \mu(E_n)$.