Measure and Integration
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Lecture-7
2.2 - Exercises

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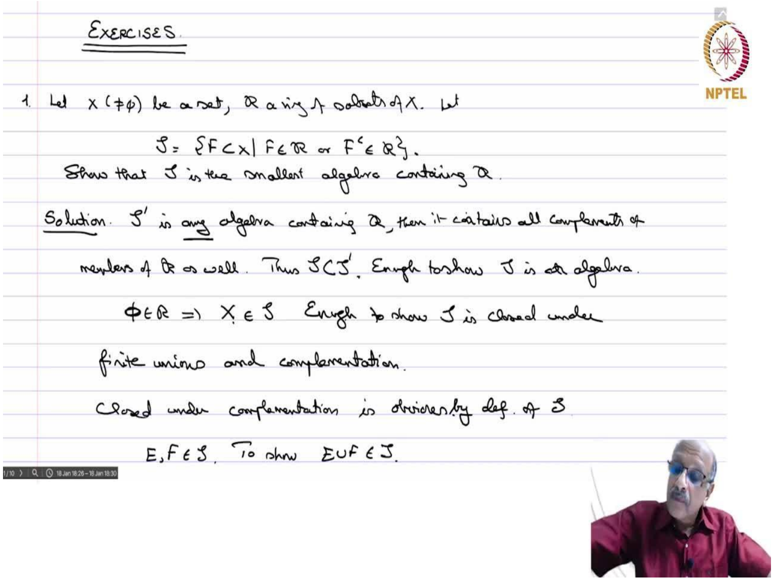

Exercises:

So, let us now do some exercises. So, the first one.

(1) Let X ($\neq$ ϕ) be a set, R ring of subsets of X let

$$S = \{F \subset X,\ F \in R\, or\, F^c \in R\}.$$

Show that S is the smallest algebra containing R.

solution: If S' is any algebra containing R, then it contains all compliments of members of R as well. Thus, $S \subset S'$. So, enough to show S is an algebra.

So, $\phi \in R \Rightarrow X \in S$. Now, enough to show we have already seen this S is closed under finite unions and complementation. So, closed under complementation is obvious by construction because if any member of S either it is a member of R in which case its complement is also in S. If it is not in R that means its complement is in R and therefore, it is also in S that member also the complement is also in S. So, this is obvious by definition. So, now we have to show that it is closed under union. So, let us take $E, F \in S$. So, to show $E \cup F \in S$.

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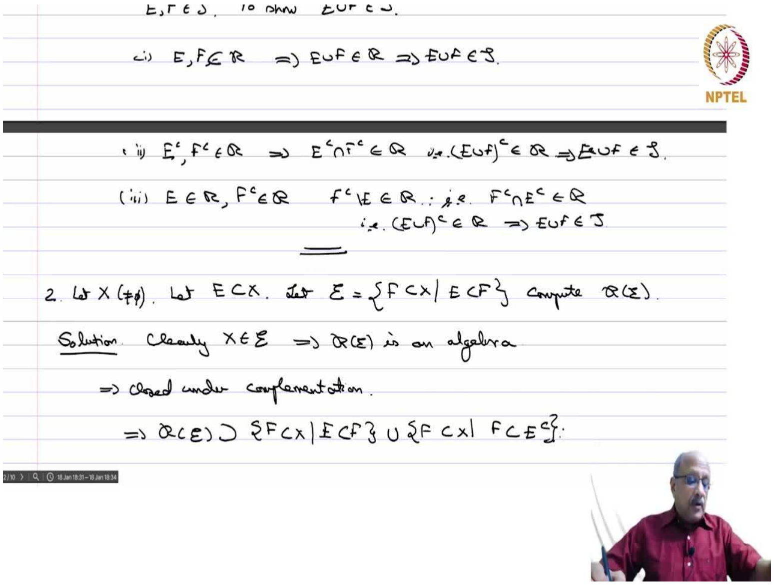

So, let us look at different cases. So, first case.

(i) if $E, F \in R \Rightarrow E \cup F \in R \Rightarrow E \cup F \in S$.

(ii) if $E^c, F^c \in R \Rightarrow E^c \cap F^c \in R,\ i.e., (E \cup F)^c \in R \Rightarrow E \cup F \in S$.

(iii) $E, F^c \in R \Rightarrow F^c \backslash E \in R,\ i.e.,\ F^c \cap E^c \in R,\ i.e.,\ (E \cup F)^c \in R \Rightarrow E \cup F \in S$.

E belongs to R and let us say F complement belongs to this is the third possibility.

Therefore, it is an algebra and it is in fact the smallest algebra containing R.

(2) Let X ($\neq \phi$) and let $E \subset X$ be given. Let $\hat{E} = \{F \subset X: E \subset F\}$. Compute $R(\hat{E})$.

solution: Clearly, $X \in \hat{E} \Rightarrow R(\hat{E})$ is an algebra $\Rightarrow$closed under the complementation.

$$\Rightarrow = \{F \subset X: E \subset F\} \cup = \{F \subset X: F \subset E^c\} \subset R(\hat{E}) = S.$$

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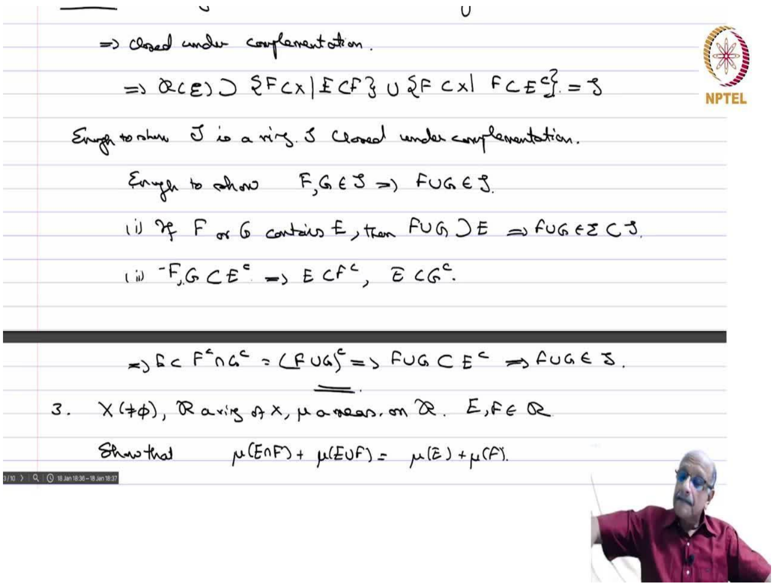

So, enough to show it is algebra whatever they want. So, it is closed under complementation and therefore, enough to show, $F, G \in S \Rightarrow F \cup G \in S$.

So, again we have to take the 3 cases.

(i) if F or G contains E, then $E \subset F \cup G \Rightarrow F \cup G \in \hat{E} \subset S$.

(ii)

$F, G \subset E^c \Rightarrow E \subset F^c,\ E \subset G^c \Rightarrow E \subset F^c \cap G^c = (F \cup G)^c \Rightarrow F \cup G \subset E^c \Rightarrow F \cup G \in S.$

So, this becomes an algebra.

(3) X non-empty set, R ring on X, μ - a measure on R, $E, F \in R$. Show that

$$\mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F). \text{ ------------- } (*)$$

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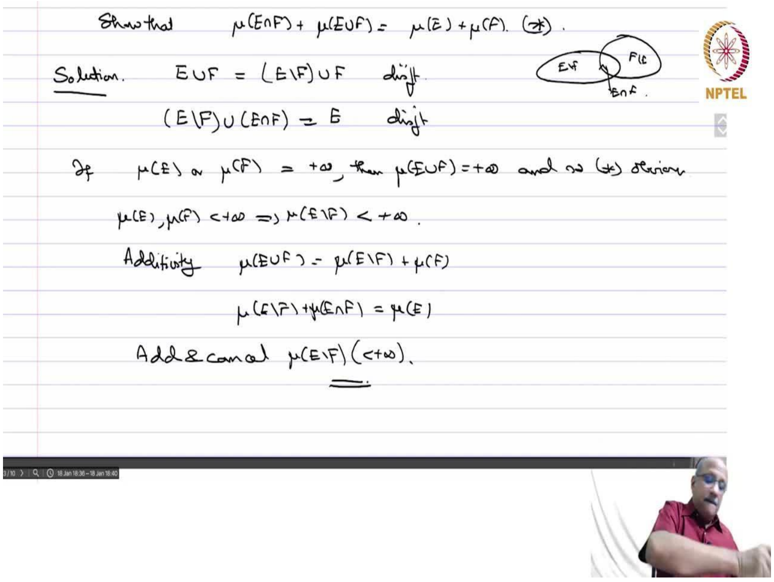

solution: So $E \cup F = (E\backslash F) \cup F$ *and* $(E\backslash F) \cup (E \cap F) = E$.

So, this can be written and this is disjoint.

So, if $\mu(E)$ *or* $\mu(F) =+ \infty$, then $\mu(E \cup F) =+ \infty$ and so (*) is obvious.

So, we can assume that both $\mu(E)$ *and* $\mu(F) <+ \infty \Rightarrow \mu(E \cup F) <+ \infty$. So, now by additivity $\mu(E \cup F) = \mu(E\backslash F) \cup \mu(F)$ *and* $\mu(E\backslash F) \cup \mu(E \cap F) = \mu(E)$.

So, add and cancel, $\mu(E\backslash F) <+ \infty$. Remember so I am allowed to cancel and everything is finite in this equation and that is the solution.

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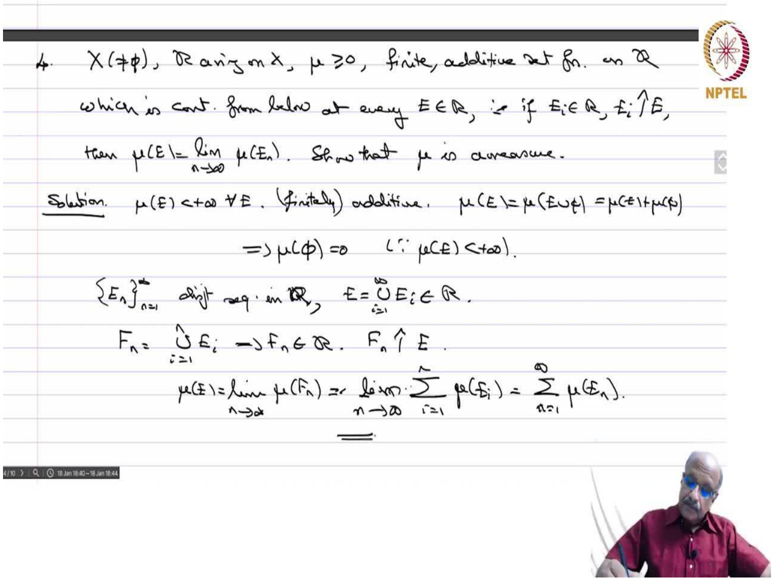

(4) X non-empty set, R ring on X, and $\mu \geq 0$, finite, additive set function on R, which is continuous from below at every $E \in R$. i.e., if $E_i \in R$, $E_i \uparrow E$, then $\mu(E) = \lim_{n \to \infty} \mu(E_i)$.

Show that μ is a measure.

Solution: $\mu < +\infty \ \forall E$, and finitely additive. Therefore, $\mu(E) = \mu(E \cup \phi) = \mu(E) + \mu(\phi)$.

$$\Rightarrow \mu(\phi) = 0.\ (as\ \mu(E) < +\infty).$$

So, we only have to show countable additivity. So $\{E_n\}_{n=1}^{\infty}$ disjoint sequence in R and

$E = \cup_{i=1}^{\infty} E_i \in R$. So, you take $F_n = \cup_{i=1}^{n} E_i \to E \in R,\ F_n \uparrow E$.

So, $\mu(E) = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(E_i) = \sum_{n=1}^{\infty} \mu(E_n)$.

So that proves the results.

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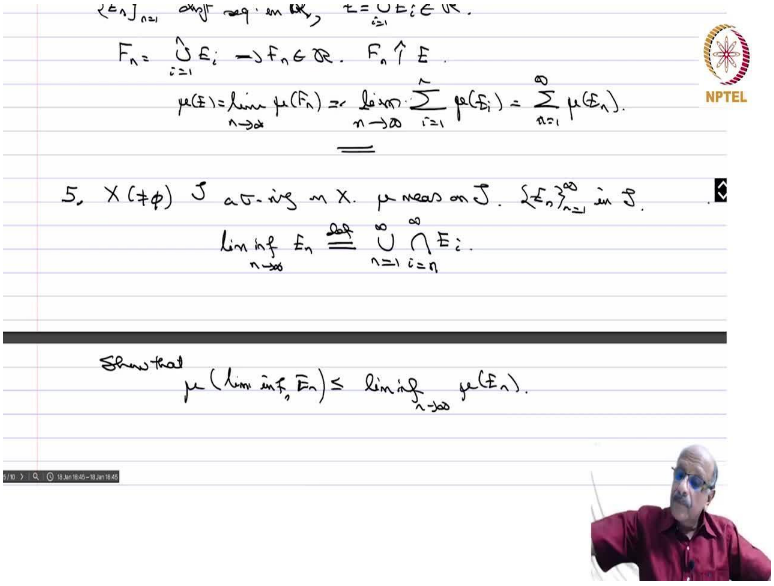

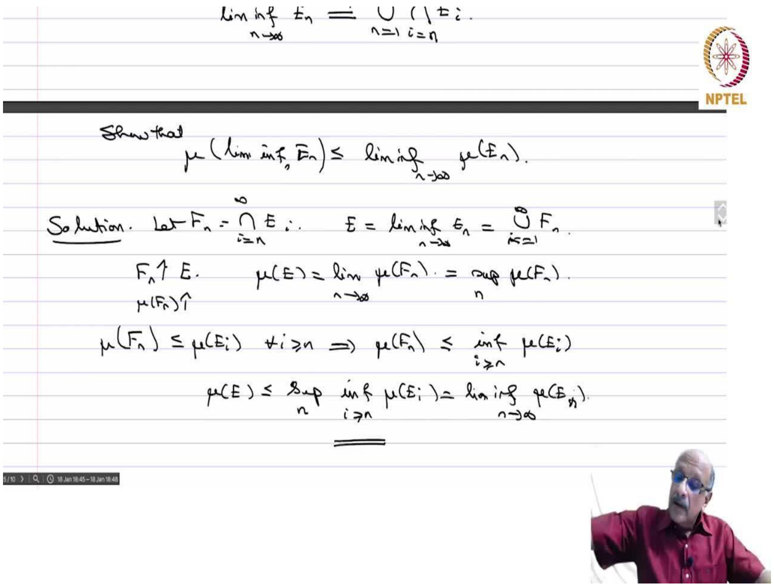

(5) X non-empty set, S a σ-ring on X, μ measure on S, $\{E_n\}_{n=1}^{\infty} \in S$. Now, you define

$\lim\inf_{n\to\infty} E_n = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} E_i$. Then show that $\mu(\lim\inf_{n\to\infty} E_n) \leq \lim\inf_{n\to\infty} \mu(E_n)$.

solution: So, let $F_n = \cap_{i=n}^{\infty} E_i$, $E = \lim\inf_{n\to\infty} E_n = \cup_{n=1}^{\infty} F_n$. So, $F_n \uparrow E$, $\mu(F_n) \uparrow$and therefore, $\mu(E) = \lim_{n\to\infty} \mu(F_n) = \sup_n \mu(F_n)$. Now,

$$\mu(F_n) \leq \mu(E_i) \; \forall\, i \geq n \Rightarrow \mu(F_n) \leq \inf_{i\geq n} \mu(E_i).$$

Therefore, $\mu(E) \leq \sup_n \inf_{i \geq n} \mu(E_i) = \liminf_{n \to \infty} \mu(E_n)$.