Measure and Integration
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Lecture No-69
11.1 - Approximation

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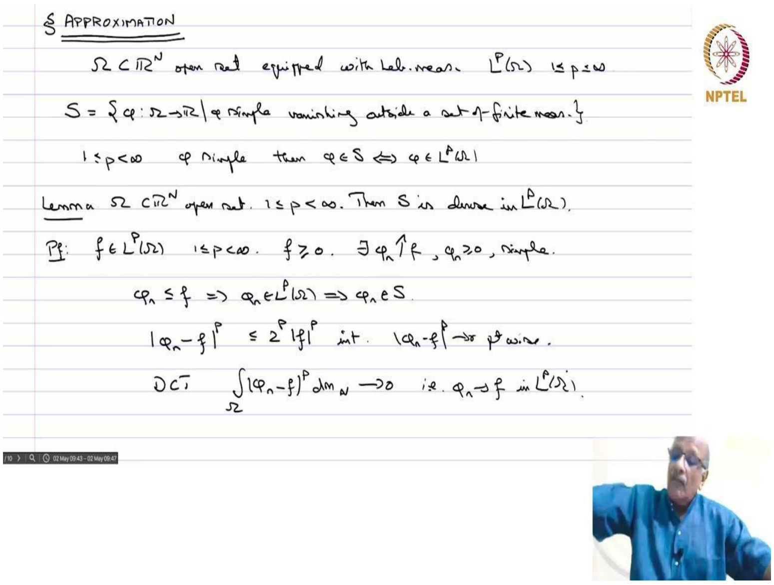

Now discuss approximation properties for LP spaces. So, by this I mean identifying sets which are dense and also some separability properties of LP spaces. So, we are dealing with $\Omega \subset \mathbb{R}^N$ open set equipped with the Lebesgue measure so, this is the measure we are going to talk off and then I said the notation for this is $L^p(\Omega),\ 1 \leq p \leq \infty$.

So, we are going to define the set S as

$$S = \{\phi : \Omega \to \mathbb{R} \mid \phi\ simple\ vanishing\ outside\ a\ set\ of\ finite\ measure\}.$$

So, if $1 \leq p < \infty$, then ϕ simple, $\phi \in S$ if and only if $\phi \in L^p(\Omega)$.

Lemma: $\Omega \subset \mathbb{R}^N$ open set, $1 \leq p < \infty$. Then S is dense in $L^p(\Omega)$.

proof: So, $f \in L^p(\Omega),\ 1 \leq p < \infty,\ f \geq 0,\ \exists\, \phi_n \uparrow f,\ \phi_n \geq 0,\ simple$ and

$$\phi_n \geq f \Rightarrow \phi_n \in L^p(\Omega) \Rightarrow \phi_n \in S.$$

Now, $|\phi_n - f|^p \leq 2^p|f|^p$ and this is integrable, then $|\phi_n - f|^p \to 0$ pointwise. Therefore, by the dominated convergence theorem, we have $\int_X |\phi_n - f|^p dm_N \to 0,\ i.e.,\ \phi_n \to f\ in\ L^p(\Omega)$.

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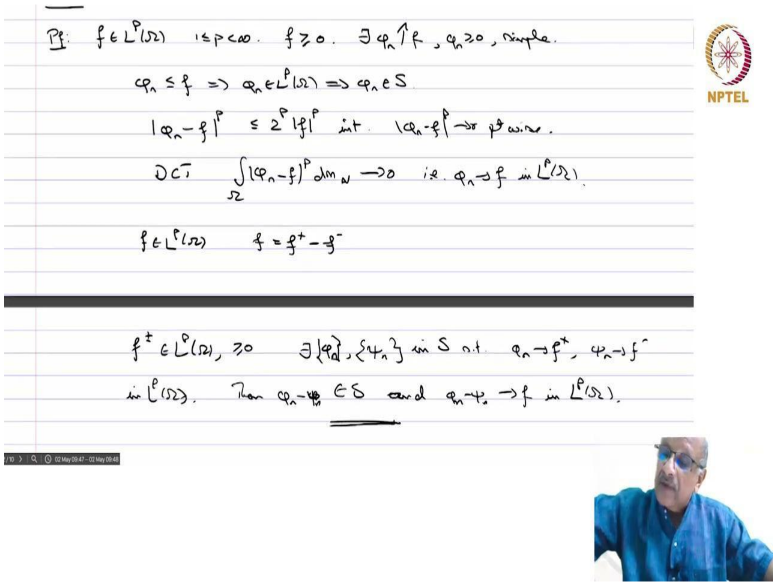

Now, if $f \in L^p(\Omega)$, you write $f = f^+ - f^-$, then $f^+, f^- \in L^p(\Omega)$, and they are non negative. Therefore, there exists a $\phi_n \in S$ and the sequence $\psi_n \in S$ such that $\phi_n \to f^+,\ \psi_n \to f^-\ in\ L^p(\Omega)$. Then $\phi_n - \psi_n \in S$ and $\phi_n - \psi_n \to f\ in\ L^p(\Omega)$.

So, this shows that S is dense in $L^p(\Omega)$.

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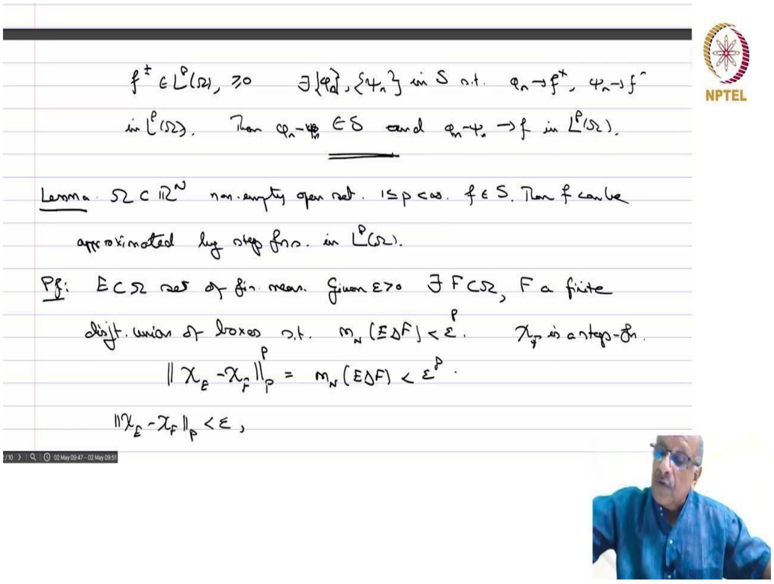

Lemma: $\Omega \subset \mathbb{R}^N$ non-empty open set now, of course, $f \in S$. Then then f can be approximated by step functions in $L^p(\Omega)$.

So, what is the step function, step function is a simple function where the chi of E i, E i's are all boxes.

proof. So, E in omega set of finite measure then we have seen given epsilon positive there exists $F \subset \Omega$, finite disjoint union of boxes such that $m_N(E\Delta F) < \epsilon^p$.

Now, $||\chi_E - \chi_F||^p_p = m_N(E\Delta F) < \epsilon^p \Rightarrow ||\chi_E - \chi_F||_p < \epsilon$.

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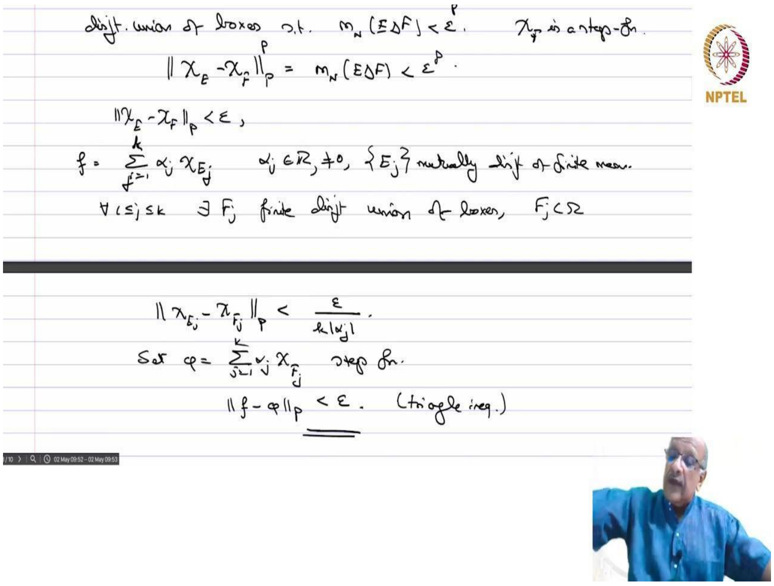

So, if you have a function $f = \sum_{j=1}^{k} \alpha_j \chi_{E_j}$, $\alpha_j \in \mathbb{R}$, $\neq 0$, $\{E_j\}$ mutually disjoint or finite measure, then for every $1 \leq j \leq k$, there exists F_j finite disjoint union of boxes $F_j \subset \Omega$, and

$$||\chi_{E_j} - \chi_{F_j}||_p < \frac{\epsilon}{k|\alpha_j|}.$$

Now you set $\phi = \sum_{j=1}^{k} \alpha_j \chi_{E_j}$. So, this is step function and by triangle inequality $||f - \phi||_p < \epsilon$.

So, this completes the proof so, every S can be every element of S can be approximated by a step function.

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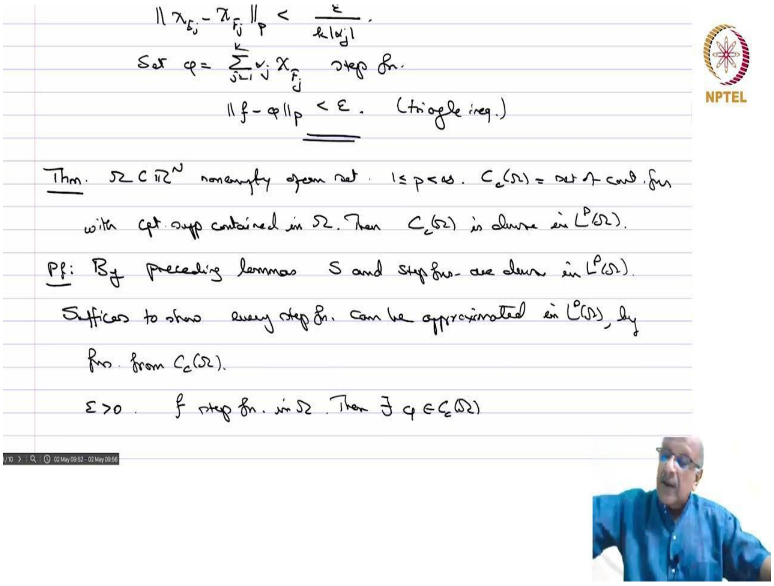

Theorem. So, $\Omega \subset \mathbb{R}^N$ non empty open set, $1 \leq p < \infty$, $C_c(\Omega)$ = set of continuous functions with compact support contained in Ω. Then $C_c(\Omega)$ is dense in $L^p(\Omega)$.

Proof: So, by proceeding lemmas S and step functions are dense in $L^p(\Omega)$.

So, we can assume to show that, so sufficient to show every step function can be approximated in $L^p(\Omega)$ by functions from $C_c(\Omega)$, so, you are given a step function and so, let $\epsilon > 0$, Then we know given a step function we have already shown that there exists a continuous function with compact support. So, f step function in Ω then there exists phi in $C_c(\Omega)$ such that the following two properties are true.

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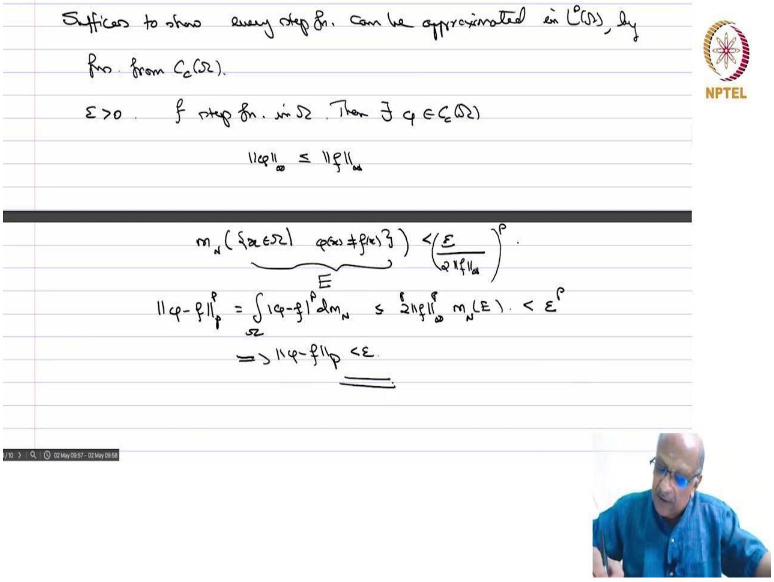

$$||\phi||_\infty \leq ||f||_\infty,\ m_N(\{x \in \Omega\colon \phi(x) \neq f(x)\}) < (\frac{\epsilon}{2||f||_\infty})^p,$$

$$||\phi - f||_p^{\ p} = \int_\Omega |\phi - f|^p dm_N \leq 2^p ||f||_\infty^{\ p} m_N(E) < \epsilon^p \Rightarrow ||\phi - f||_p < \epsilon.$$

That proves that every step function can be approximated by the C infinity function of compact support.

So, you take LP function approximate by a function in S, take the S function and approximate it by a step function, step function can be approximated by C infinity function with compact support and therefore, every element in LP 1 less than p less than infinity can be approximated by a continuous function with compact support.

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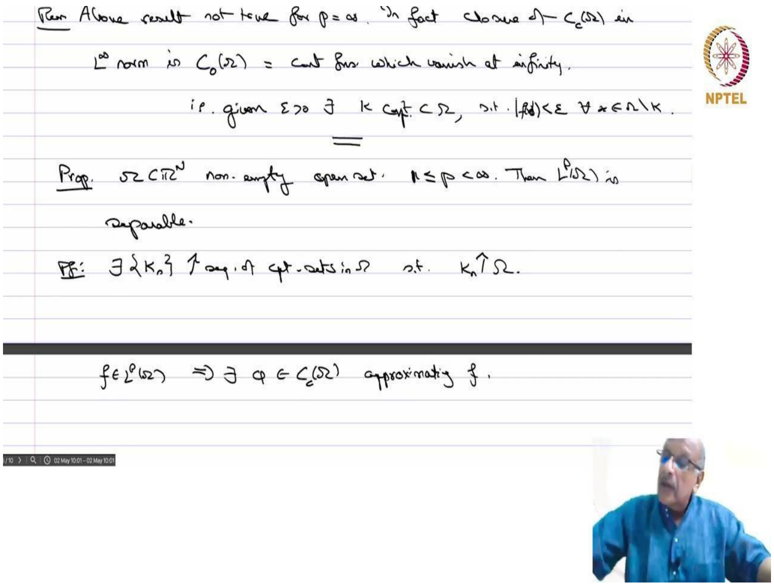

Remark: Above result is not true for $p = \infty$. In fact closure of $C_c(\Omega)$ in L^{∞} norm is C_0 of omega equals continuous functions which vanish at infinity that means, given epsilon there exists F K compact contained in omega such that mod f is less than epsilon mod f x for all x in omega minus K. So, such functions are functions which vanish at infinity and that is again a continuous function and therefore, you cannot approximate L infinity, LP, L infinity functions by means of this.

So, there are several interesting applications of this result. We will see them subsequently but to start with we have the following proposition.

Proposition: So, $\Omega \subset \mathbb{R}^N$ non empty open set $1 \leq p < \infty$. Then $L^p(\Omega)$ is separable, separable means there exists a countable dense set.

proof: There exists Kn increasing sequence of compact sets in omega such that Kn increases to omega that means omega is the union of all the Kn's. And now $f \in L^p(\Omega)$ implies there exists $\phi \in C_c(\Omega)$ approximating f if you can make it as close to it as possible.

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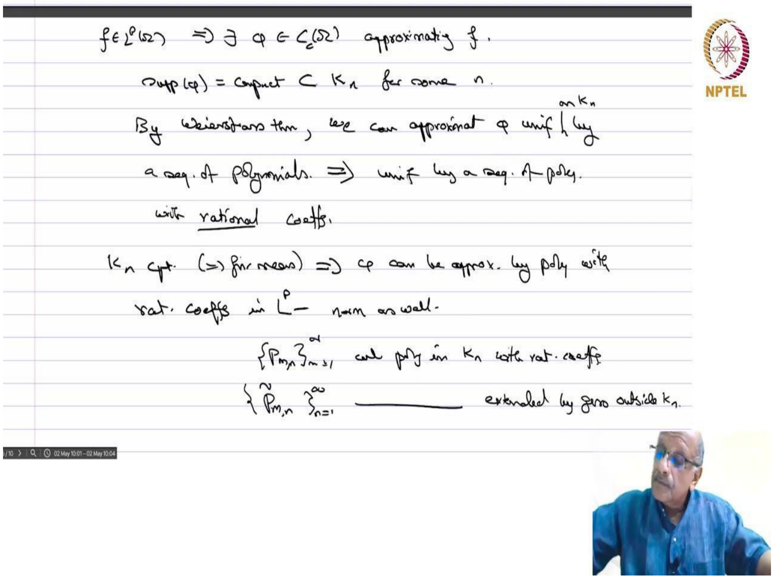

Then $supp(\phi)$ is compact and therefore they should be contained in Kn For some n because kn's are increasing compact sets and they ultimately exhaust all of it. Now, by Weierstrass approximation theorem, we can approximate ϕ uniformly by a sequence of polynomials and since the coefficients of the polynomials are real numbers. So, you can approximate it by rational numbers and therefore implies uniformly by a sequence of polynomials with rational coefficients.

But Kn is compact on Kn, yes, Kn is compact implies finite measure and therefore L infinity is continuously embedded in any L P this implies phi can be approximated by polynomials with rational coefficients in Lp norm, now set of all polynomials with rational coefficients is countable. So, let us say so, Pmn m equals 1 to infinity are all polynomials in Kn with rational coefficients so, Pmn tilde m equals 1 to infinity extended by zero outside Kn.

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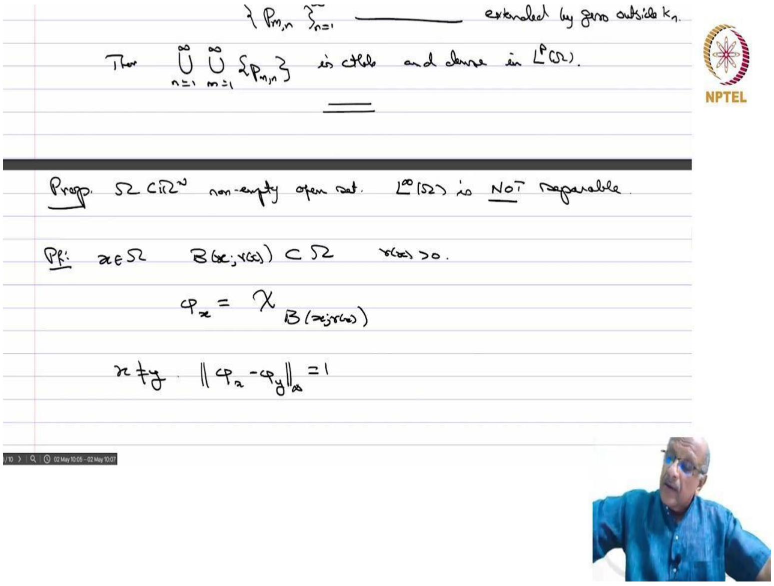

Then union n equals 1 to infinity union m equals 1 to infinity Pmn is countable and dense in Lp of omega because given any f in Lp you can approximate it by Cc continuous function with compact support and each continuous function with compact support the support will be in some Kn and they can be approximated in the Lp norm by the Pmns and therefore, phi can itself be approximated by Pmn tilde in all of omega because outside omega In both phi and P mn are 0.

So, you have this and this shows that Lp is separable, countable dense set. So, Proposition Omega in RN non empty open set L infinity of omega is not separable. Proof, so, let x belong to omega then there exists a ball center x, radius r x which is contained in omega so, r x is positive this a ball, center x radius r x. So, let us take phi x equals chi of B x r x so, this is a function phi f x.

Now, let x not equal to y, then if you take phi x minus phi y. So, these are differences of two characteristic functions on the intersection; they will be 0, outside it will be plus 1 or minus 1 or 0 depending where you are and therefore, this norm infinity is always equal to 1 if x is not equal to y.

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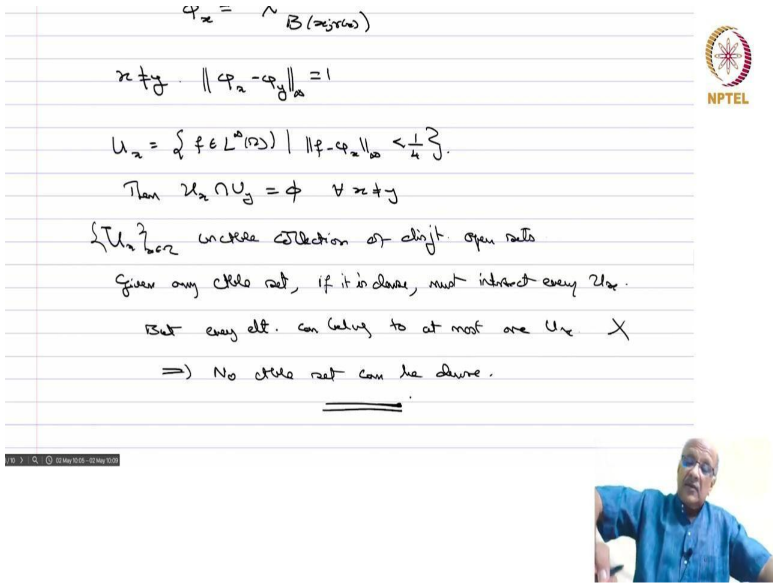

So, now you take $U_x = \{f \in L^{\infty}(\Omega) : ||f - \phi||_{\infty} < \frac{1}{4}\}$. Then $U_x \cap U_y = \phi \ \forall \, x \neq y$.

Therefore $\{U_x\}_{x \in \Omega}$ is an uncountable set collection, uncountable collection of disjoint open sets. Now, so, given any countable set U_x must intersect every one of them or rather if it is dense must intersect every U_x but every element can belong to at most 1, U_x because it cannot belong to more than 1 because U_x intersection U_y is empty when x is not equal to y. This implies that, so, this is a contradiction. Because, you have only a countable number of 1 which will be, so there will be uncountable numbers which are not intersecting. So, this implies that no countable set can be dense.

And therefore, L infinity is not separable, it is a very useful idea to prove (())(26:54) if you can show that you have an uncountable collection of open sets, which are all mutually disjoint then that such a space can never be countable. So we will continue with the applications next time.