Measure and Integration
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Lecture No-68

10.6 – Convergence in L^p

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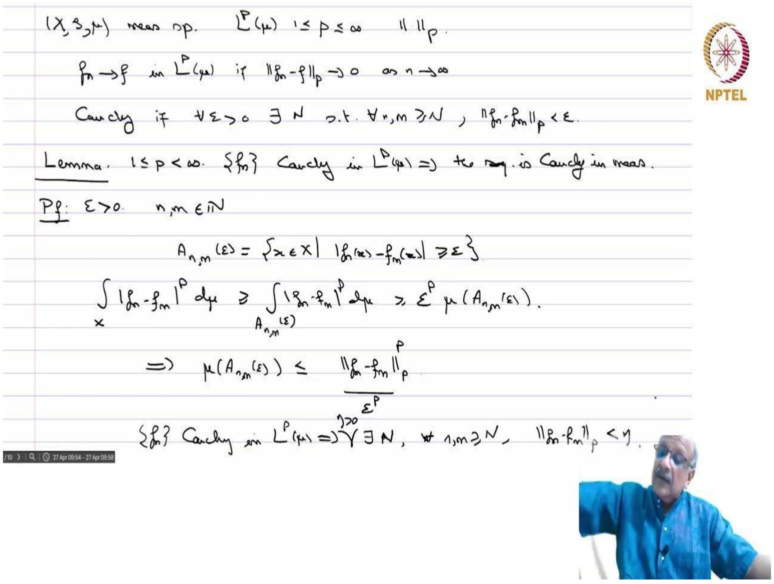

So, (X, S, μ) measure space. So, $L^p(\mu)$ are the spaces, $1 \leq p \leq \infty$ with the norm norm p. So, we say $f_n \to f$ in $L^p(\mu)$ if $||f_n - f||_p \to 0$ as n tends to infinity. So, Cauchy if for every epsilon positive there exists a capital N such that for all $n, m \geq n$ we have $||f_n - f_m|| < \epsilon$.

So, we want to show that these spaces are complete, namely this symmetric space empty Cauchy sequence converges to its metric, then it is a complete space and a complete norm linear space is called a Banach space. So, we want to show you all these L p spaces are Banach spaces.

So, before that let us start with the following lemma.

Lemma. So, $1 \leq p < \infty$. So, $\{f_n\}$ Cauchy in $L^p(\mu)$ implies the sequence is Cauchy in measure.

proof: so, let $\epsilon > 0$, then for all n, m positive integers, we define

$$A_{n,m}(\epsilon) = \{x \in X: |f_n(x) - f_m(x)| \geq \epsilon\}.$$

So, then $\int_X |f_n - f_m|^p dx \geq \int_{A_{n,m}(\epsilon)} |f_n - f_m|^p dx \geq \epsilon^p \mu(A_{n,m}(\epsilon))$.

So, this implies that $\mu(A_{n,m}(\epsilon)) \leq \frac{||f_n - f_m||^p}{\epsilon^p}$.

So, fn Cauchy in $L^p(\mu)$ implies of course, that mu so, there exists a N such that for all n, m greater than equal to so, given $\eta > 0$ and greater than equal to capital N we have

$$||f_n - f_m|| < \eta.$$

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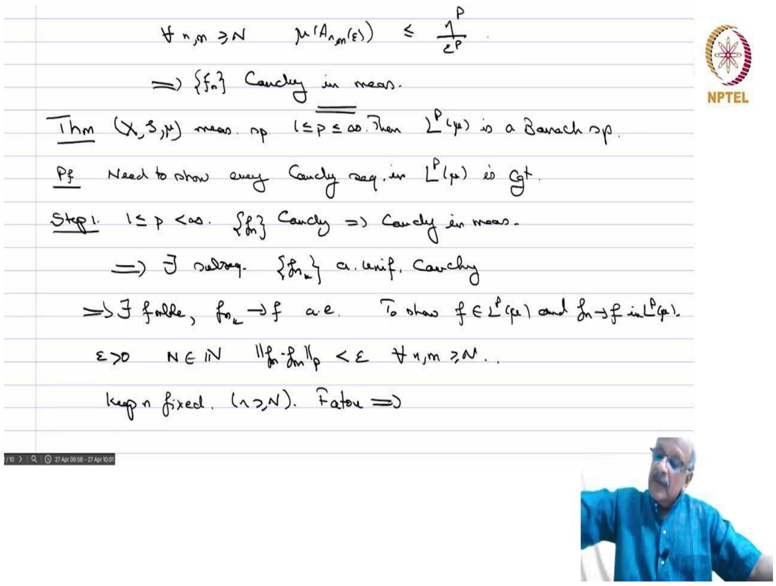

Therefore, for all $n, m \geq N$, we have $\mu(A_{n,m}(\epsilon)) \leq \frac{\eta^p}{\epsilon^p}$ and therefore, you have $\{f_n\}$ is Cauchy in measure.

So, now, we have the following important theorem, sometimes called the Riesz Fischer theorem also.

Theorem: So, let (X, S, μ) measure space, $1 \leq p \leq \infty$. Then $L^p(\mu)$ is a Banach space. So that is all we need to show.

proof: **Step 1**: $1 \leq p < \infty$. So, then $\{f_n\}$ Cauchy implies Cauchy in measure by the lemma above and that implies there exists a subsequence $\{f_{n_k}\}$, which is almost uniformly Cauchy. We have seen this before given a sequence which is Cauchy in measure, then there is a sub sequence which is almost uniformly Cauchy.

And, that implies that $f_{n_k} \to f$, almost everywhere. So, there implies, there exists f measurable such that fn k converges to f pointwise almost everywhere. So, let epsilon be, we now have a candidate for the limit, so, we have to show this so, to show f belongs to L p mu and fn goes to f in L p mu.

So, if we can show these two things, our theorem is proved so, let epsilon be positive and let N belong to the natural numbers such that norm fn minus fm p is less than epsilon for all n, m greater than equal to capital N. So, keep n fixed of course, n is greater than N.

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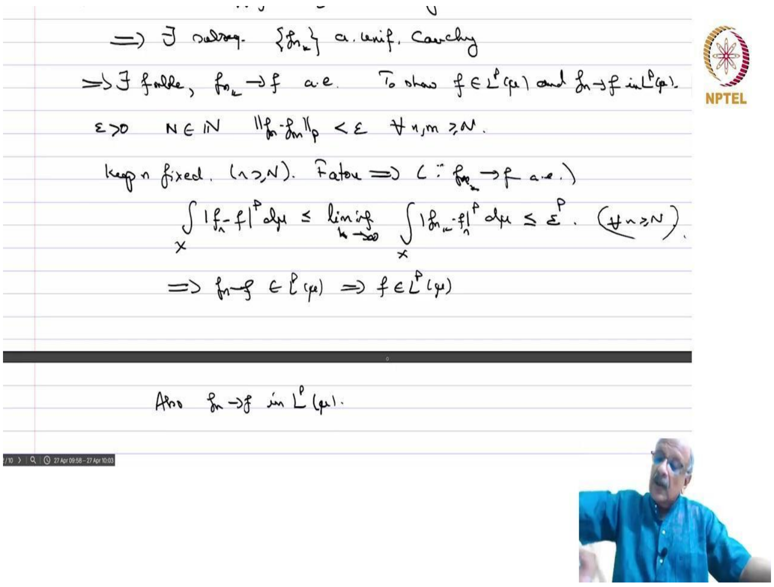

So, by Fatou lemma, since fm converges fm converges to f almost everywhere and therefore, you have $\int_X |f_n - f|^p d\mu \leq \lim_{n\to\infty} \inf \int_X |f_{n_k} - f|^p d\mu \leq \epsilon^p$ $(\forall\, n \geq N)$.

$$\Rightarrow f_n \to f \in L^p(\mu) \Rightarrow f \in L^p(\mu).$$

Also, $f_n \to f\ \ in\ L^p(\mu)$.

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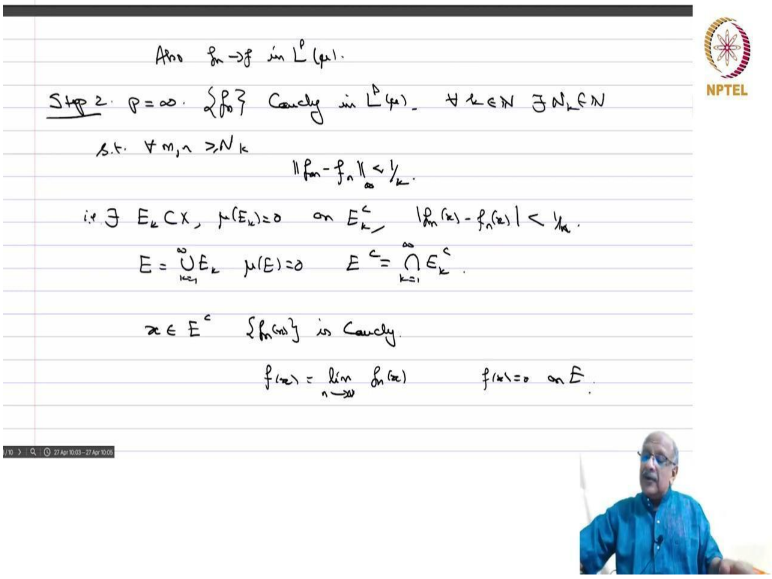

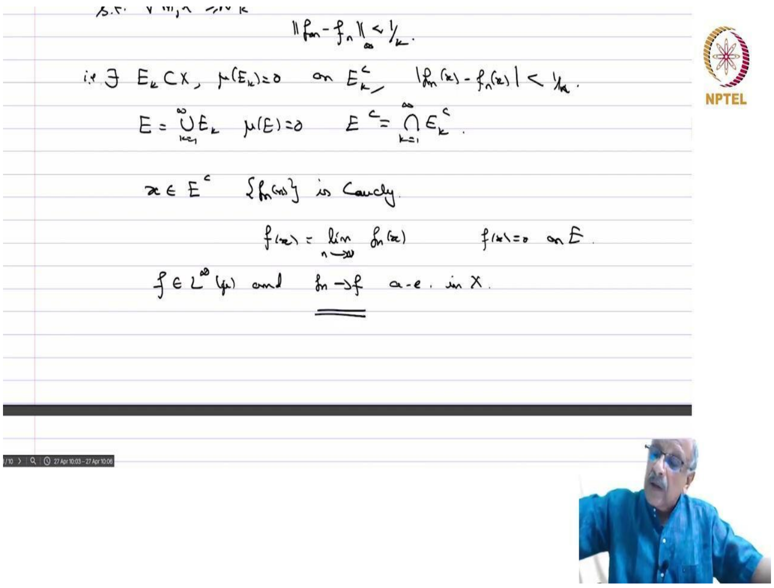

Step 2: $p = \infty$. So, $f_n \, Cauchy \; in \, L^p(\mu)$, so, there exists for every k in N there exists $N_k \in \mathbb{N}$ such that for all $m, n \geq N_k$, we have $||f_n - f_m|| < \frac{1}{k}$.

i.e., there exists $E_k \subset X$, $\mu(E_k) = 0 \; on \; E_k^c$, and $|f_n(x) - f_m(x)| < \frac{1}{k}$. So, let $E = \cup_{k=1}^{\infty} E_k$, $\mu(E) = 0$, $E^c = \cap_{k=1}^{\infty} E_k^c$. So, if $x \in E^c$, it is in every Ek compliment and therefore, $f_n(x)$ is Cauchy. So, you take $f(x) = \lim_{n \to \infty} f_n(x)$, $f(x) = 0 \; on \; E$.

And, you have that $f \in L^{\infty}(\mu)$ and $f_n \to f \; a.e. \; X$.

So, that proves that this is a Banach space. So, this completes the proof on both (())(11:58).

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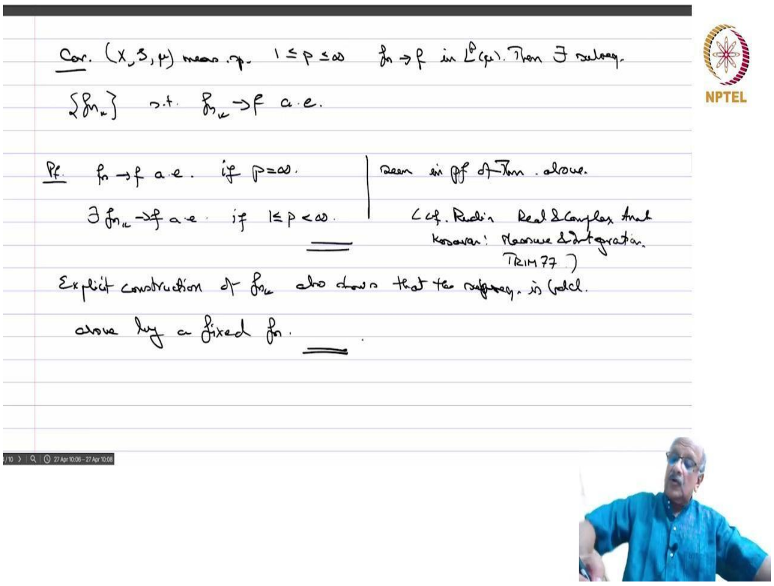

Corollary: (X, S, mu) measure space, $1 \leq p \leq \infty$ and $f_n \rightarrow f$ in $L^p(\mu)$. Then, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow f$ a.e.

proof: $f_n \rightarrow f$ almost everywhere if $p = \infty$, we have already seen and there exists $f_{n_k} \rightarrow f$ a.e. if $1 \leq p \leq \infty$. We have seen this.

So, seen in the proof of the theorem above and therefore, this. Now, you can also explicitly construct the subsequence in case of 1 less than equal to p less than infinity and then you can find in almost any textbook on measure theory in particular I will say compare Rudin Real and Complex Analysis or the book I am following Measure and Integration Trim77.

So, you can find it in either of the books and explicit construction now. There is an additional advantage in this proof that in fact explicit construction of fn k also shows that the sub sequence is bounded above by a fixed function. Anyway, that is not very important for the moment and so, we will not give the proof we have used the convergence in measure arguments which is an easier argument to do.

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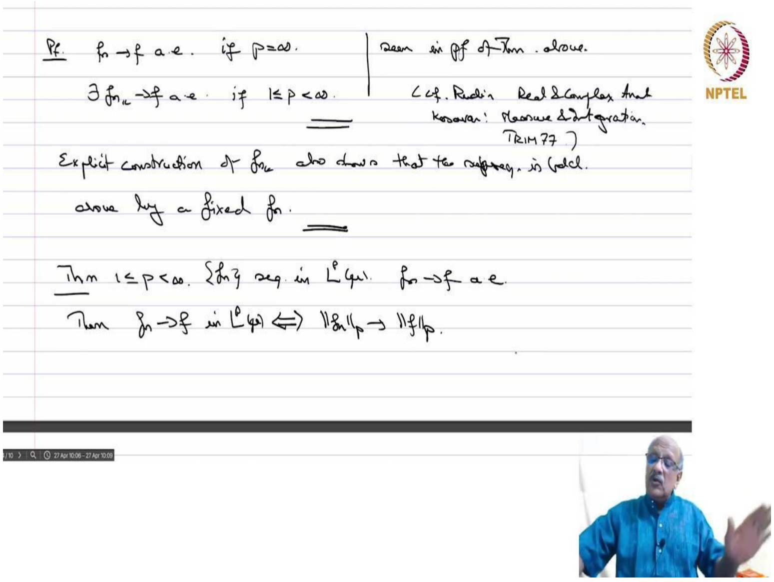

Theorem: $1 \leq p < \infty$, $\{f_n\}$ sequence in $L^p(\mu)$, $f_n \to f$ almost everywhere. Then $f_n \to f \ in\ L^p(\mu)$ if and only if $||f_n||_p \to ||f||_p$.

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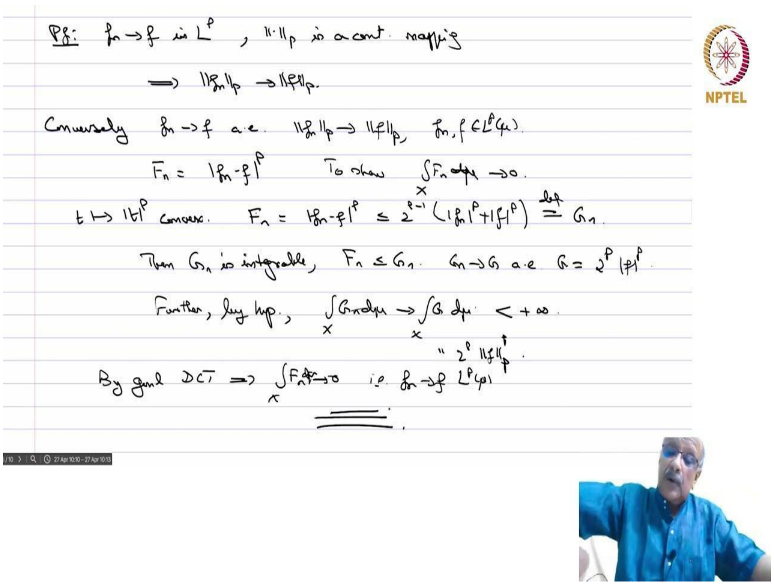

proof: so, $f_n \to f \ in\ L^p(\mu)$ is a continuous mapping and therefore, this implies that $||f_n||_p \to ||f||_p$. So, that is obvious and therefore, you do not have to do anything here. Conversely, $f_n \to f$ almost everywhere and $||f_n||_p \to ||f||_p$ $f_n, f \in L^p(\mu)$.

So, for that we take $F_n = |f_n - f|^p$. So, to show the integral fn dm over x goes to 0. So, that is what we want to show. Now, $t \to |t|^p$ is convex. Therefore,

$$F_n = |f_n - f|^p \leq 2^{p-1}(|f_n|^p + |f|^p) = G_n.$$

Then G_n is integrable, $F_n \leq G_n$, and $F_n \to G_n$ almost everywhere. Further by hypothesis we have $\int_X G_n d\mu \to \int_X G d\mu < +\infty$.

integral Fn, sorry integral Gn d mu over x goes to the integral over G sorry x G d mu, this is wrong Gn goes to G almost everywhere and where G is equal to 2 power p mod f p and this is also finite because what is this integral this is 2 power p norm f p power p.

So, this, so, by generalized dominated convergence theorem we have seen this, this implies that $\int_X F_n d\mu \to 0$, *i.e.*, $f_n \to f$ in $L^p(\mu)$.

So, that completes. So, we have shown that it is a Banach space we have seen about convergence. So, next time we will take up some other special properties of L p spaces like density and then separability and such topological properties.