Measure and Integration
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No-67

Examples. Inclusion Questions

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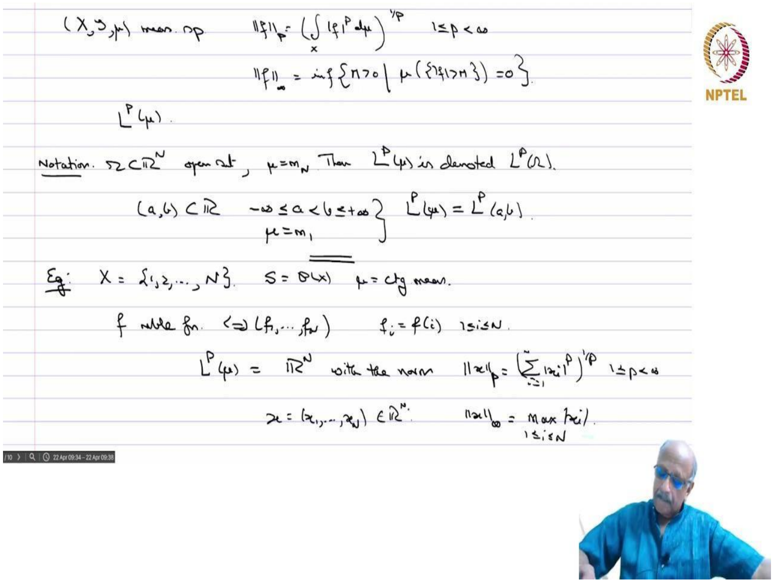

We were looking at L p spaces. So, $(X, S,\ \mu)$ measure space and you have L p. So, you have the $||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}},\ 1 \leq p < \infty,\ ||f||_\infty = \inf\{M > 0\colon\ \mu(\{|f| > M\}) = 0\}$.

Notation: if $\Omega \subset \mathbb{R}^N$ open set , $\mu = m_N$ Lebesgue measure, then $L^p(\mu)$ is denoted $L^p(\Omega)$. And similarly, if $(a, b) \subset \mathbb{R},\ -\infty \leq a < b \leq +\infty,\ \mu = m_1,\ L^p(\mu) = L^p((a, b))$.

Example: Take $X = \{1, 2, .., N\}$, then S is the power set of X and mu is the counting measure. Then what is a measurable function f measurable function means is the same as saying f1, fN where $f_i = f(i),\ 1 \leq i \leq N$. So, measurable function is equal to M triple and we have L p of mu in this case is the same as R N with the $||x||_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}},\ 1 \leq p < \infty,\ x = (x_1, x_2, ..., x_N),\ ||x||_\infty = \max_{1 \leq i \leq N} |x_i|$.

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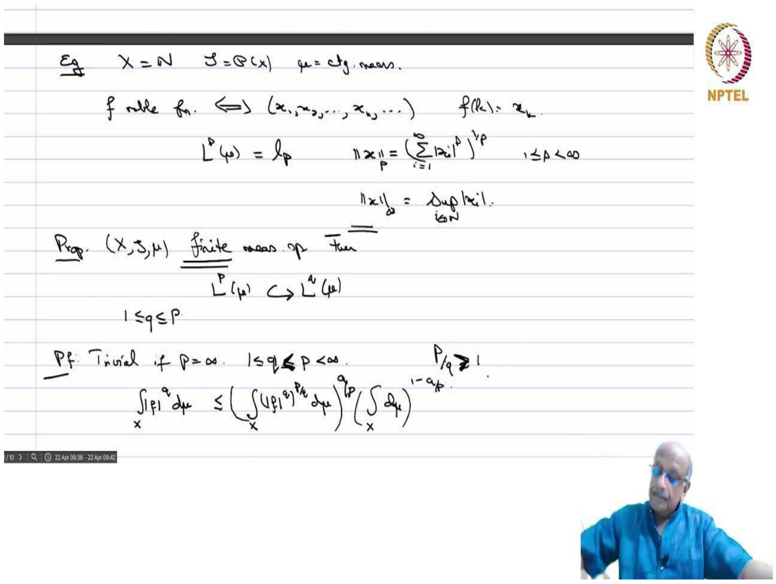

So, in this case equivalence classes are Singleton's because equality almost everywhere because a set of measure 0 is only the empty set and therefore equality almost everywhere is the same as equality everywhere and therefore equivalence class Singleton's so, you really do not have anything special.

So, for example again now you take X equal to N and then S is equal to power set of x and mu equals counting measure then what is the measurable function? f measurable function so, this is the same as infinite sequence x1, x2, xk and so on. Where f of x k, f of k equals x k and therefore, you have that L p of mu is the same as this space little l p of all sequences. So, x equals norm x p equals sigma mod x i power p i equals 1 to infinity power 1 by p if 1 less than equal to p less than infinity and norm x infinity is sup of mod xi in N.

So, these are the familiar spaces, the sequence spaces and the L p spaces. So, again in this case measure sets of measure 0 are only the empty sets and therefore, the equivalence classes are on Singleton's.

proposition. $(X, S,\ \mu)$ finite measure space then $L^p(\mu)$ is continuously embedded in $L^q(\mu)$. The symbol means that it is a set inclusion and the inclusion map is a continuous map between these normed linear spaces for all 1 less than equal to q less than equal to p. So, the bigger L p spaces are contained in the smaller L p spaces.

proof: trivial if p equals infinity, because bounded sets are integrable any power is integrable and so on. So, now, let us assume that $1\ \leq\ q\ <\ p\ <\ \infty$. So,

$$\int_X |f|^q d\mu \le \left(\int_X (|f|^q)^{\frac{p}{q}} d\mu\right)^{\frac{q}{p}} \left(\int_X d\mu\right)^{1-\frac{q}{p}} = \left(\int_X |f|^p d\mu\right)^{\frac{q}{p}} \mu(X)^{1-\frac{q}{p}} = ||f||_p{}^q \mu(X)^{1-\frac{q}{p}}.$$

$$\Rightarrow ||f||_q \le (\mu(X))^{\frac{1}{q}-\frac{1}{p}} ||f||_p.$$

So, this is integral x the function you are taking two functions mod f to the q and 1. So, there are two functions we are using the Holder's inequality, this is of course, 1 minus q by p.

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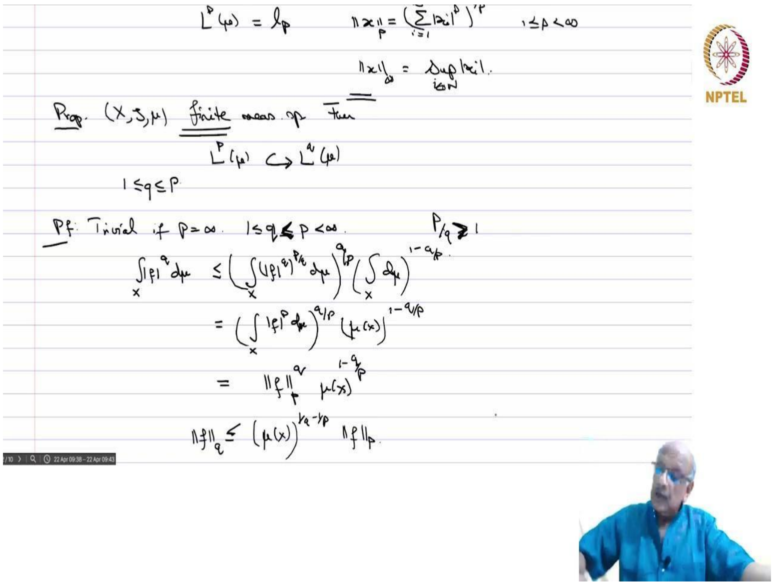

And therefore, you have that it is a continuous inclusion. So, that is.

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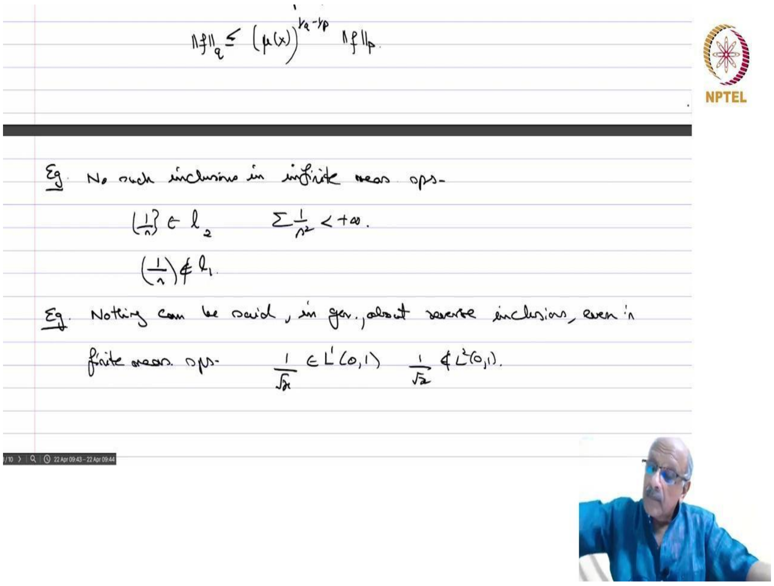

Example: no such inclusions in infinite measure spaces for instance 1 by N the sequence belonging to l2\ sigma 1 by N square is finite, but 1 by N does not belong to l1 . So, this is to be a nice example.

Example: nothing can be said in general about reverse inclusions even in finite measure spaces. Then since 1 over root x belongs to l1 of 0, 1, but 1 by root x does not belong to l2 of 0, 1.

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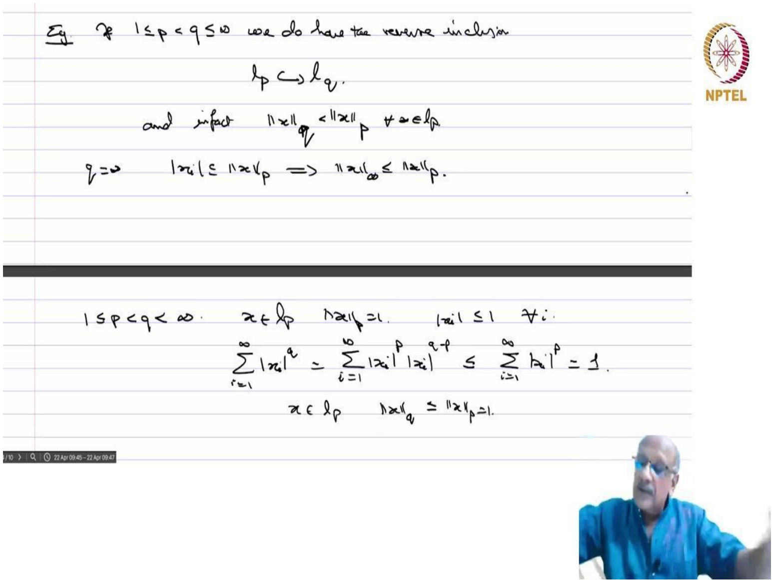

Example: again if $1 \leq p < q \leq \infty$, we do have the reverse inclusion and lp contained in lq. So, if it is in a smaller if it is some of them with respect to a smaller exponent than it is of course, some will and in fact $||x||_q \leq ||x||_p \ \forall\, x \in l^p$. So, if q equals infinity then of course, $|x_i| \leq ||x||_p$ and this implies that $||x||_\infty \leq ||x||_p$.

So, now, let us assume that $1 \leq p < q < \infty$. So, let x in lp norm x p equal to 1 then mod x i is less than or equal to 1 for all i. So, sigma mod xi to the power of q i equals 1 to infinity is equal to sigma mod xi power p i equals 1 to infinity mod xi to the q minus p and q minus p is a positive exponent x is less than xi so all less than to sigma i equals 1 to infinity mod xi power p it is equal to 1. So, this is equal to 1 and therefore, you have that x belongs to lq and norm x q is less than equal to norm x p which is equal to 1.

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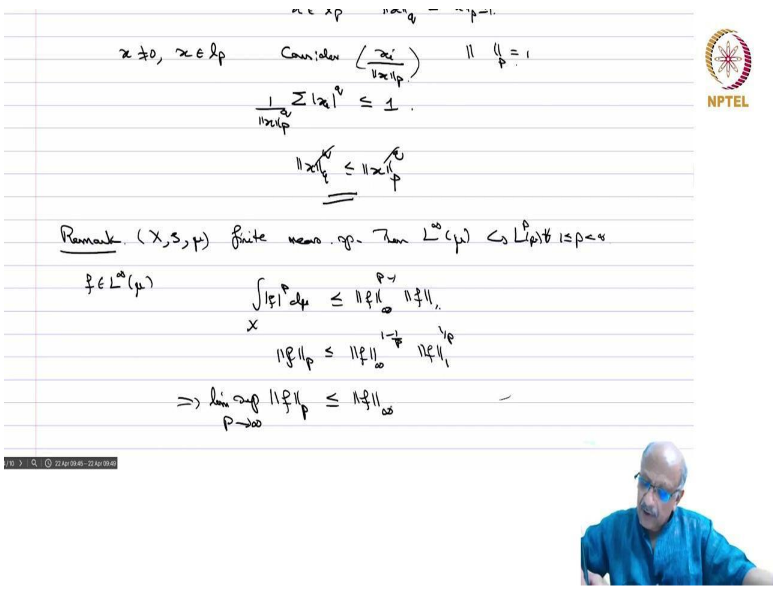

Now, for the general case what you do is, if x of course is not 0 then x is in lp then you have consider xi by norm x power p, than norm for this is equal to 1 and therefore, you have 1 by norm x power p into sigma power q into mod xi power q is less than equal to 1. So, you have norm x q power q less than equal to norm x p power q and therefore, the q can cancel and you have the result. So, you have this is also true.

Remark: Let (X, S, μ) be a finite measure space then $L^\infty(\mu)$ is contained in $L^p(\mu)$ for all $1 \leq p < \infty$. Now, $\int_X |f|^p d\mu \leq ||f||_\infty^{\,p-1} ||f||_1$, $||f||_p \leq ||f||_\infty^{\,1-\frac{1}{p}} ||f||_1^{\,\frac{1}{p}}$.

$$\Rightarrow \ \lim\ \sup_{p\to\infty} ||f||_p \leq ||f||_\infty.$$

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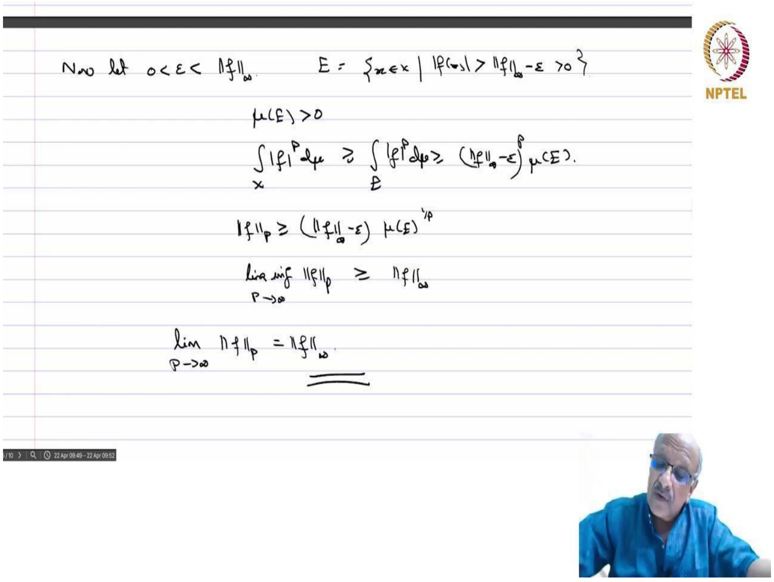

Now, let $0 < \epsilon < ||f||_{\infty}$, $then\ E = \{x \in X: |f(x)| > ||f||_{\infty} - \epsilon > 0\}$. So, if you look at this then mu of E is positive because, if you this is less than the L infinity norm, which is the smallest number such that this set will have measure 0 and therefore, this is the set has positive measure and

$$\int_X |f|^p d\mu \geq \int_E |f|^p d\mu \geq (||f||_{\infty} - \epsilon)^p \mu(E).$$

So, $||f||_p \geq (||f||_{\infty} - \epsilon)\mu(E)^{\frac{1}{p}}$.

Therefore, you have the $\lim_{p \to \infty} \inf ||f||_p \geq ||f||_{\infty}$. Then $\lim_{p \to \infty} ||f||_p = ||f||_{\infty}$.

So, next we want to look at sequences in L p spaces and their convergence and so on, which we will do next time.