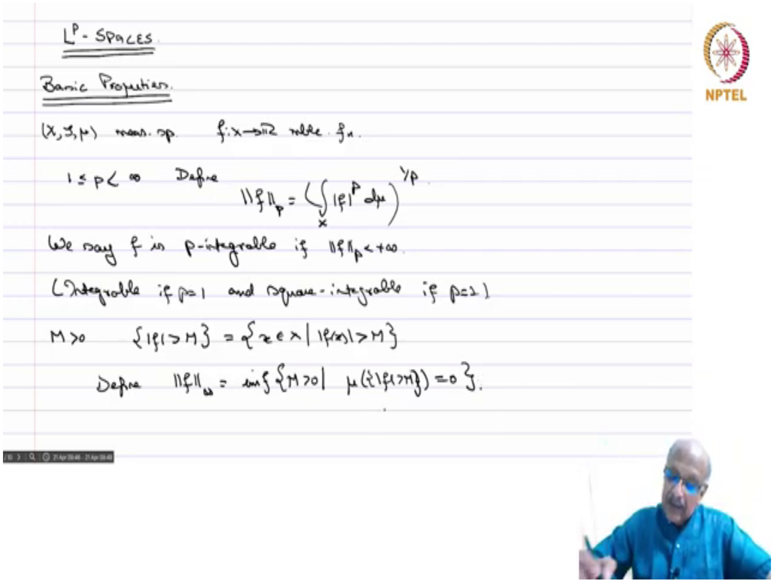
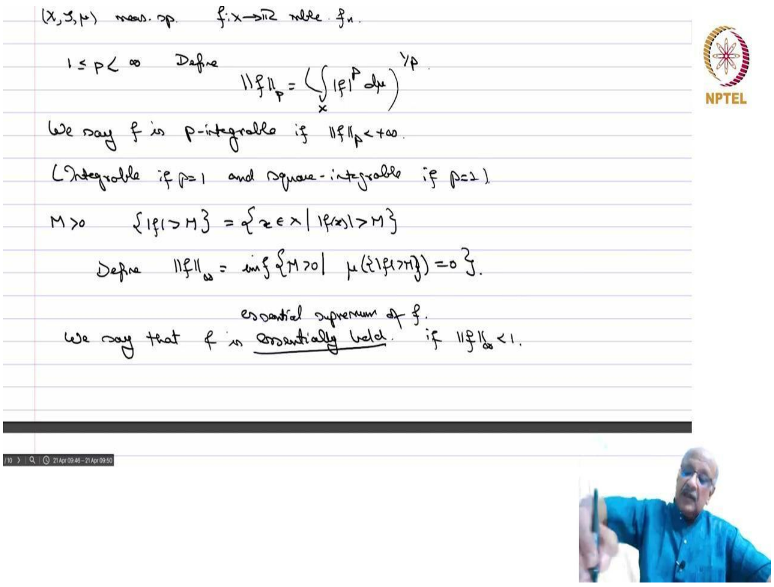
**Measure and Integration**
**Professor S. Kesavan**
**Department of Mathematics**
**The Institute of Mathematical Sciences**
**Lecture No-66**

# Lebesgue Spaces

(Refer Slide Time: 0:17)




So, we now start a new chapter $L^p$ spaces. The $L^p$ Spaces or the Lebesgue spaces are Banach spaces, which are a very rich source of examples and counterexamples in functional analysis, and they also very naturally occur in many applications of mathematics, especially the study of partial differential equations. And so, these spaces are very, very important and they have such interesting properties.

So, first we will look at some basic properties. So, $(X, S, \mu)$ measure space and $f: X \to \mathbb{R}$ measurable function. We are dealing with real valued functions with many things. Almost everything that I say we carry over to complex valued functions also. So, $1 \le p < \infty$. Define

$$||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}.$$

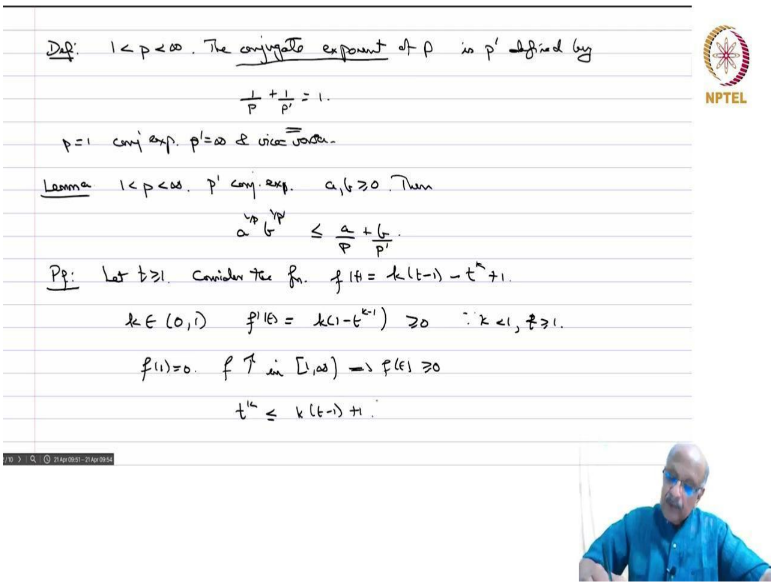
So, we say f is p-integrable if $||f||_p < \infty$. (So, integrable if p equals 1, so, when p equals 1 is just a notion of integrability and square integrable if p equals to 2, otherwise is called the p integral).

Then let $M > 0$ and we said $\{|f| > M\} = \{x \in X: |f(x)| > M\}$. Now, define

$$||f||_\infty = \inf \{M > 0: \mu(\{|f| > M\}) = 0\}.$$

So, this is called the essential supremum of f and we say that f is essentially bounded if $||f||_\infty$ is finite.

(Refer Slide Time: 4:36)




**Definition:** $1 < p < \infty$, the conjugate exponent of p is $p'$ defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

So, if p=1, conjugate exponent $p' = \infty$ and vice versa.

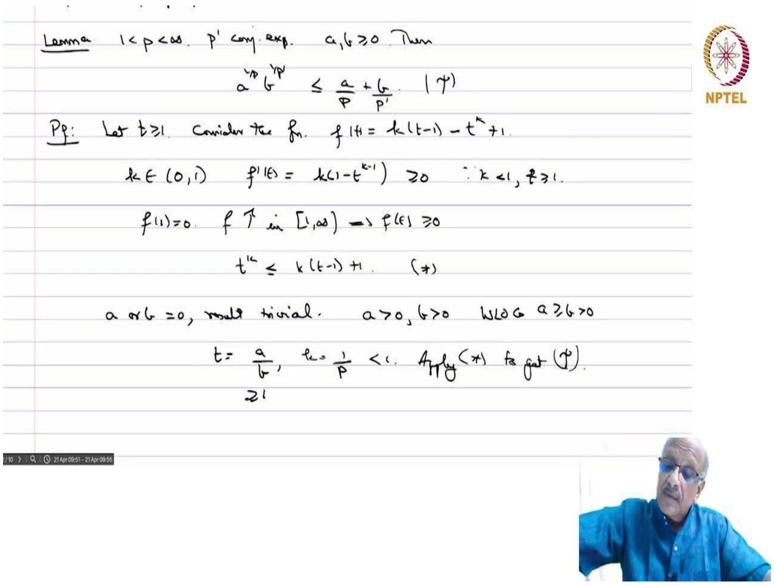
**Lemma:** $1 < p < \infty$, $p'$ conjugate exponent, then a, b greater than or equal to 0 then

$$a^{\frac{1}{p}} b^{\frac{1}{p'}} \leq \frac{a}{p} + \frac{b}{p'}.$$

proof: So, let $t \geq 1$. Consider the function $f(t) = k(t - 1) - t^k + 1$. And, when k belongs to 0, 1. So, k is between 1.

So, what is f dash t, f dash t is equal to k into 1 minus t power k minus 1 and this is greater than or equal to 0. Since k is less than 1 and t is greater or equal to 1 and therefore, since f of 0 equal to 0, sorry f of 1 and f is increasing in one infinity and consequently you have the t power k is less than equal. So, this function has to be always non-negative f of t so, in place f t is greater than equal to 0. So, t power k is less than equal to k times t minus 1 plus 1.

(Refer Slide Time: 8:08)

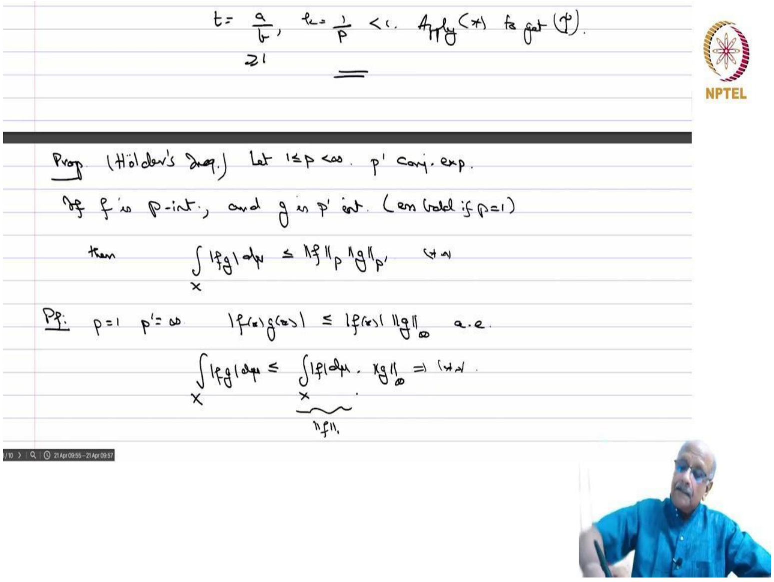



Now, if a or b equal to 0 results are trivial. So, you can assume $a > 0$, $b > 0$. So, without loss of generality you can assume that $a \geq b > 0$. So, you take

$$t = \frac{a}{b} \geq 1, \; k = \frac{1}{p} < 1$$

and then apply (*). So, (*) this inequality and you will get to get that.

(Refer Slide Time: 9:10)



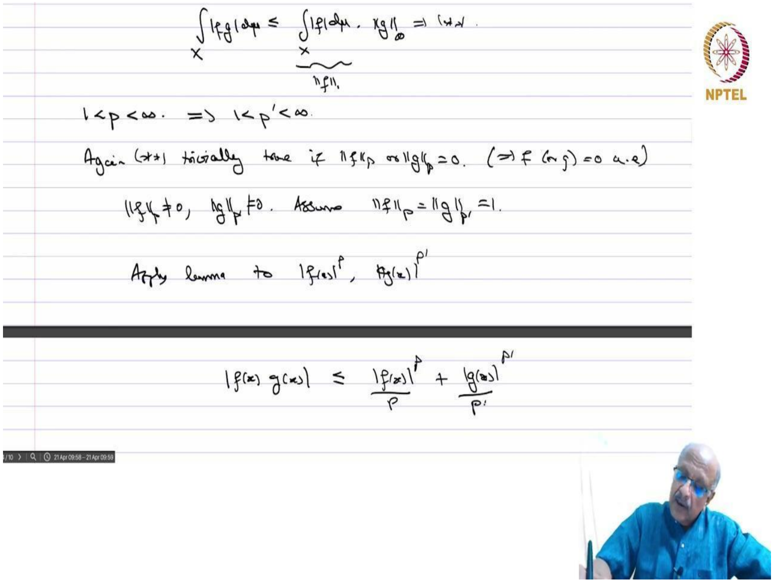

**Proposition.** (Holder's inequality). Let $1 \leq p < \infty$, and is $p'$ conjugate exponent. If f is p integrable and g is $p'$ integrable (essentially bounded if p = 1), then

$$\int_X |fg|d\mu \leq ||f||_p ||g||_{p'}. \qquad (**)$$

proof: so, let us take $p = 1,\ p' = \infty$. Then $|f(x)g(x)| \leq |f(x)| ||g||_\infty\ a.e.$

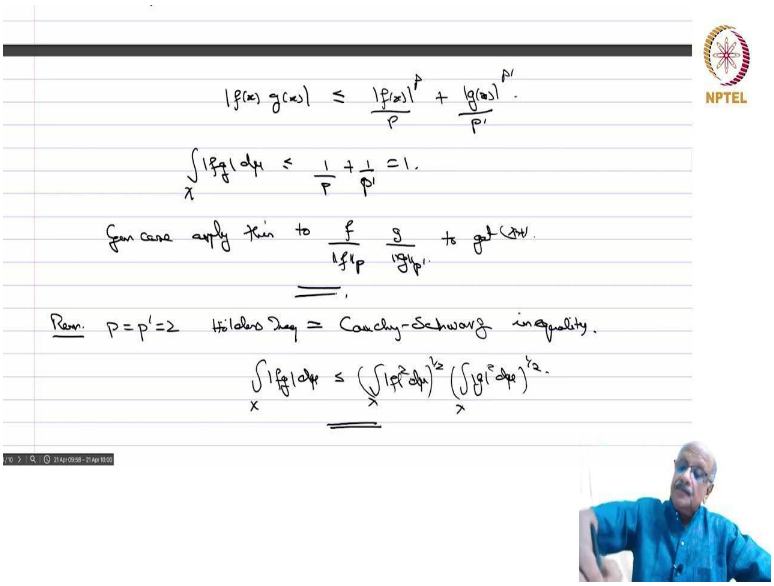
So, now, if you integrate, so, $\int_X |fg|d\mu \leq \int_X |f|d\mu\ .\ ||g||_\infty \Rightarrow$ (**).

(Refer Slide Time: 11:14)




So, now, we assume that $1 < p < \infty \Rightarrow 1 < p' < \infty$. And then again (**) trivially true if $||f||_p$ $or$ $||g||_p = 0$. Because in that case f of g equal to 0 almost everywhere. So, we can assume that $||f||_p \neq 0$ $or$ $||g||_p \neq 0$. So, assume $||f||_p = ||g||_p = 1$. Then, apply lemma to $|f(x)|^p$, $|g(x)|^{p'}$, then what will you get $|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'}$.

(Refer Slide Time: 12:54)



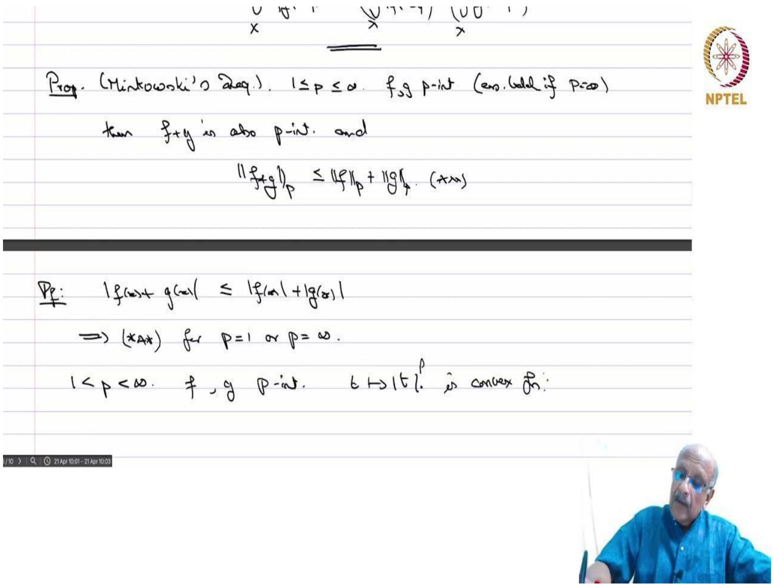

So, then $\int_X |fg| d\mu \leq \frac{1}{p} + \frac{1}{p'} = 1$.

For the general case apply this to $\frac{f}{||f||_p}$, $\frac{g}{||g||_{p'}}$ to get (**).

*Remark:* When $p = p' = 2$, Holder's inequality is the same as the Cauchy Schwarz inequality that is

$$\int_X |fg| d\mu \leq \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}} \left(\int_X |g|^2 d\mu\right)^{\frac{1}{2}}.$$

(Refer Slide Time: 14:28)



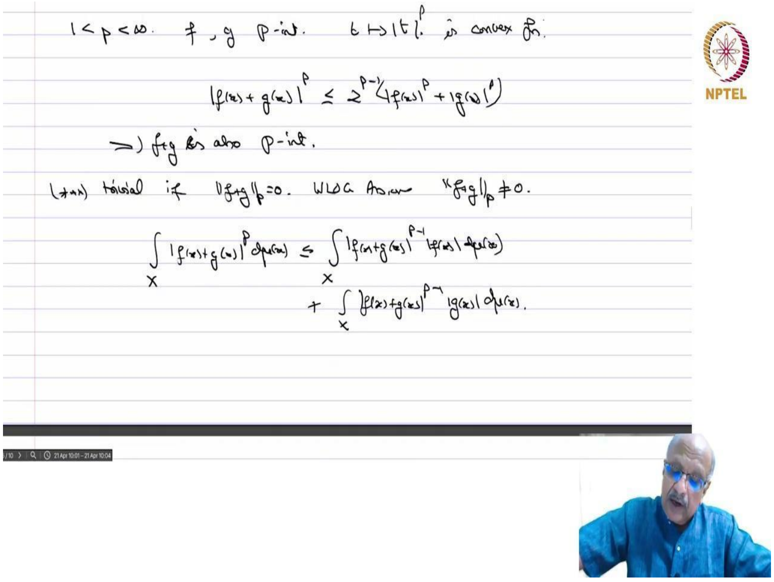

So, then we have the next proposition.

**Proposition:** (Minkoski's inequality) $1 \leq p \leq \infty$ f, g, p integrable essentially bounded if p equals infinity then f+g is also p integrable and

$$||f + g||_p \leq ||f||_p + ||g||_p \qquad (**)$$

proof: so, mod f x plus mod gx plus gx sorry is less than or equal to mod f x plus mod gx. So, this implies that let us call this triple star for p equals 1 or p equals infinity, because that is obvious from this inequality. So, we can assume that 1 less than p less than infinity. Again, the result is trivial if the norm f plus g p equal to 0, so, let us I am jumping a bit, so, let me not say that. So, now f, g p integrable and t going to t power p is a convex function.

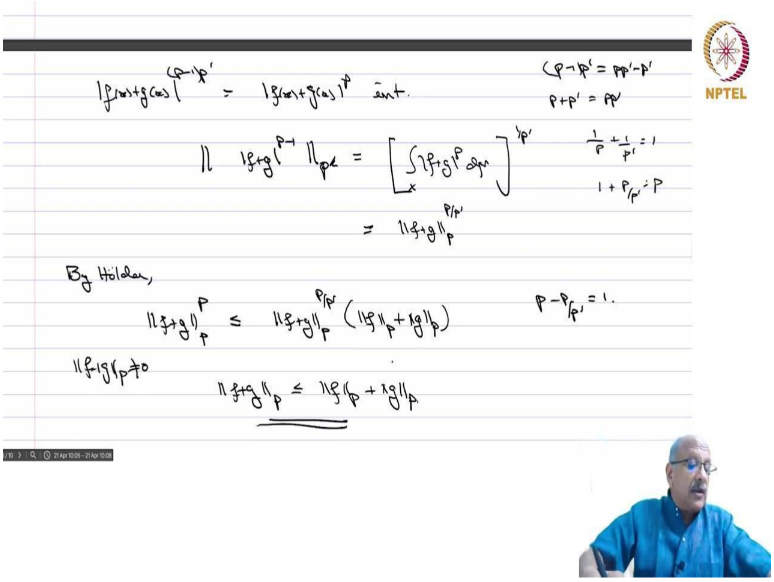
(Refer Slide Time: 16:57)

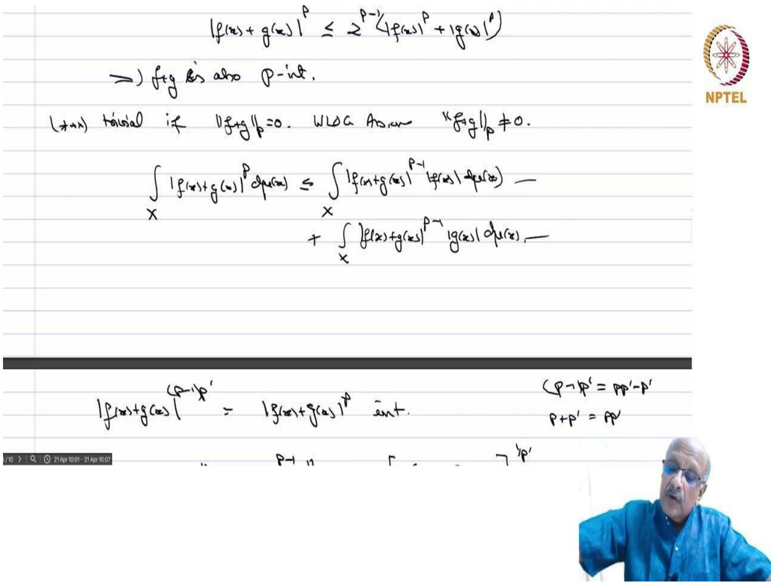



So, by definition of a convex function, you get that mod f x plus gx power p by 2 power p should come is less than equal to 2 power p minus 1 of mod f x power p plus mod gx power p. So, this implies that f plus g is also p integrable. Just integrate both sides you will get this. So, triple star trivial if norm f plus g p is 0. So, without loss of generality assume norm f plus g p is not equal to 0.

So, we now going to write mod f of x plus gx power p d mu x this is less than or equal to integral over x using the triangle inequality f x plus gx power p minus 1 mod f x d mu x plus integral over x mod fx plus gx p minus 1 gx d mu x. So, we want to apply Holder's inequality to each of these terms.

(Refer Slide Time: 18:42)



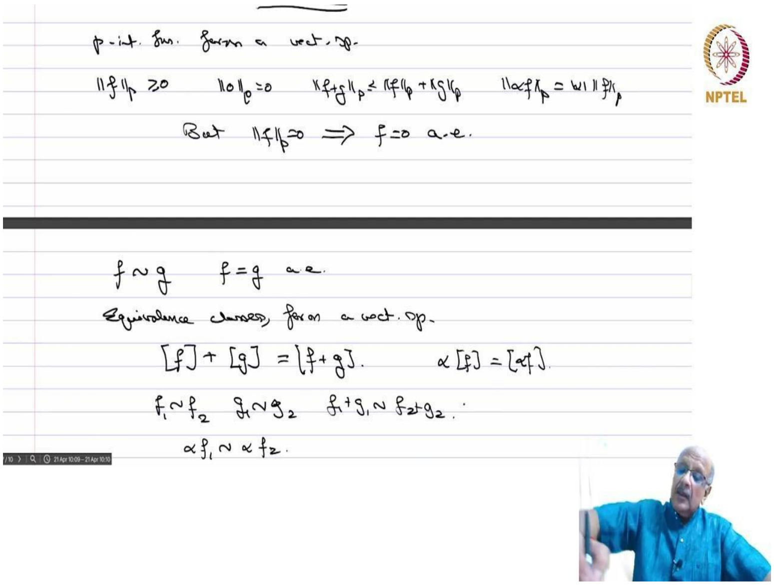

So, we will for f is in p integrable therefore, I can apply here so, I want to know if this the first term is p dash integrable. So, let us take mod f x plus gx power p minus 1. So, into p dash so, we want to know if this function is integrable. But what is this is p minus 1 p dash is nothing but pp dash minus p dash but p plus p dash equals pp dash and therefore, pp dash, means p dash is nothing but p and therefore this is equal to mod fx plus gx power p and that is integrable.

And so, what is norm fx f plus g power, sorry norm of mod f plus g power p minus 1, what is its norm, its p dash norm? So, the p dash norm is nothing but power 1 by p dash into the integral over x of the function here, so, f plus g power p d mu, because that is what this p dash and that is equal to nothing but norm f plus g p over p by p dash.

Because this thing is nothing but the pth power of the norm and therefore, you have that this is f plus g power p by p dash. So, now, having done all this work, now, we apply Holder's inequality to each of the terms here by Holder’s. So, what is the left-hand, side left hand side is f plus g p whole power p. So, norm f plus g p whole power p is less than equal to, now we want p dash norm of this function so norm f plus g p p by p dash into norm fp.

And similarly, the other term would give your normal gp, this will give you norm fp and this would have given you norm gp. Now, you can because f plus g p is not 0 so, you can divide and then p minus p by p dash is nothing but 1. Because 1 by p plus 1 by p dash equal to 1, so, 1 plus p by p dash equal to p. So, p minus p by p dash is nothing but 1 so, you get norm f plus g p is less than equal norm f p plus norm gp. So, this proves the Minkoski’s inequality.

(Refer Slide Time: 22:12)

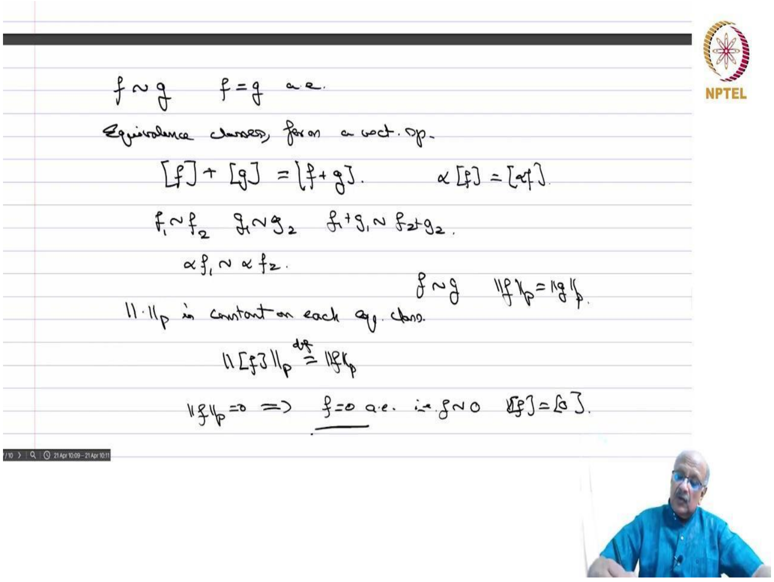



So, now, if you look at so, p integrable functions form a vector space and if you take norm f p is of course, greater than equal to 0 norm 0 p is 0 norm of f plus g p is less than or equal to norm f p plus norm gp. And of course, the norm of alpha f p this distributed check is mod alpha times norm fp.

So, it shows everything which is similar to a norm. But norm f p equal to 0 only implies that f equal to 0, almost every pair it is not f equal to 0. So, this is a problem, so, you cannot make it a norm. So, what we are going to do is to do the usual thing we do in mathematics, namely question the difficulty.

So, we say f is equal to g if f equals g almost everywhere. Now, if you take the equivalence classes from a vector space why? So, if you take f is a representative. So, if you take two equivalence classes say f and g you take a representative listing because if f is f 1 is equivalent to f2, g 1 is equal to g2 then f1 plus g1 is equivalent to f2 plus g2. And alpha f1 and then alpha times f equals alpha f1 is equivalent to alpha f2 and therefore, any representative you take through point (())(24:31) and then take the equivalence class then you will get the. So, this forms a vector space.

(Refer Slide Time: 24:38)

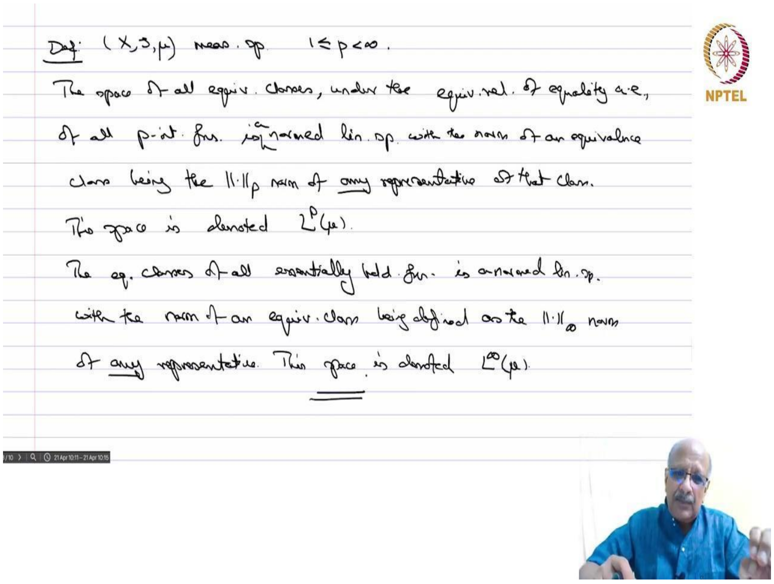


And, $||.||_p$ is constant on each equivalence class because if f and g that is f equivalent to g, then norm fp is the same as norm gp. So, you do not have to worry, you can define the norm as the norm. So, $||[f]||_p = ||f||_p$.

This implies that $||f||_p = 0 \Rightarrow f = 0 \; a.e.,\; i.e.,\; f \sim 0,\; [f] = [0].$

So, using these equivalence classes we will define a nonlinear space.

(Refer Slide Time: 25:38)



**Definition:** (X, S, μ) measure space, $1 \leq p < \infty$. The space of all equivalence classes under the equivalence relation of equality almost everywhere of all p integrable functions is a normed linear space with the norm of an equivalence class being the $||.||_p$ of any representative of that class this space is denoted $L^p(\mu)$. The equivalence class classes of all essentially bounded functions is a normed linear space with the norm of an equivalence class being defined as the $||.||_\infty$ of any representative this space is denoted $L^\infty(\mu)$.

So, we are talking of equivalence classes, but actually we will be working with a representative of every equivalence class. So, whenever we will say $L^p$ function, but what we are really talking of is not a function but an equivalence class of functions, but all the functions in the same equivalence class are equal to each other almost everywhere. So, any computation you do, especially integration related computations, it does not matter which representative you take because if two functions are equal almost everywhere, then the integrals are all preserved.

And therefore, we will remember that we are talking of equivalence classes, but we will not make a fuss we will just say function in $L^p$ it means a function which is an equivalence class of a p integrable function or essentially bounded function. So, this is the notion. So, now that we have defined the L p spaces, next time we will take up some of its properties.