Measure and Integration
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Lecture No-65
Exercises

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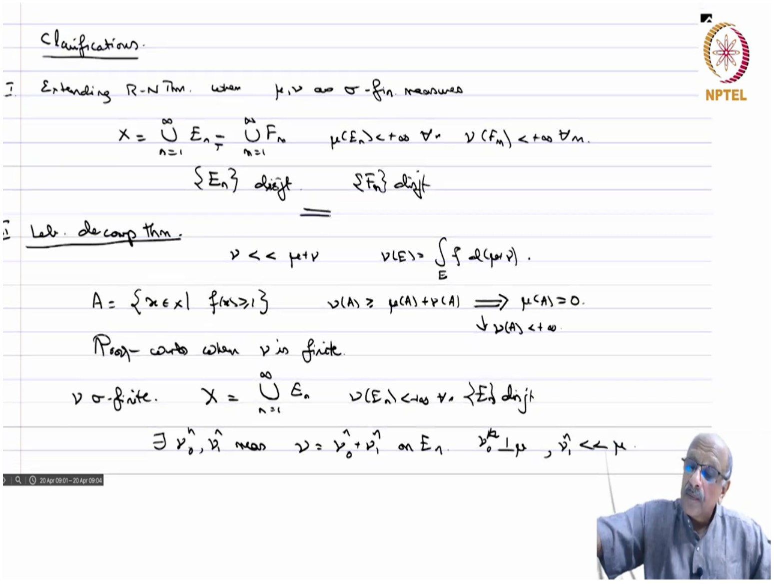

Before starting the exercises, I want two clarifications about the previous lecture. So when extending the Radon-Nikodym theorem when μ, ν are σ-finite measures, then I said that X can be written as $X = \cup_{n=1}^{\infty} E_n = \cup_{m=1}^{\infty} F_n$, $\mu(F_n) < \infty \, \forall \, n$, $\nu(F_m) < \infty \, \forall \, m$, $\{E_n\}, \{F_m\}$ all disjoint. We split up into disjoint sets and then of course, we took En intersection Fm and then extended it to the, from the finite case to the sigma-finite case. So this is the first clarification.

The second 1 is about the Lebesgue decomposition theorem. This is more serious. So what we did was we said that $\nu << \nu + \mu$ and we then saw $\nu(E) = \int_E f d(\nu + \mu)$.

And then we took this at $A = \{x \in X : f(x) \geq 1\}$, $\nu(A) \geq \nu(A) + \mu(A) \Rightarrow \mu(A) = 0$.

But this only because when nu(A) is finite.

So it works, so this about the proof works when ν is finite because I have been harping on this a long time, when you subtract and cancel things on either side, then they just you can only do it if it is a finite quantity.

So how do you do? So assume now nu is sigma-finite and then of course, we rewrite $X = \cup_{n=1}^{\infty} E_n$, $\nu(E_n) < \infty \, \forall \, n$, $\{E_n\}$ disjoint.

So there exists ν_0^n, ν_1^n, $\nu = \nu_0^n + \nu_1^n$ *on* E_n, and $\nu_0^n \perp \mu$, $\nu_1^n << \mu$.

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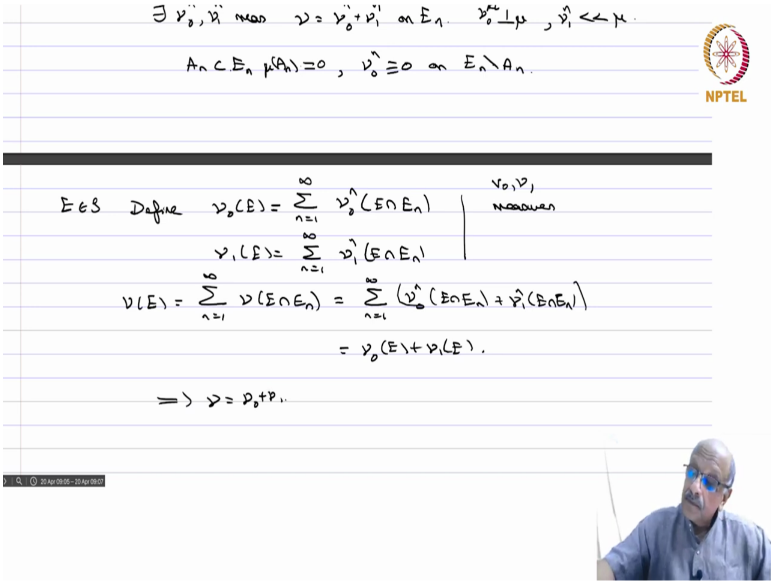

And then you take $A_n \subset E_n$, $\mu(A_n) = 0$, $\nu_0^n \equiv 0$ *on* $E_n \backslash A_n$. Let me be more precise En minus An because everything is now defined only on En.

So now, if E is in S, you define $\nu_0(E) = \sum_{n=1}^{\infty} \nu_0^n(E \cap E_n)$, $\nu_1(E) = \sum_{n=1}^{\infty} \nu_1^n(E \cap E_n)$

and $\nu\ (E) = \sum_{n=1}^{\infty} \nu(E \cap E_n) = \sum_{n=1}^{\infty} (\nu_0^n(E \cap E_n) + \nu_1^n(E \cap E_n)) = \nu_0(E) + \nu_1(E)$.

And therefore, we have $\nu = \nu_0 + \nu_1$.

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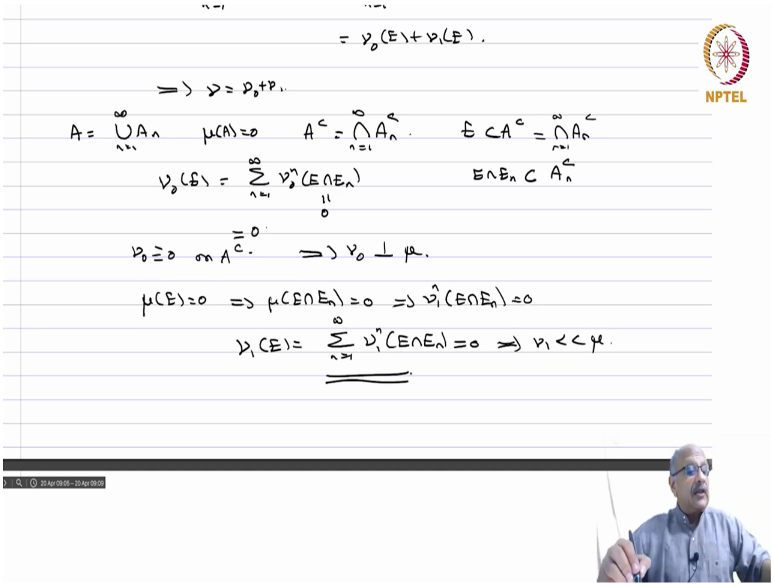

Now you said $A = \cup_{n=1}^{\infty} A_n$, $\mu(A) = 0$, $A^c = \cap_{n=1}^{\infty} A_n^c$, $E \subset A^c = \cap_{n=1}^{\infty} A_n^c$.

Then what is nu naught of E? $\nu_0(E) = \sum_{n=1}^{\infty} \nu_0^n (E \cap E_n)$. And then but what is $E \cap E_n$? E intersection En is E is contained in A complement which is the intersection of all this. So this is contained in An complement because of the definition of this. So this is equal to intersection A_n^c.

So E intersection An complement and therefore each one of them is 0 and therefore this is equal to 0. So $\nu_0 \equiv 0 \; on \; A_n^c \Rightarrow \nu_0 \perp \mu$. The other one is more easy. Of course, if $\mu(E) = 0 \Rightarrow \mu(E \cap E_n) = 0 \Rightarrow \nu_1^n (E \cap E_n) = 0$.

Therefore, you have $\nu_1(E) = \sum_{n=1}^{\infty} \nu_1^n (E \cap E_n) = 0 \Rightarrow \nu_1 << \mu$.

And this completes the proof of the thing in the sigma-finite case. So the proof which I gave in the previous video is only applicable to finite measures when nu is finite but then it is a very simple extension to the sigma-finite case.

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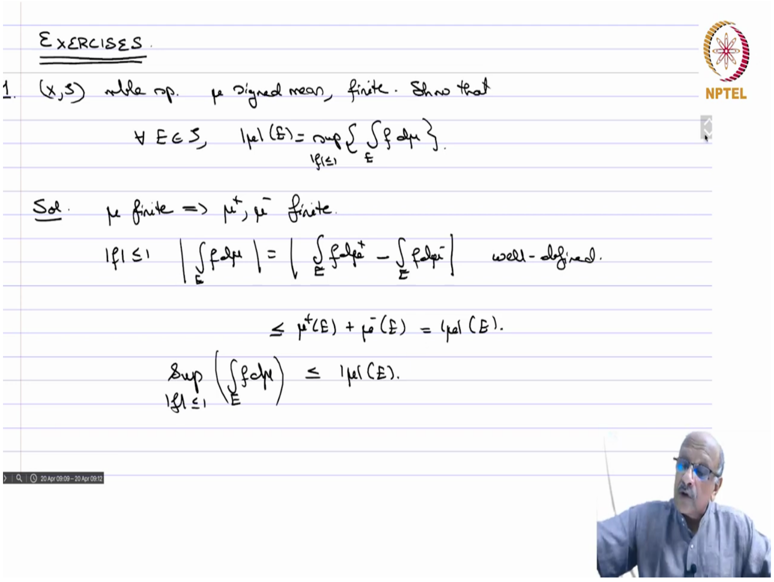

Okay so now let us do some exercises.

(1) : (X, S) measurable space, μ signed measure which is finite. Show that for every E in S, we have $|\mu|(E) = \sup_{|f|\leq 1}\{\int_E f d\mu\}$.

sol: so μ is finite $\Rightarrow \mu^+, \mu^-$ are finite. And therefore, if $|f| \leq 1$, then

$$|\int_E f d\mu| = |\int_E f d\mu^+ - \int_E f d\mu^-| \leq \mu^+(E) - \mu^-(E) = |\mu|(E).$$

Therefore $\sup_{|f|\leq 1}\{\int_E f d\mu\} \leq |\mu|(E)$.

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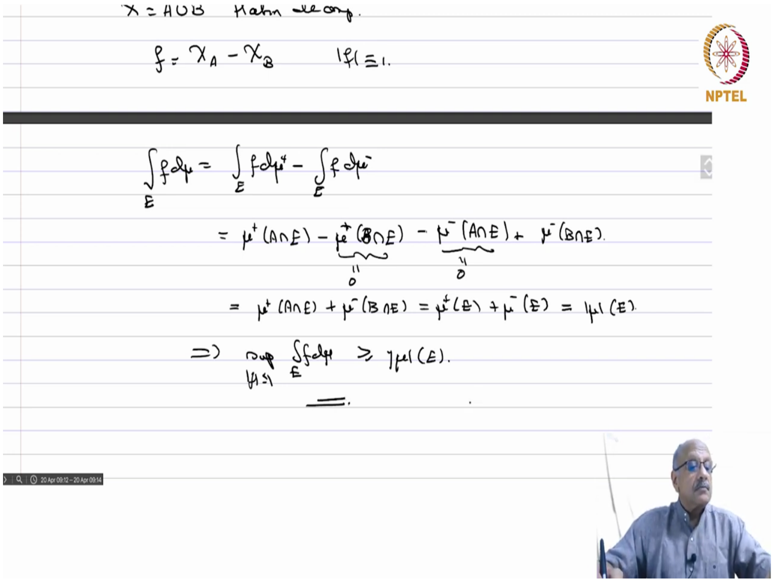

Now we have to show the reverse inequality. So you take $X = A \cup B$, Hahn decomposition and you take $f = \chi_A - \chi_B$. So $|f| \equiv 1$ and if you take

$$\int_E f d\mu = \int_E f d\mu^+ - \int_E f d\mu^- = \mu^+(A \cap E) - \mu^+(B \cap E) - \mu^-(A \cap E) + \mu^-(B \cap E)$$

$$= \mu^+(A \cap E) + \mu^-(B \cap E) = \mu^+(E) + \mu^-(E) = |\mu|(E).$$

This says that $\sup_{|f| \leq 1}\{\int_E f d\mu\} \geq |\mu|(E)$.

And so we have both the inequalities and consequent and hence the result.

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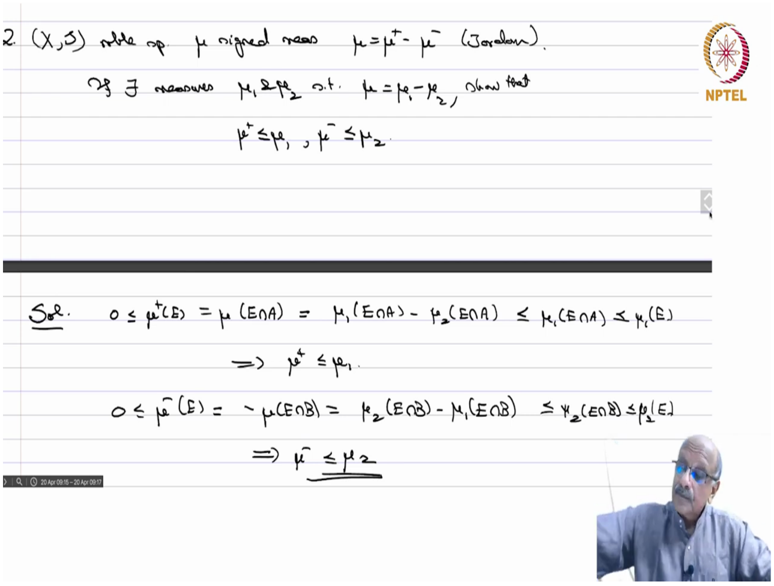

(2). (X, S) measurable space, μ signed measure and $\mu = \mu^{+} - \mu^{-}$. It is the Jordan decomposition. If there exists measures μ_1, μ_2 such that $\mu = \mu_1 - \mu_2$, show that

$$\mu^{+} \leq \mu, \ \mu^{-} \leq \mu_2.$$

solution. $0 \leq \mu^{+}(E) = \mu(E \cap A) = \mu_1(E \cap A) - \mu_2(E \cap A) \leq \mu_1(E \cap A) \leq \mu_1(E).$

$$\Rightarrow \mu^{+} \leq \mu_1.$$

Similarly,

$$0 \leq \mu^{-}(E) =- \mu(E \cap B) = \mu_2(E \cap B) - \mu_1(E \cap B) \leq \mu_2(E \cap B) \leq \mu_2(E).$$

$$\Rightarrow \mu^{-} \leq \mu_2.$$

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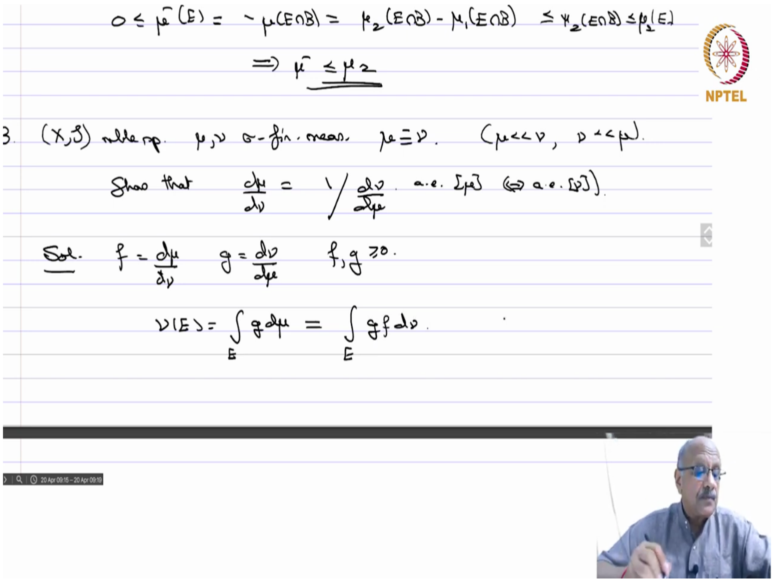

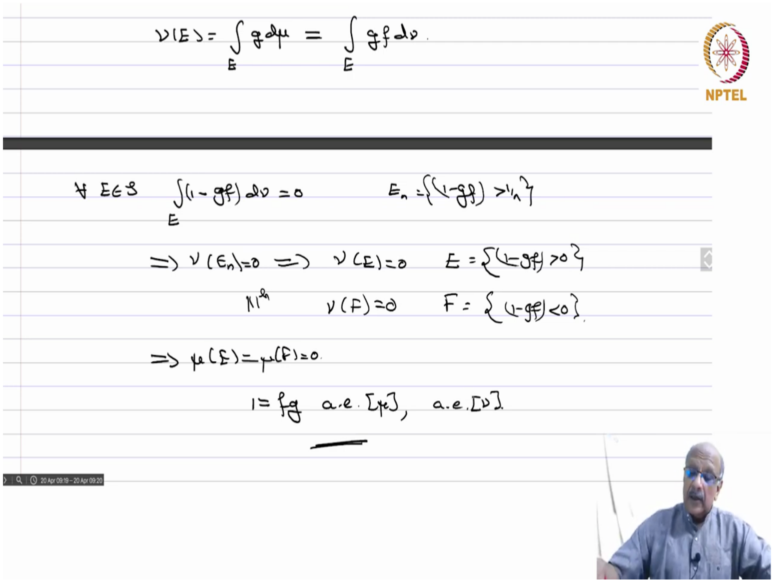

(3). (X, S) measurable space, μ, ν σ-finite measures and $\mu \equiv \nu$ (*i.e.*, $\mu << \nu$, $\nu << \mu$). Show that $\frac{d\mu}{d\nu} = \frac{1}{\frac{d\nu}{d\mu}}$ $a.e.$ $[\mu]$.

solution. So let us say $f = \frac{d\mu}{d\nu}$, $g = \frac{d\nu}{d\mu}$, $f, g \geq 0$. So

$$\nu(E) = \int_E g d\mu = \int_E gf d\mu.$$

Therefore, for every E in S, you have $\int_E (1 - gf)d\mu = 0$. So now, you take $E_n = \{(1 - gf) > \frac{1}{n}\}$. Then this will imply that

$$\nu(E_n) = 0 \Rightarrow \nu(E) = 0, \ where\ E = \{(1 - gf) > 0\}.$$

Similarly, $\nu(F) = 0,\ where\ F = \{(1 - gf) < 0\}$.

This implied that $\mu(E) = \mu(F) = 0$.

Okay, so and so you have that $1 = gf\ \ a.e.\ [\mu],\ \ a.e.\ [\nu]$.

And that is exactly what we wanted to prove. Okay so with this we conclude this chapter. So next time we will start with an important topic namely Lebesgue spaces or L^p spaces.