Measure and Integration

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Lecture N0-64

Radon-Nikodym Theorem

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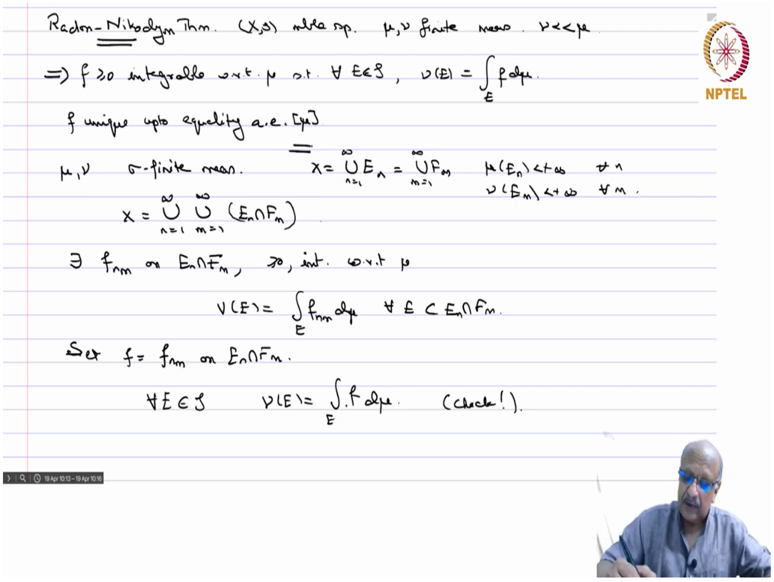

So we proved the Radon-Nikodym theorem. So (X, S) measurable space, μ, ν finite measures, ν << μ, then there exists f- non-negative, integrable with respect to μ such that for every E in S, $\nu(E) = \int_E f d\mu$ and f unique up to equality almost everywhere with respect to μ.

So now we want to extend this to the general cases. So let us assume that μ, ν σ-finite measure, then you have $X = \cup_{n=1}^{\infty} E_n = \cup_{m=1}^{\infty} F_m$, $\mu(E_n) < \infty$, $\nu(F_n) < \infty$. So then $X = \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} (E_n \cap F_m)$. So there exists $f_{n,m}$ on $E_n \cap F_m$ non-negative, integrable with respect to μ such that $\nu(E) = \int_E f_{n,m} d\mu$ for all $E \subset E_n \cap F_m$.

So set $f = f_{n,m}$ on $E_n \cap F_m$. Then for all E in S, we have $\nu(E) = \int_E f d\mu$. Check! This f need not be integrable now and because that is because nu is no longer a finite measure. If it is a finite measure, f has to be integrable, otherwise it need not be integral.

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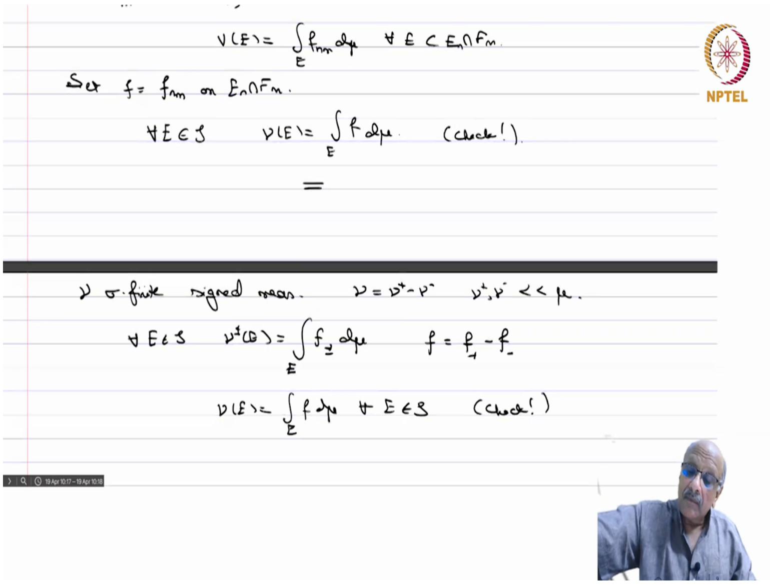

So then next is so if mu and nu are sigma-finite measures, we still have the Radon-Nikodym theorem holding true. So now assume that nu is a sigma-finite, sigma-finite signed measure, then you can write nu equals nu plus minus nu minus and these two are sigma-finite measures. So nu plus, nu minus are also absolutely continuous with respect to mu.

And therefore, for every E in S, you have nu plus of E, nu plus minus of E equals integral f plus minus dm over E. So now you take f equals f sub plus minus f minus. This is not the positive and negative parts but simply the difference of these two functions which we have here. And then you have that nu(E) equals the integral over E f d mu for every E in S. Again check! Again these are all very trivial checkings which you can do.

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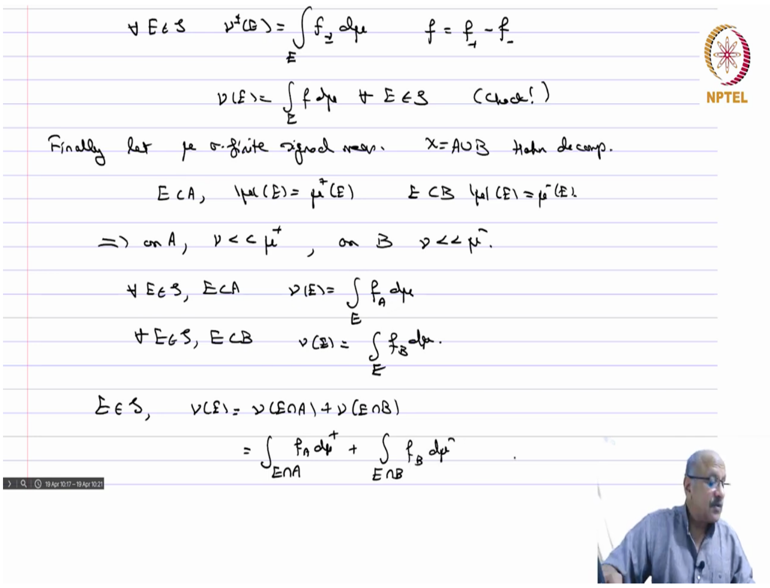

So finally, let μ also be σ-finite signed measure, let $X = A \cup B$, a Hahn decomposition. So if E is contained in A, then $|\mu|(E) = \mu * (E)$, *and if* $E \subset B$, $|\mu|(E) = \mu^{-}(E)$.

$$\Rightarrow on\ A,\ \nu << \mu *;\ on\ B\ \ \nu << \mu^{-}.$$

So then for every E in S, E contained in A, you have that $\nu(E) = \int_E f_A d\mu$. Similarly, for every E in S, E contained in B you have $\nu(E) = \int_E f_B d\mu$. Okay, so now if E belongs to S, then you have $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \int_{E \cap A} f_A d\mu^{+} + \int_{E \cap B} f_A d\mu^{-} = \int_E (f_A \chi_A - f_B \chi_B) d\mu.$

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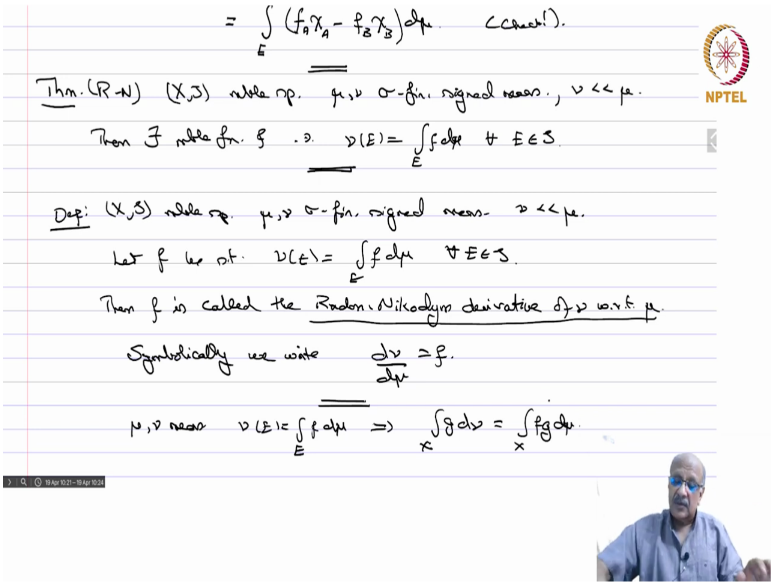

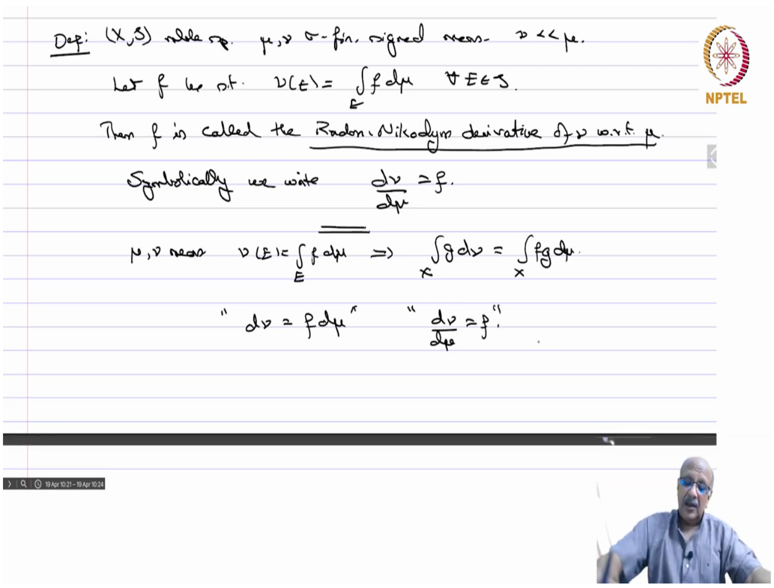

Finally, we have the theorem, Radon-Nikodym in its full generality.

Theorem: (Radon-Nikodym). So (X, S) measure space, μ, ν σ-finite signed measures and $\nu << \mu$, then there exists a measurable function f such that $\nu(E) = \int_E f d\mu \ , \forall E \in S$.

So this question of integrability does not come now because you do not have finiteness of the measures.

Definition: (X, S) measure space, μ, ν σ-finite signed measures and $\nu << \mu$. Let f be such that $\nu(E) = \int_E f d\mu \ , \forall E \in S$. Then f is called the Radon-Nikodym derivative of ν with respect to μ.

And symbolically we write the $\frac{d\nu}{d\mu} = f$. So why this notation? So let us recall that if you had mu, nu measures and nu(E) equal to integral f d mu over E, then this implies that integral g d nu over X is equal to integral over X fg d mu.

We have seen this thing. We proved it first for characteristic functions, then for simple functions, then for some theorem we monotone or dominated, we proved it for the general case.

So this exercise which we or proposition which we have already seen, so symbolically we can write d nu equals f d mu and again symbolically we write d nu by d mu is equal to f. These are all symbolic calculations so do not take them too seriously. Otherwise, it is just notation and therefore, this is the reason why we call it the Radon-Nikodym derivative.

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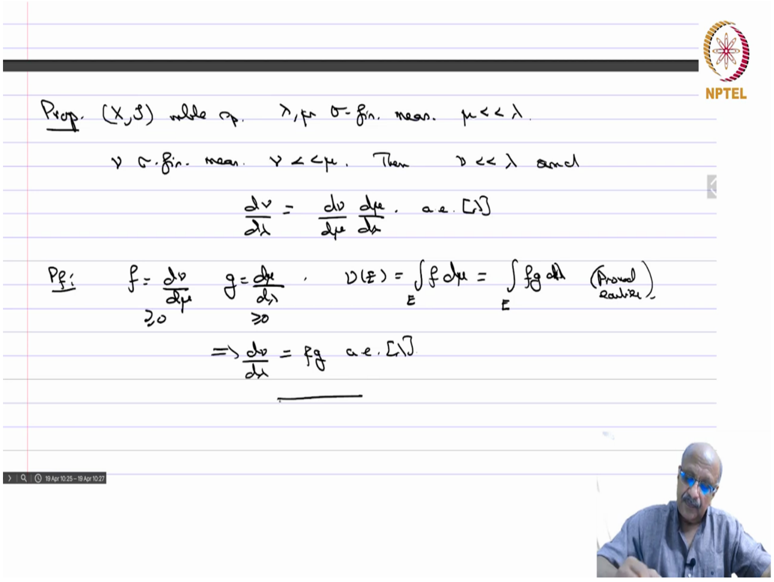

Proposition: (X, S) measurable space, λ, μ σ-finite measures defined on S and μ << λ, then ν σ-finite measure ν << μ, then ν << λ, and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \; a.e. \; [\lambda].$$

proof. So let $f = \frac{d\nu}{d\mu}$, $g = \frac{d\mu}{d\lambda}$. Then $\nu(E) = \int_E f d\mu = \int_E fg d\mu$.

All these are non-negative because we are having measures and then we have seen this. Okay.

So you have this proved earlier. And by the uniqueness with respect to lambda therefore, you have $\frac{d\nu}{d\lambda} fg$ a.e. [λ].

And that proves this proposition.

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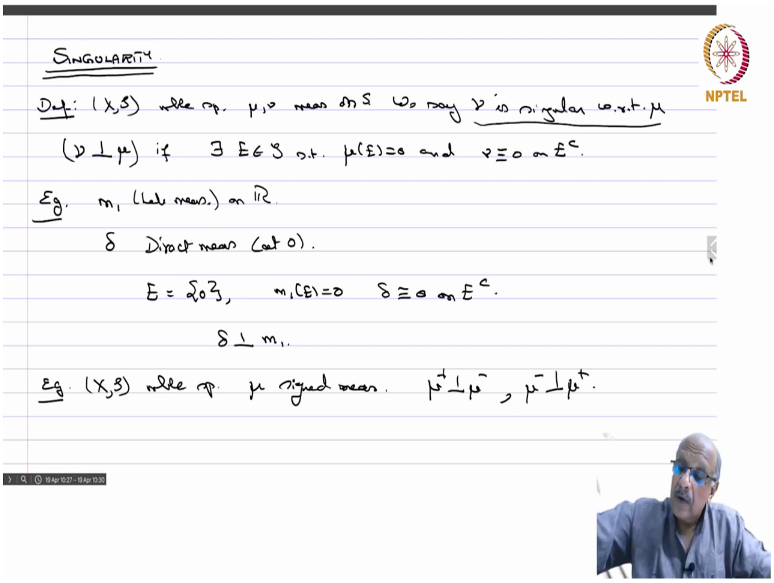

Okay, so now we are going to start a new topic called singularity.

Definition. (X, S) measurable space, μ, ν measures. So, we say ν is singular with respect to μ and the notation is $\nu \perp \mu$, if there exists a E in S such that $\mu(E) = 0$ and $\nu \equiv 0$ $on\ E^c$.

So this is the opposite. So if you have absolute continuity, then mu(E) equals 0 implies nu(E) equals 0 and for all subsets whereas here nu is 0 where mu is not 0, okay, and vice-versa. So this is the notion of singularity.

So, example. So let us take m1 (Lebesgue measure) on R and delta (Dirac measure) concentrated at the origin 0. So now if you take E equal to singleton 0 and then you have a m1 of E equal to 0 and delta is identically 0 on E^c. So the delta is singular with respect to m1.

Example again: so (X, S) measurement space, mu is a signed measure, okay, then $\mu^+ \perp \mu^-$ and $\mu^- \perp \mu^+$ and the corresponding set which you are looking at will be A or B in the Hahn decomposition according to it. Okay.

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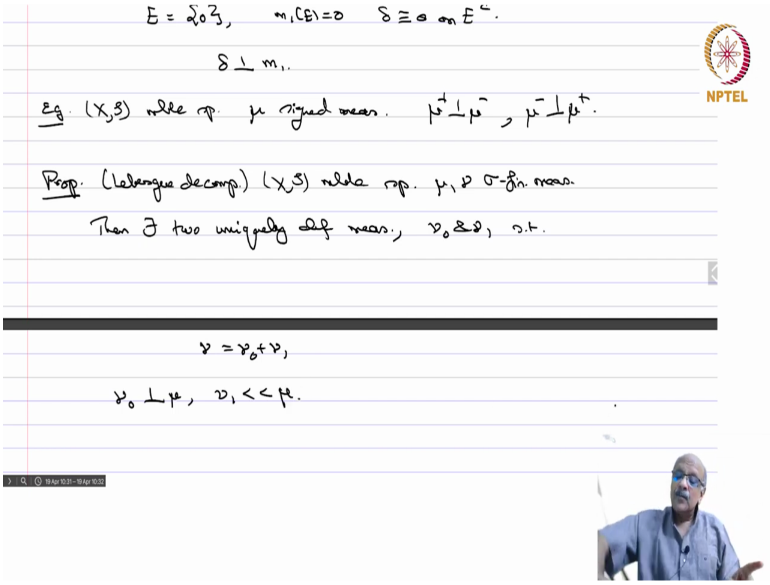

Proposition. This is called the Lebesgue decomposition. (X, S) measurable space, μ, ν σ -finite measures. Then there exists two uniquely defined measures ν_0 and ν_1 such that

$$\nu = \nu_0 + \nu_1, \ \nu_0 \perp \mu, \ \nu_1 << \mu.$$

So I can decompose nu into two measures- nu naught and nu1and nu naught will be singular with respect to mu and nu1 will be absolutely continuous with respect to mu. So every measure can be decomposed into two parts, one which is singular with respect to another given measure and the other one will be absolutely continuous with respect to the same measure.

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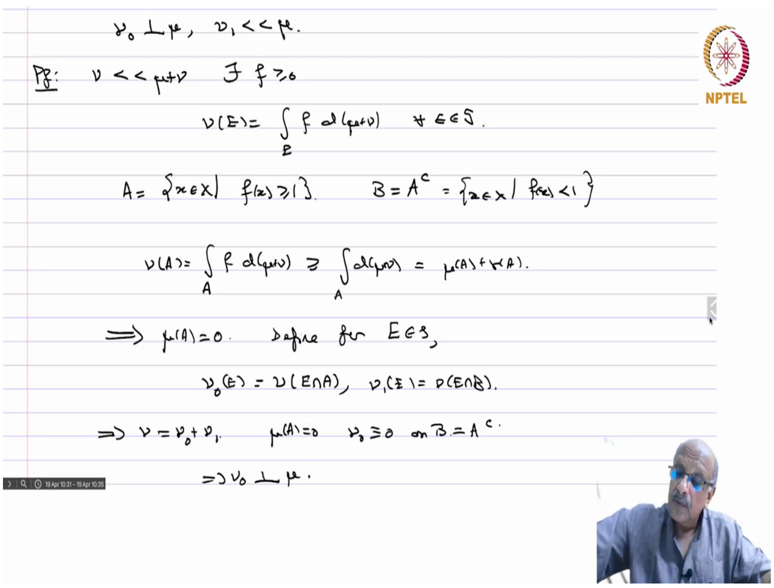

proof. We have $\nu << \mu + \nu$. This is obvious and therefore, there exists f (non-negative) such that $\nu(E) = \int_E f d\mu(\mu + \nu)$, $\forall E \in S$. So you take $A = \{x \in X: f(x) \geq 1\}$. And let $B = A^c = \{x \in X: f(x) < 1\}$. And f is of course non-negative; we already know. Okay.

So $\nu(A) = \int_A f d(\mu + \nu) \geq \int_A d(\mu + \nu) = \mu(A) + \nu(A)$.

$$\Rightarrow \mu(A) = 0.$$

So define for E in S, $\nu_0(E) = \nu(E \cap A)$, $\nu_1(E) = \nu(E \cap B)$.

Then that of course, obviously implies that $\nu = \nu_0 + \nu_1$, $\mu(A) = 0, \nu_0 \equiv 0\ on\ B = A^c$.

This means that $\nu_0 \perp \mu$.

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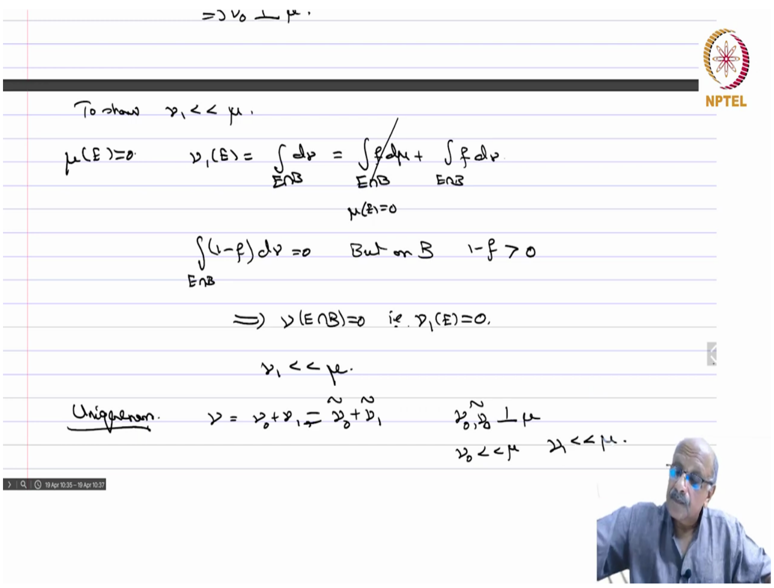

So, to show nu1 is absolutely continuous with respect to mu. So let mu(E) equal to 0, then nu1 of E equals integral E intersection B d nu which is equal to integral f d mu over E intersection B plus integral E intersection B f d nu.

But mu(E) is equal to 0. So this term will disappear and therefore, you have that integral E intersection B of 1 minus f d nu equal to 0. But on B, 1 minus f is strictly positive and therefore, this implies that nu of E intersection B equal to 0 that is nu1 of E equal to 0.

So mu(E) equals 0 implies nu(E), nu1 of is 0 and therefore, you have nu1 is absolutely continuous with respect to mu.

Uniqueness. So let us take $\nu = \nu_0 + \nu_1 = \widetilde{\nu}_0 + \widetilde{\nu}_1$, $\nu_0, \widetilde{\nu}_0 \perp \mu, \nu_0 << \mu, \nu_1 << \mu$.

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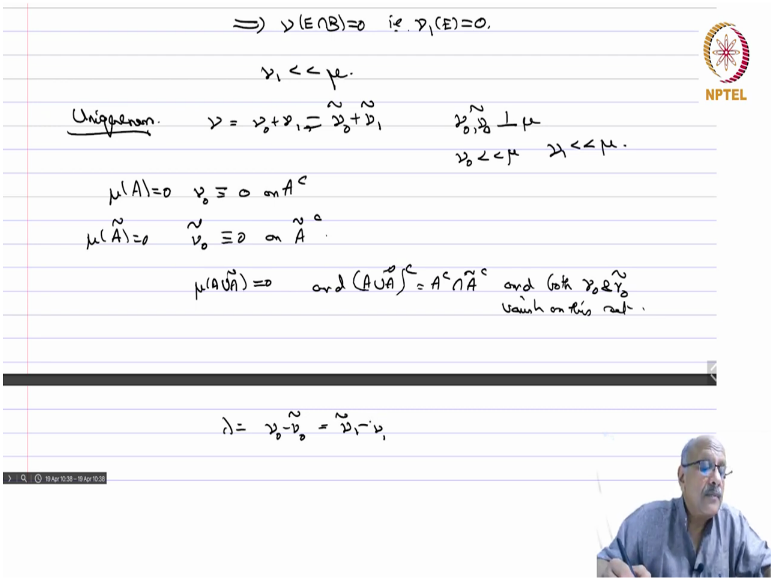

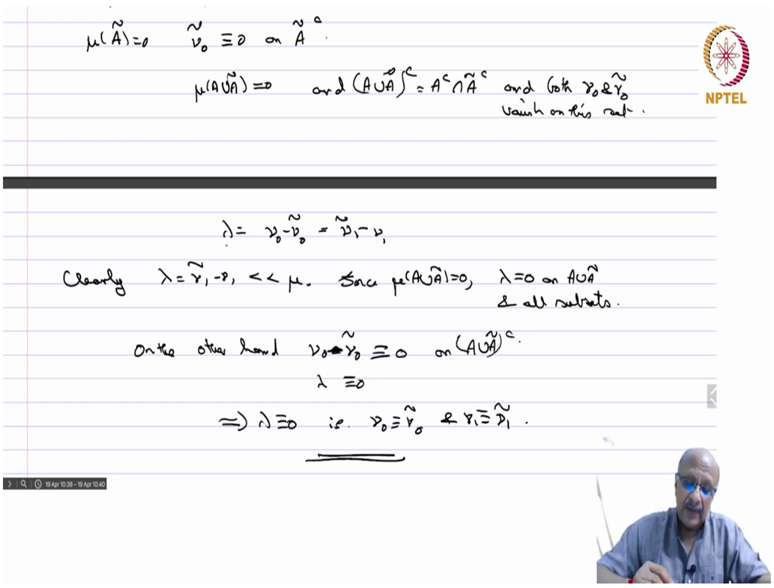

So let A be the set such that $\mu(A) = 0$ and $\nu_0 \equiv 0$ on A compliment and A tilde, $\mu(\widetilde{A}) = 0$; nu naught tilde identically 0 on A tilde complement. So we just applied the definition so far. Okay.

So $\mu(A \cup \widetilde{A}) = 0$ and $(A \cup \widetilde{A})^c = A^c \cap \widetilde{A}^c$.

So you said, $\lambda = \nu_1 - \widetilde{\nu}_1 << \mu$

Since $\mu(A \cup \widetilde{A}) = 0,\ we\ have\ \lambda = 0\ on\ A \cup \widetilde{A}.$

Because of all subsets, mu is a measure. So it is true for all subsets also because of the absolute continuity not because lambda is a measure.

Now on the other hand, we have that $\nu_0 - \widetilde{\nu}_0 = 0\ on\ (A \cup \widetilde{A})^c,\ i.e.\ \lambda \equiv 0.$

That is $\nu_0 = \widetilde{\nu}_0\ and\ \nu_1 = \widetilde{\nu}_1.$

So the decomposition, Lebesgue decomposition, is uniquely defined here. So this means here we have that lambda is both singular and absolutely continuous with respect to mu. That is what we have had and that shows that it has to be truly a zero-measure. So this completes. So we will do some exercises before we conclude this chapter.