Measure and Integration

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Lecture No-63

Radon-Nikodym theorem

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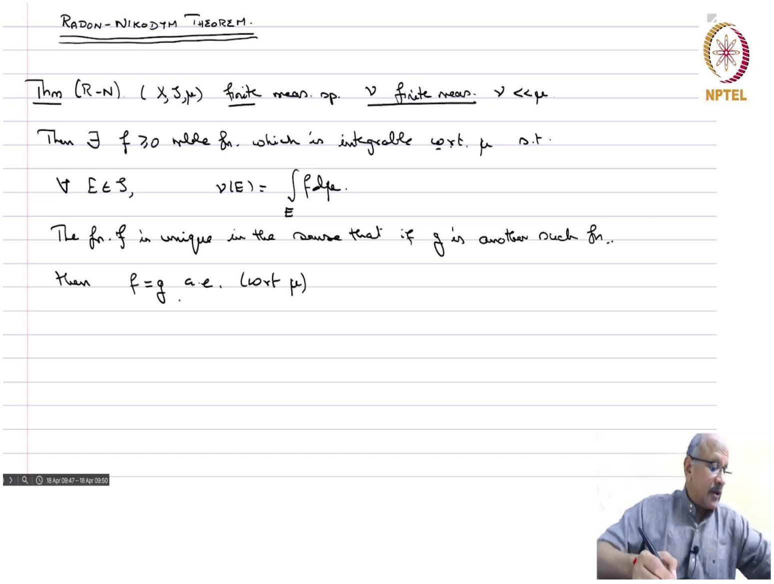

So today we look at the Radon-Nikodym theorem. So this is one of the important theorems in measure theory.

Theorem: (Radon-Nikodym). So (X, S, μ) finite measure space, ν finite measure, $\nu << \mu$. Then there exists f - non-negative measurable function which is integrable with respect to μ such that for every E in S, $\nu(E) = \int_E f d\mu$.

And the function f is unique in the sense that if g is another such function, then f equals g almost everywhere with respect to mu. So there is essentially only one function because you are integrating with respect to mu any other function which has the same property should be equal almost everywhere then the integrals will be the same for every set E.

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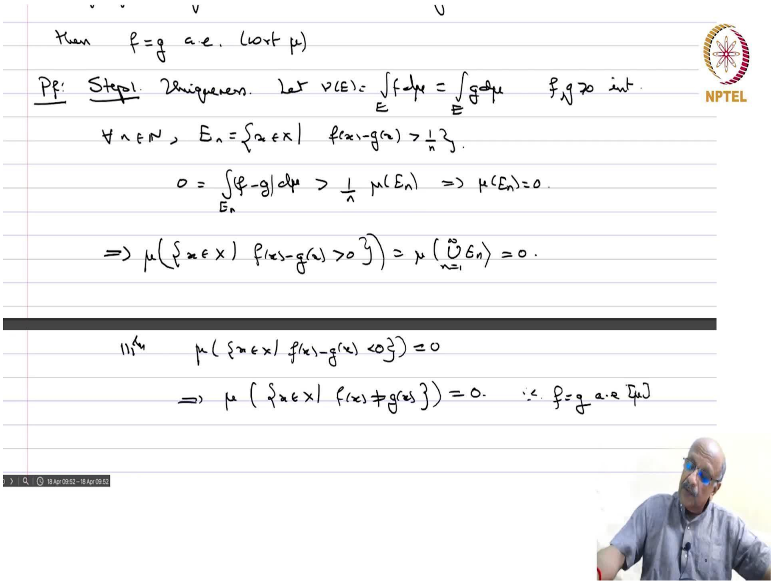

proof: *step 1*: (uniqueness). So let $\nu(E) = \int_E f d\mu = \int_E g d\mu$ Then for every $n \in \mathbb{N}$, let us take $E_n = \{x \in X \colon f(x) - g(x) > \frac{1}{n}\}$.

Then $0 = \int_{E_n} (f - g) d\mu > \frac{1}{n}\mu(E_n) \Rightarrow \mu(E_n) = 0.$

$$\Rightarrow \mu(\{x \in X\colon f(x) - g(x) > 0\}) = \mu(\cup_{n=1}^{\infty} E_n) = 0.$$

Similarly, $\mu(\{x \in X\colon f(x) - g(x) < 0\}) = 0 \Rightarrow \mu(\{x \in X\colon f(x) - g(x) \neq 0\}) = 0.$

That is f=g, almost everywhere with respect to μ. So we have dispensed with the uniqueness. So now we want to find this function. So the first idea is to find the candidate.

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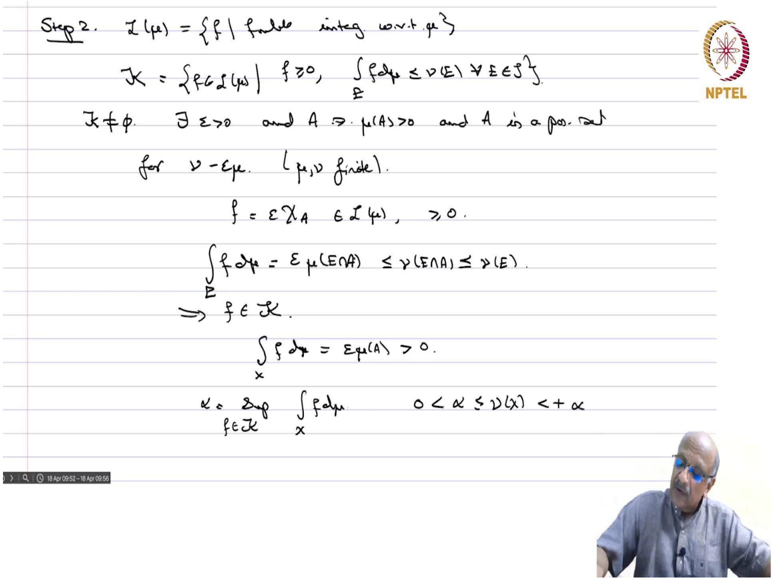

step 2. So $L(\mu) = \{f:\ f\ measurable\ and\ integrable\ with\ respect\ to\ mu\}$.

$$K = \{f \in L(\mu):\ f \geq 0,\ \int_E f d\mu \leq \nu(E)\ \forall E \in S\}.$$

So first of all, K is non-empty. So why is that non-empty? To see this, there exists an epsilon positive and A such that mu A is positive and A is a positive set for nu minus epsilon mu. This was one of the last propositions which we proved, because everything is now finite and therefore, this theorem can be applied. So mu, nu are all finite and therefore, we can apply this one. And therefore, now you put $f = \epsilon\chi_A$. So then $f \in L(\mu)$ and then it is also non-negative. It is integrable because the measure of A is finite and therefore, it is integrable.

Now, $\int_E f d\mu = \epsilon\mu(E \cap A) \leq \nu(E \cap A) \leq \nu(E) \Rightarrow f \in K$.

So now, $\int_X f d\mu = \epsilon\mu(A) > 0$. So now, you set $\alpha = \sup_{f \in K} \int_X f d\mu,\ 0 < \alpha \leq \nu(X) <+ \infty$.

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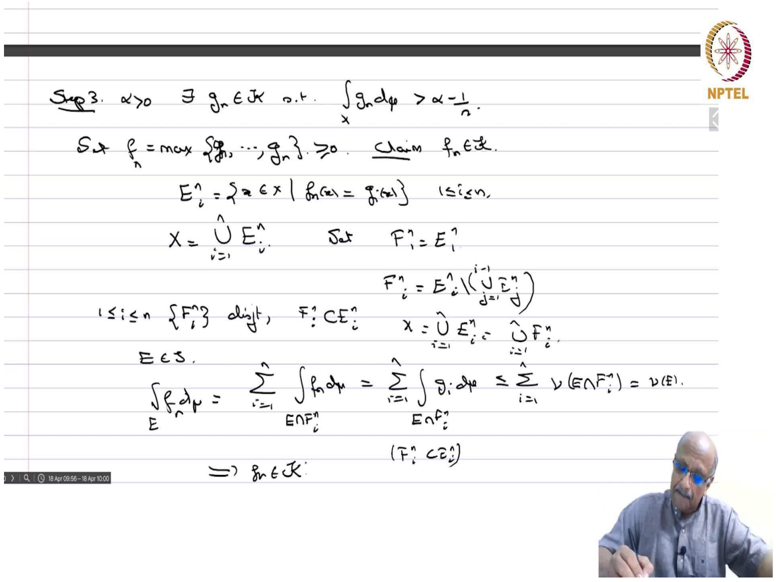

Step 3. So $\alpha > 0$, therefore there exists $g_n \in K$ such that $\int_X g_n d\mu > \alpha - \frac{1}{n}$. So set f equals max f 1, sorry, g 1 to g n, f n. So again this is going to be equal to 0 and claim, f n also belongs to K. So we want to show this. So En i equals the set of all x in X, so set f n x equal to g i x, 1 less than or equal to i less than equal to n. Then capital X is equal to union i equals 1 to n En i, because E, f n must be equal to some g i, it is only a maximum, so it is equal to some g i for every X, f n x is equal to g i x for some I, and therefore, x is equal to this.

So now, you set Fn 1 equals En 1 and Fn i equal to En i minus union g equals 1 to i minus 1 En j. Then, for 1 less than I less than equal to n Fn i are disjoint. Fn i is contained in En i and x equals union i equals 1 to n En i equals union i equals 1 to n Fn i. It is the usual way we write the union as a disjoint union. So if A belongs to S, let us take the integral over E f n d mu.

This is equal to sigma i equals 1 to n integral E intersection Fn find mu, but this is equal sigma i equals 1 to n integral E intersection Fn i on Fn i, Fn i is contained in En i, remember, and on En i f n and g i are the same. So this is g i d mu. So this is less than or equal to sigma i equals 1 to n nu of E intersection Fn i. And that is equal to nu E. And again, so this implies that f n belongs to K.

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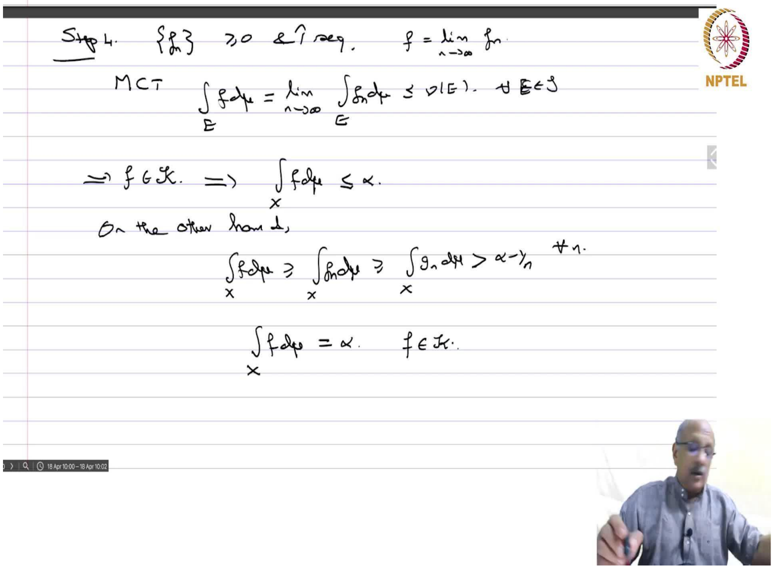

step 4: so f_n is a non-negative and increasing sequence.. So let $f = \lim_{n\to\infty} f_n$. Then by the monotone convergence theorem, you have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \le \nu(E),\ \forall\, E \in S.$$

$$\Rightarrow f \in K \Rightarrow \int_X f d\mu \le \alpha.$$

On the other hand, you have $\int_X f d\mu \ge \int_X f_n d\mu \ge \int_X g_n d\mu > \alpha - \frac{1}{n}$, $\forall\, n$.

And therefore, you have that $\int_X f d\mu = \alpha,\ f \in K$.

So we have found a maximal element.

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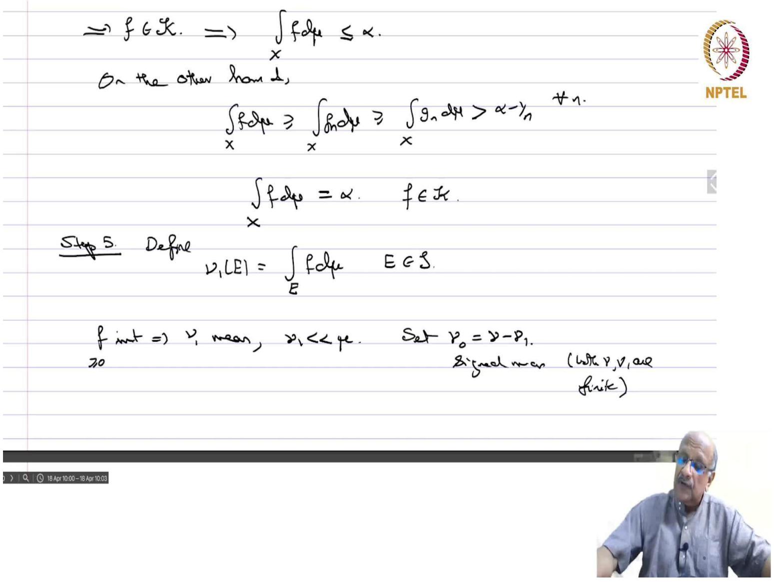

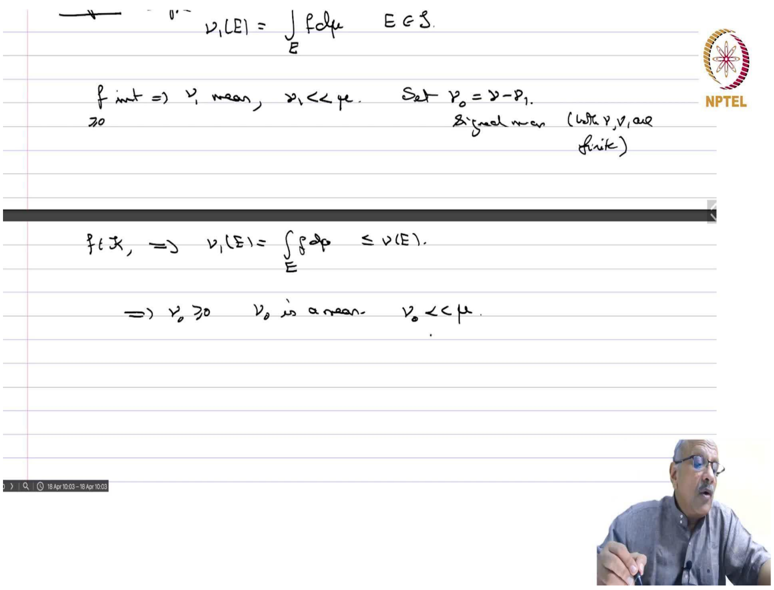

Step 5. So define $\nu_1(E) = \int_E f d\mu$, $E \in S$. So f integrable implies ν_1 measure, and $\nu_1 << \mu$. So set $\nu_0 = \nu - \nu_1$. So this is a signed measure both nu, nu 1 are finite So the difference is well defined, and therefore, it is a signed measure.

So f belongs to K, so this implies that $\nu_1(E) = \int_E f d\mu \leq \nu(E)$.

$\Rightarrow \nu_0 \geq 0$, ν_0 measure, $\nu_0 << \mu$.

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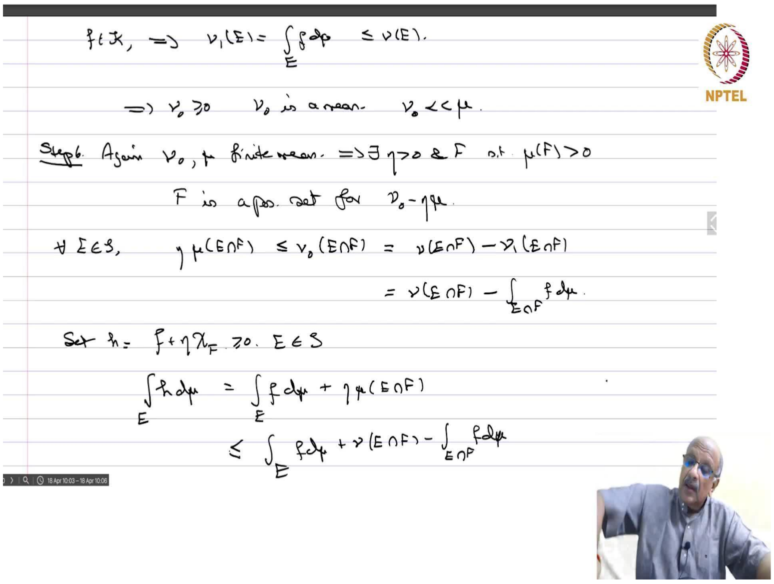

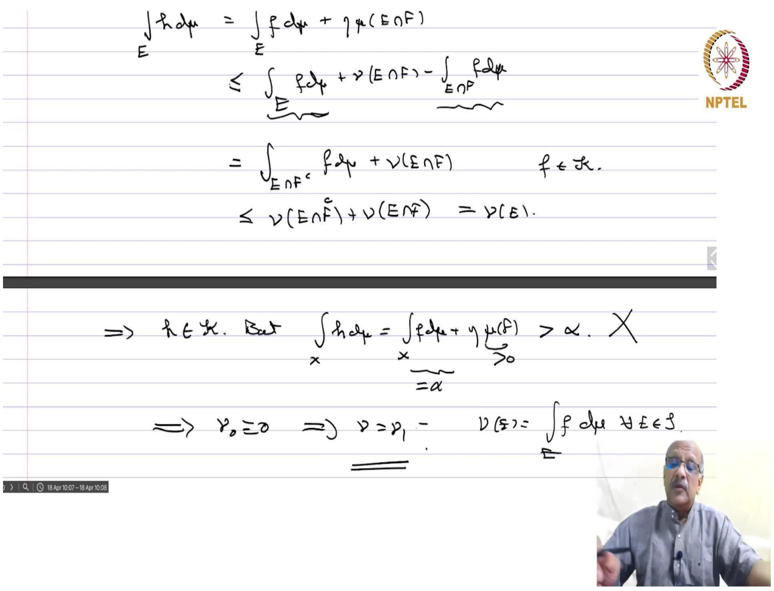

Step 6. Again, ν_0, μ finite measures implies there exists any eta greater than 0 and F such that $\mu(F) > 0$ and F is a positive set for $\nu_0 - \eta\,\mu$. So now, for every E in S you have

$$\eta\mu(E \cap F) \leq \nu_0(E \cap F) = \nu\,(E \cap F) - \nu_1(E \cap F) = \nu\,(E \cap F) - \int_{E\cap F} f d\mu.$$

Set $h = f + \eta\chi_F \geq 0$. So if E belongs to S, then you have

$$\int_E h\,d\mu = \int_E f\,d\mu + \eta\mu(E \cap F) \leq \int_E f\,d\mu + \nu(E \cap F) - \int_{E\cap F} f\,d\mu.$$

$$= \int_{E\cap F^c} f\,d\mu + \nu(E \cap F) \leq \nu(E \cap F^c) + \nu(E \cap F) = \nu(E).$$

So this implies that h is also in K. But,

$$\int_X h\,d\mu = \int_X f\,d\mu + \eta\mu(F) > \alpha,$$

and that is a contradiction because h is in K and alpha is a supremum of all those integrals.

Therefore, this implies that $\nu_0 \equiv 0 \Rightarrow \nu = \nu_1,\ \nu(E) = \int_E f\,d\mu\ ,\ E \in S.$

And that completes our proof of the Radon-Nikodym theorem. So as I said earlier, we will also have, we will now generalize this to other cases, namely when finally, when mu and nu are both sigma finite sign measures. So that is the most general form. So right now, we have proved the theorem for every finite pair of finite measures, with one which is absolutely continuous with respect to the others.