Measure and Integration
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Lecture No- 62
Absolute continuity

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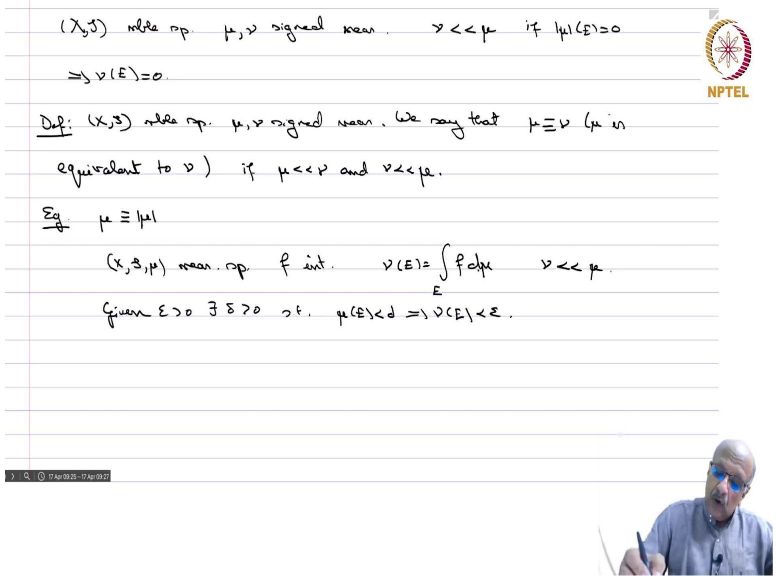

We were looking at absolute continuity. So, (X, S) measurable space and μ, ν signed measures, we say that $\nu << \mu$ if $|\mu|(E) = 0 \Rightarrow \nu(E) = 0$.

And then, so now, we have the following definition:

Definition: (X, S) measurable space and μ, ν signed measures. We say that $\mu \equiv \nu$, (mu is equivalent to nu), if $\nu << \mu$ *and* $\mu << \nu$.

Example: so, you have that $\mu \equiv |\mu|$, because that is trivial because mu is absolutely continuous with respect to mod nu and mod mu is to truly absolutely continuous with respect to mu, from the definition which we have given.

So, now, when we were looking at (X, S, μ) measure space, f integrable and we define

$$\nu(E) = \int_E f d\mu.$$

And so, $\nu << \mu$. Now, we also proved the following that if you have that given $\epsilon > 0$, there exists a delta positive such that $|\mu|(E) < \delta \Rightarrow \nu(E) < \epsilon$.

So, we saw this, and I also called this absolute continuity. So, I have mentioned absolute continuity with respect to this particular measure: nu E equals integral f of f over E dm in two ways. So, we reconcile both these things in the following proposition. Notice that this is a finite measure because f is integrable.

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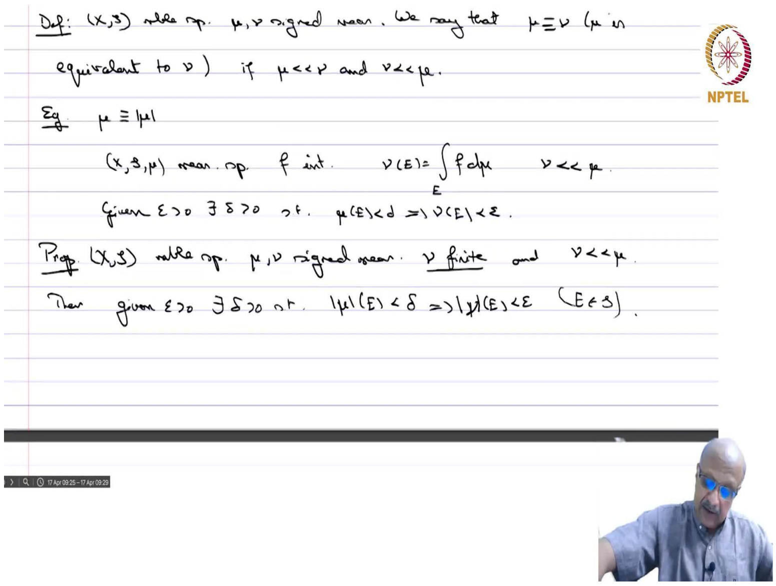

Proposition: (X,S) measurable space μ, ν signed measures, ν is finite and absolutely continuous with respect to μ. Then given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\mu|(E) < \delta \Rightarrow |\nu|(E) < \epsilon, \ (E \in S).$$

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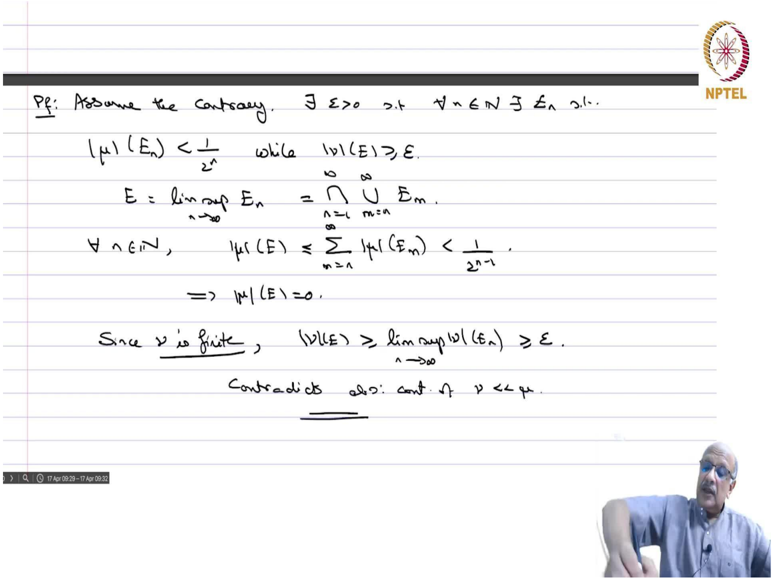

proof: Assume the contrary. That means given any, that means there exists $\epsilon > 0$, s.t. for every n in N there exists E_n such that $|\mu|(E_n) < \frac{1}{2^n}$ while $|\nu|(E) \geq \epsilon$. So, now, you take

$$E = \lim_{n\to\infty} \sup E_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_n.$$

So, now, so, for every n in capital N, you have that $|\mu|(E) \leq \sum_{m=n}^{\infty} |\mu|(E_n) < \frac{1}{2^{n-1}}$.

$$\Rightarrow |\mu|(E) = 0.$$

On the contrary, mu E is lim inf lim sup and we have seen since ν, we have

$$|\nu|(E) \geq \lim_{n\to\infty} \sup |\nu|(E_n) \geq \epsilon.$$

We have a contradiction absolutely, absolute continuity of nu with respect to mu. So that reconciles the two kinds of definitions.

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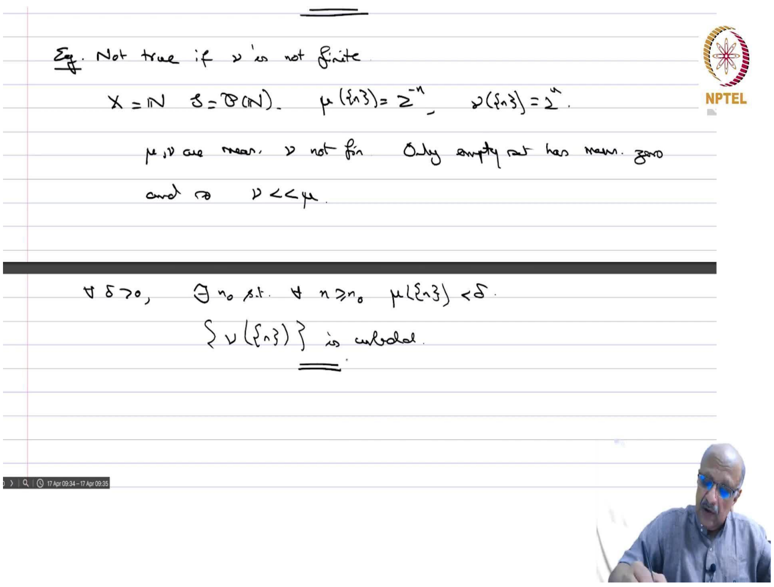

Example: Not true if ν is not finite. So, you take

$$X = \mathbb{N},\ S = P(\mathbb{N}),\ \mu(\{n\}) = 2^{-n},\ \nu(\{n\}) = 2^{n},$$

And from this you can generate a measure because you define it over each atom, each point, and then by countable additivity, you can expand it, find the measure for everything. So, then μ, ν are measures, ν not finite. And so, $\nu << \mu$. There is no doubt about it, because the only set with measure 0 is that. So now, what about for any $\delta > 0$, there exists n_0 such that for all $n \geq n_0$ you have that $\mu(\{n\}) < \delta$. But $\{\nu(\{n\})\}$ is unbounded. And therefore, this, you cannot have the epsilon delta definition which you had earlier.

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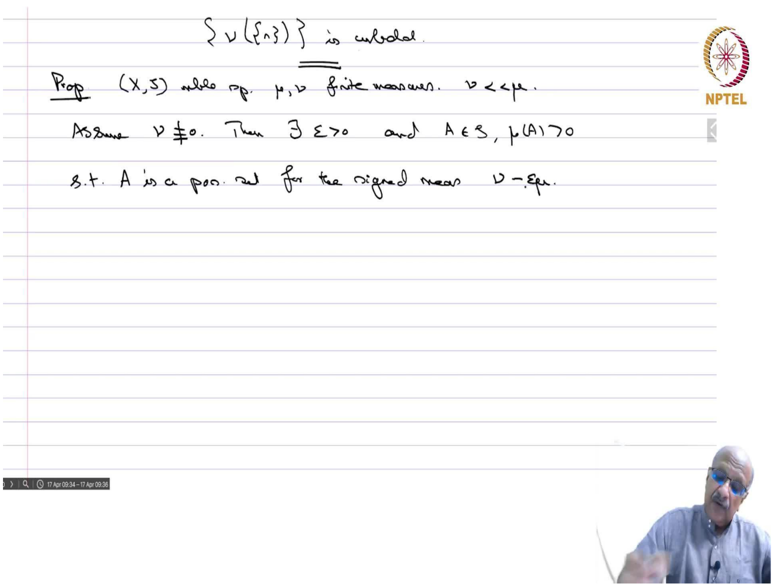

Proposition: (X, S) measurable space, μ, ν finite measures, and $\nu << \mu$. Assume, $\nu \neq 0$. Then there exists an $\epsilon > 0$ and $A \in S,\ \mu(A) > 0$ such that A is a positive set for the signed measure $\nu - \epsilon\mu$.

So, you have two finite measures and then you are taking nu minus epsilon mu and we say that it has a positive set, whose measure is also strictly positive for some epsilon.

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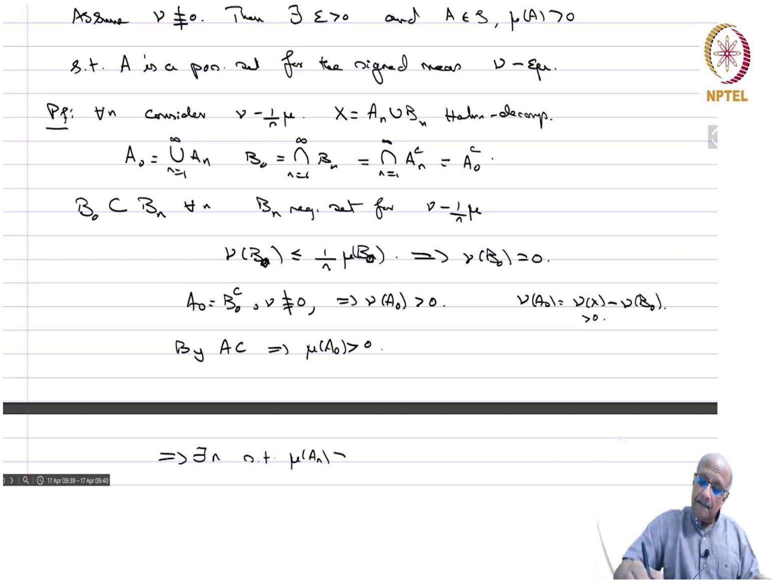

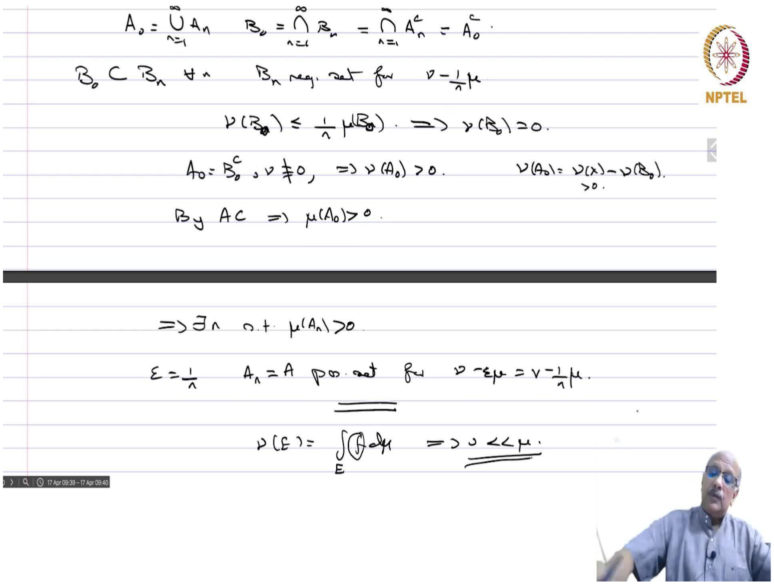

proof, for each n, consider $\nu = \frac{1}{n}\mu$. Then $X = A_n \cup B_n$ Hahn decomposition. So write

$$A_0 = \cup_{n=1}^{\infty} A_n,\ B_0 = \cap_{n=1}^{\infty} B_n = \cap_{n=1}^{\infty} A^c_n = A^c_0.$$

Now, $B_0 \subset B_n$ for all n, because it is the intersection. And B_n is a negative set for $\nu - \frac{1}{n}\mu$. That means $\nu(B_n) \le \frac{1}{n}\mu(B_0) \Rightarrow \nu(B_0) = 0$. So, $A_0 = B_0^{\ c} = \nu \neq 0, \Rightarrow \nu(A_0) > 0$.

$$\Rightarrow \exists\, n\, s.t.\ \mu(A_n) > 0.$$

So, now you take $\epsilon = \frac{1}{n}$, then $A_n = A$, which is a positive set for $\nu - \epsilon\mu = \nu - \frac{1}{n}\mu$. And therefore, that proves the proposition.

So, we have done all the preliminary work necessary. So, an important theorem which we will now come across is to show that, just as we saw that nu E equals integral f d mu over E, then implies the nu is absolutely continuous with respect to Mu, we will show in the sigma finite case, whenever nu is less than or equal to, absolutely continuous with respect to mu, you can always find the function f such that nu E equals integral E f d mu. So, this is the only way in which absolutely continuous measures occur, and that is the famous Radon-Nikodym theorem, which we will prove next time.