Measure and Integration
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Lecture No-61
Upper, lower and total variations of a signed measure; Absolute continuity

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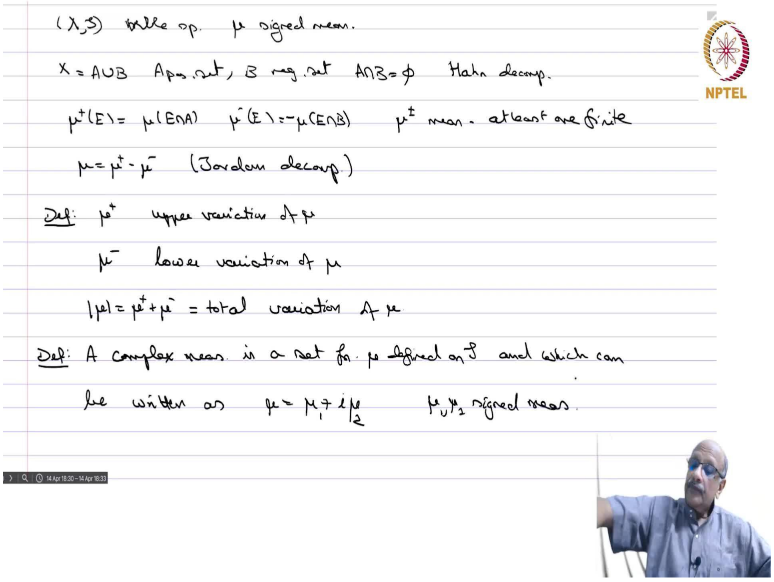

So, we have (X, S) measurable space, μ signed measure, and then we could write X equals A union B, A positive set, B negative set, $A \cap B = \phi$. So, this is called the Hahn Decomposition. Then we define $\mu^{+}(E) = \mu(E \cap A)$, $\mu^{-}(E) = -\mu(E \cap B)$, μ^{+-} measures, at least one finite, and then you have $\mu = \mu^{+} - \mu^{-}$. And this is called the Jordan Decomposition.

So, the Hahn Decomposition may not be unique, but the Jordan decomposition is unique. And we saw examples of these things.

Definition: μ^{+} is called the upper variation of μ, μ^{-} is called the lower variation of μ, and $|\mu| = \mu^{+} + \mu^{-}$ is called the total variation of μ.

Definition: A complex measure is a set function μ defined on S and which can be written as $\mu = \mu_1 + i\mu_2$, μ_1, μ_2 signed measures. So, this is called a complex measure.

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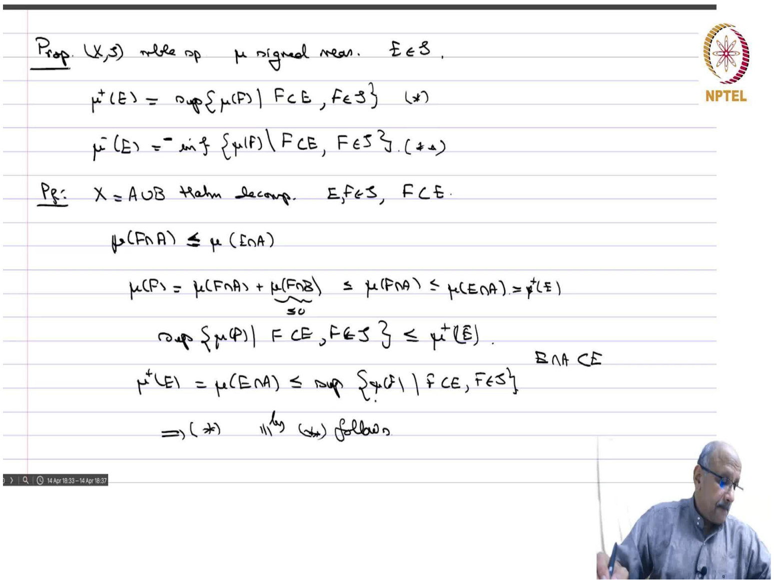

Proposition. X S, measurable space, μ signed measure, E in S. Then

$$\mu^{+}(E) = \sup\{\mu(F):\ F \subset B,\ F \in S\} \qquad (*)$$

$$\mu^{-}(E) =- \inf\{\mu(F):\ F \subset B,\ F \in S\} \qquad (**)$$

proof. Let $X = A \cup B$ - Hahn Decomposition. So, $E, F \in S,\ F \subset E$. So,

$$\mu(F \cap A) \leq \mu(E \cap A).$$

And so, you have $\mu(F) \leq \mu(F \cap A) + \mu(F \cap B) \leq \mu(F \cap A) \leq \mu(E \cap A) = \mu * (E)$.

Therefore, $\sup\{\mu(F):\ F \subset B,\ F \in S\} \leq \mu * (E)$.

sup of mu F, F contained in E, F in S is of course less than equal to mu plus of E. But mu plus of E is mu of E intersection A, and E intersection A is contained in E and therefore, this is less than or equal to sup mu F, F contained in E and F in S. And therefore, this implies star. So, in the same way, a double star follows.

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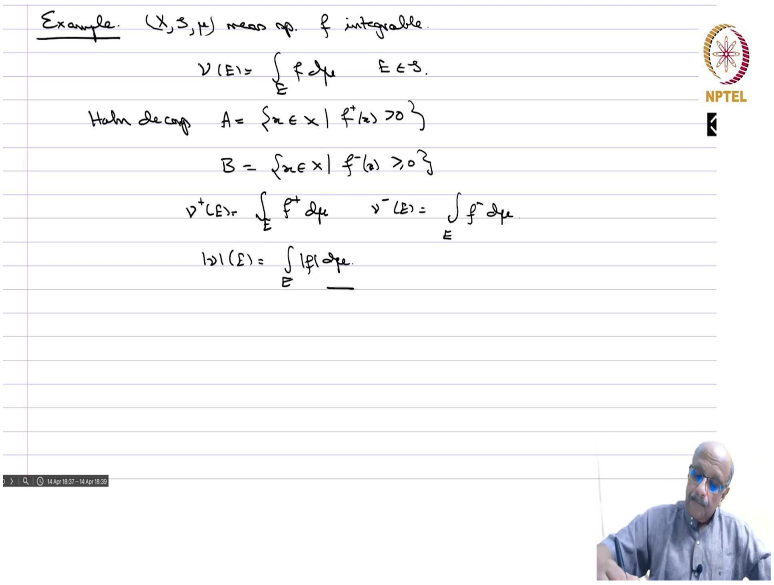

Example. So (X, S, μ) measure space, and f integrable. Then you consider

$$\nu(E) = \int_E f d\mu, \ E \in S.$$

Then the Hahn Decomposition $A = \{x \in X\colon f^{+}(x) > 0\}$, $B = \{x \in X\colon f^{-}(x) \geq 0\}$.

$$\nu^{+}(E) = \int_E f^{+} d\mu, \quad \nu^{-}(E) = \int_E f^{-} d\mu, \ |\nu|(E) = \int_E |f| d\mu.$$

You can put the equality sign in either one but since we have already seen Hahn Decomposition is not unique, so, this gives you a Hahn Decomposition because this is a positive set because if you integrate anything in a, on a subspace of this then since f plus is positive, so f equals f plus and therefore, the integral will be positive and so the measures will be positive.

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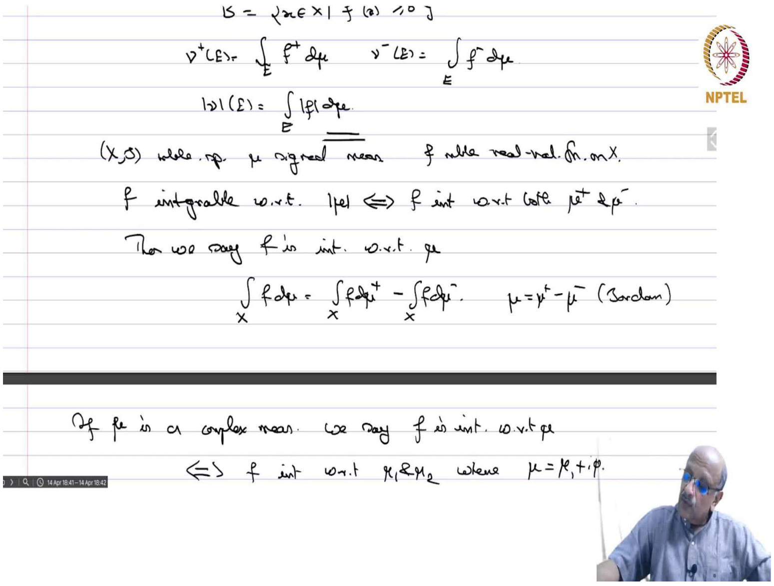

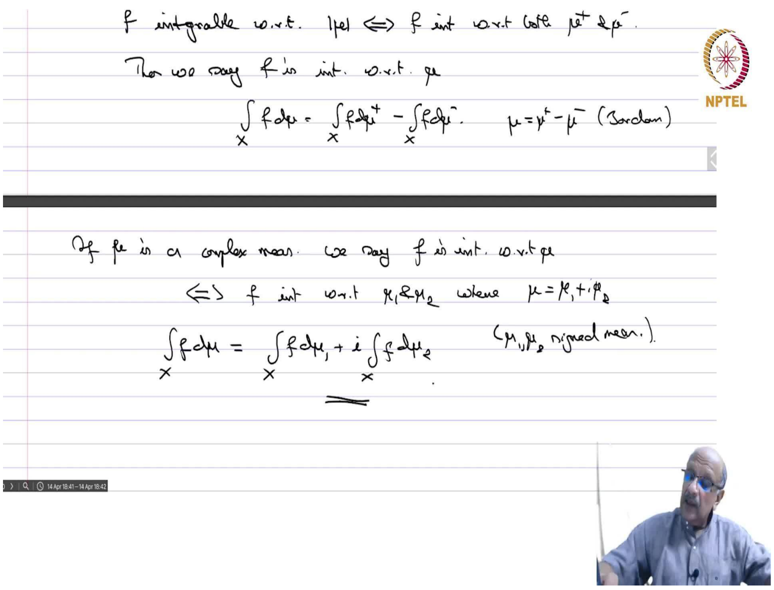

So, if (X, S) is a measurable space and μ is a signed measure and f measurable real valued function on X, then we can define, so f integrable with respect to |μ| is the same as saying f integrable with respect to both μ^+ and μ^-. Only then that integral will be finite. And then we can define the integral over X. So, then you define,

$$\int_X f \ d\mu = \int_X f \ d\mu^+ - \int_X f \ d\mu^-, \ \ \mu = \mu^+ - \mu^- \text{ (Jordan)}$$

So, if μ is a complex measure, we say f is integrable with respect to mu if and only if f integrable with respect to μ_1 and μ_2, where $\mu = \mu_1 + i\mu_2$.

And you say $\int_X f \ d\mu = \int_X f \ d\mu_1 \ + i\int_X f \ d\mu_2$.

So, mu 1 mu 2 signed measures. And then you know so, we know how to integrate with respect to sign measure and so, if it is integrable with respect to both of them, then you say you can define the measure. So, these are about the integral.

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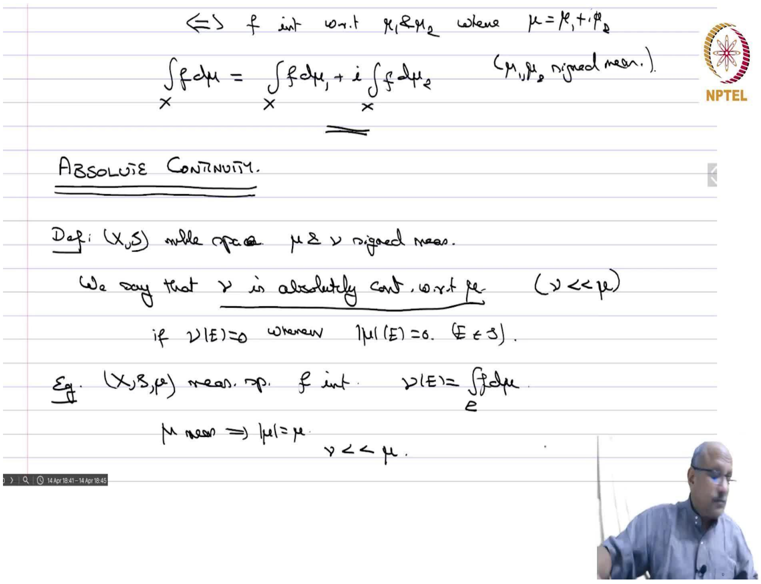

So now, we come to a different notion. So this is the notion of absolute continuity. We already encountered absolutely continuous measurements of one measure with respect to another. And we said that integral, if you have nu E was integral f d mu over E, where mu is a measure, then we say that measure was absolutely continuous.

Definition: so (X,S) is measurable spaces, μ, ν signed measures. So we say that ν is absolutely continuous with respect to μ and the symbol is $\nu << \mu$. And if $\nu(E) = 0$, whenever $|\mu|(E) = 0,\ E \in S$.

Example: (X, S, μ) measure space, f integrable and $\nu(E) = \int_E f d\mu$. And since μ is a measure, $|\mu| = \mu$. So there is no need for the Hahn, the whole set is a positive set. So, $\nu << \mu$.

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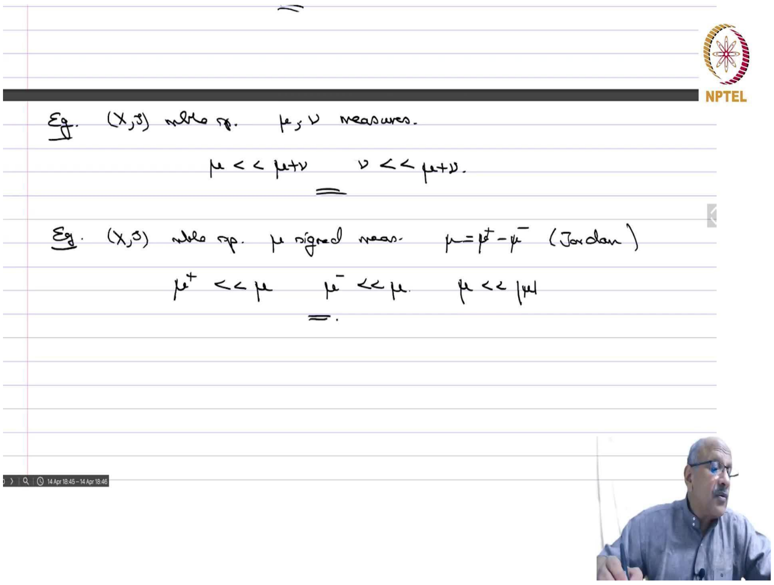

Example: (X, S, μ) measurable space and μ, ν measures. That means, they are positive. Then $\mu << \mu + \nu$, $\nu << \mu + \nu$. An example again X S, measurable space, mu signed measure. So, mu equals mu plus minus mu minus, Jordan Decomposition. And then you have my plus is absolutely continuous with this respect to mu, mu minus is absolutely continuous with respect to mu. Because you have, and you also have, mu is absolutely continuous with respect to mod mu because if mod mu is 0 then mu plus is 0, mu minus is 0 and therefore, mu is also 0. So, all these things are trivial examples.

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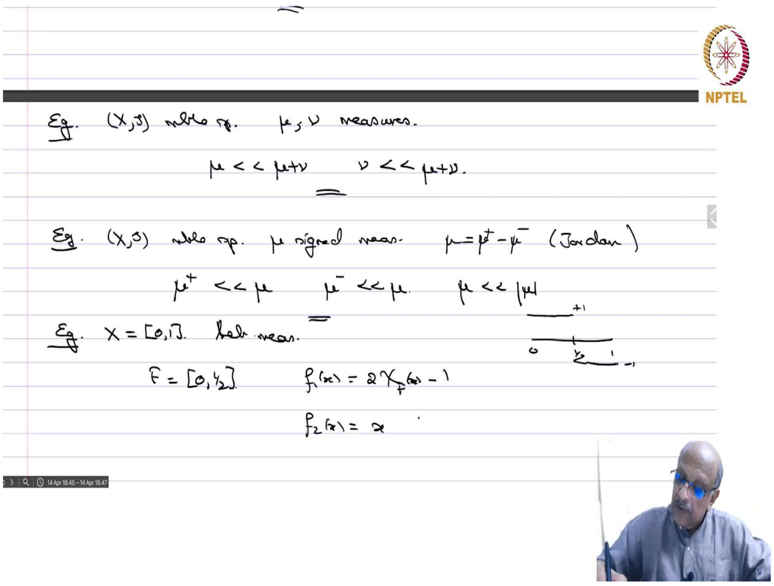

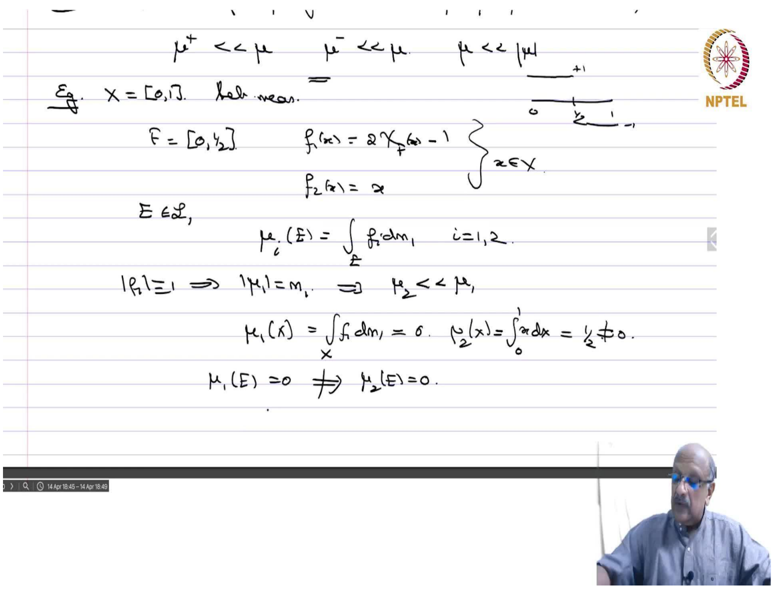

So now, let us take a different example, which tells you more about this definition.

Example: $X = [0, 1]$, Lebesgue measure. Take $F = [0, \frac{1}{2}]$. And then you define

$$f_1(x) = 2\chi_F(x) - 1;\ \ f_2(x) = x,\ x \in X.$$

So, if E is Lebesgue measurable set, then you define $\mu_i(E) = \int_E f_i dm_1,\ i = 1, 2.$

Now, $|f_1| \equiv 1$, so this means $|\mu_1| = m_1 \Rightarrow \mu_2 << \mu_1$. But

$$\mu_1(X) = \int_X f_1 dm_1 = 0, \;\; \mu_2(X) = \int_0^1 x dm_1 = \frac{1}{2} \neq 0.$$

So, $\mu_1(E) = 0$ does not imply $\mu_2(E) = 0$.

So, when you have absolute continuity, it is only the mod which matters not the function itself. So, let us tick this example you have here.

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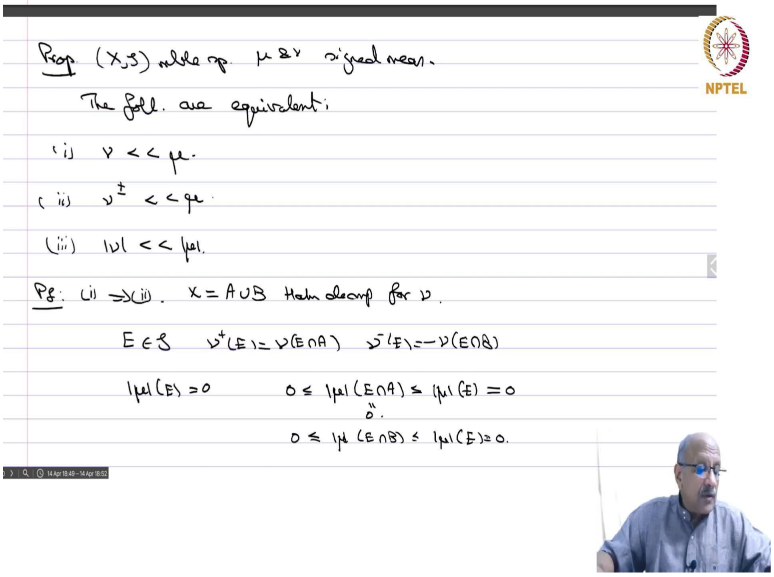

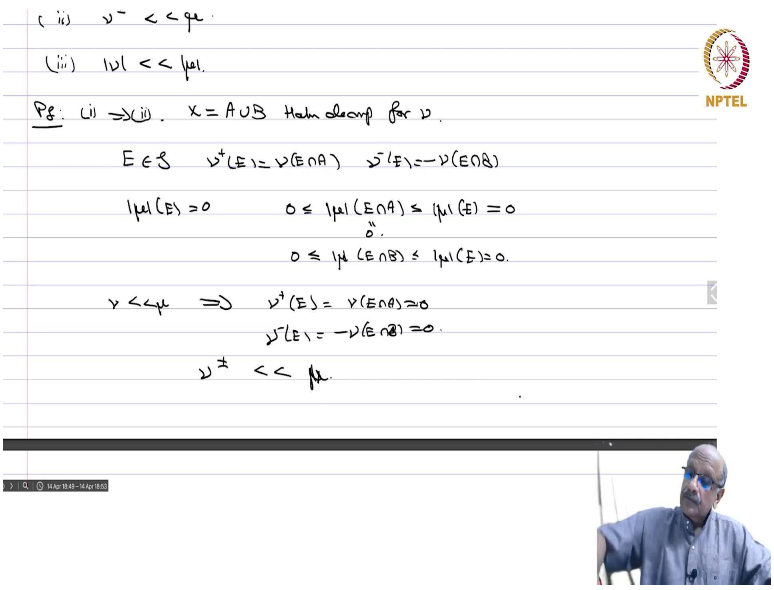

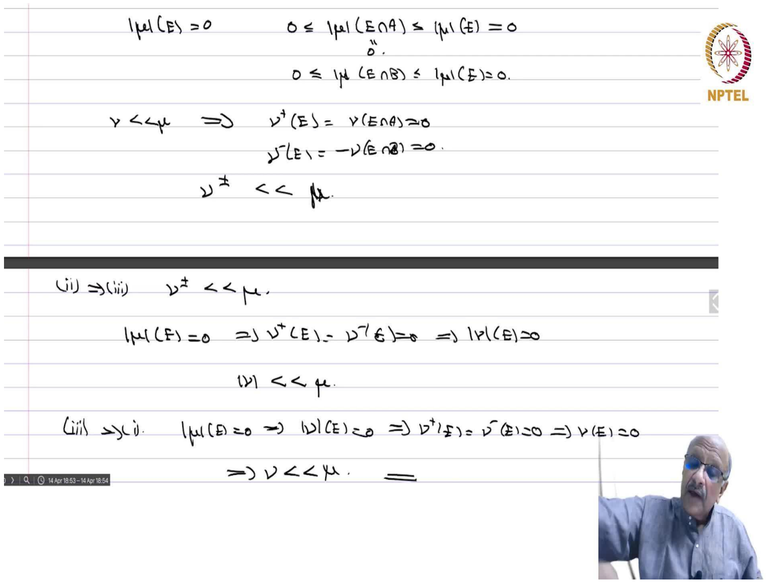

Now, proposition.

Proposition: (X, S) measurable space μ, ν signed measures. The following are the equivalent.

(i) $\nu << \mu$. (ii) $\nu^{+-} << \mu$. (iii) $|\nu| << |\mu|$.

proof: (i) implies (ii):- So $X = A \cup B$ Hahn Decomposition for ν. So E in S, so

$$\nu^{+}(E) = \nu(E \cap A),\ \ \nu^{-}(E) =- \nu(E \cap B).$$

So, if $|\nu|(E) = 0$, then $0 \leq |\mu|(E \cap A) \leq |\mu|(E) = 0$. And $0 \leq |\mu|(E \cap B) \leq |\mu|(E) = 0$.

And therefore, since $\nu << \mu$, this means that $\nu^{+}(E) = \nu(E \cap A) = 0$, $\nu^{-}(E) =- \nu(E \cap B) = 0$. And this implies $\nu^{+-} << \mu$.

(ii) implies (iii): So if $\nu^{+-} << \mu$, so $|\mu|(E) = 0 \Rightarrow \nu^{+}(E) = \nu^{-}(E) = 0 \Rightarrow |\nu|(E) = 0$. So $|\nu| << \mu$.

And then, (iii) implies (i): So $|\mu|(E) = 0 \Rightarrow |\nu|(E) = 0 \Rightarrow \nu^{+}(E) = \nu^{-}(E) = 0 \Rightarrow \nu(E) = 0$. Therefore, $\nu << \mu$.

So, this completes the proof. So we will continue with absolute continuity next time.