Measure and Integration
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Lecture 60
Hahn and Jordan decompositions

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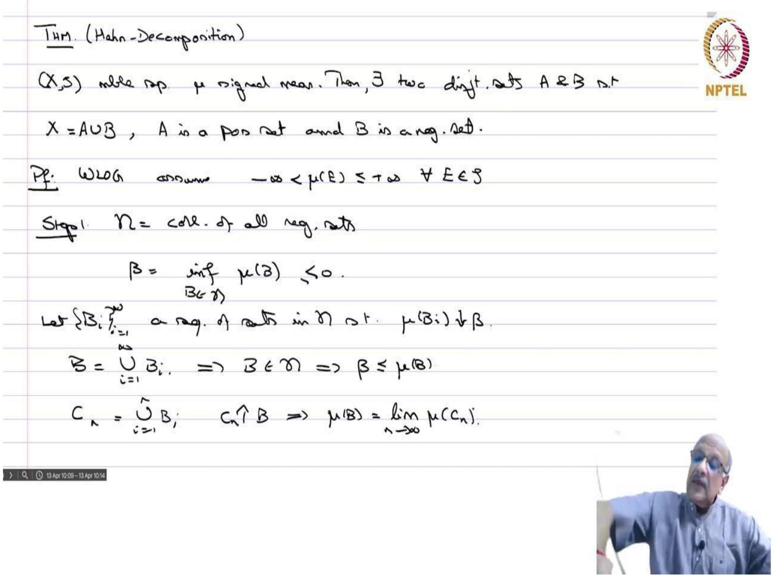

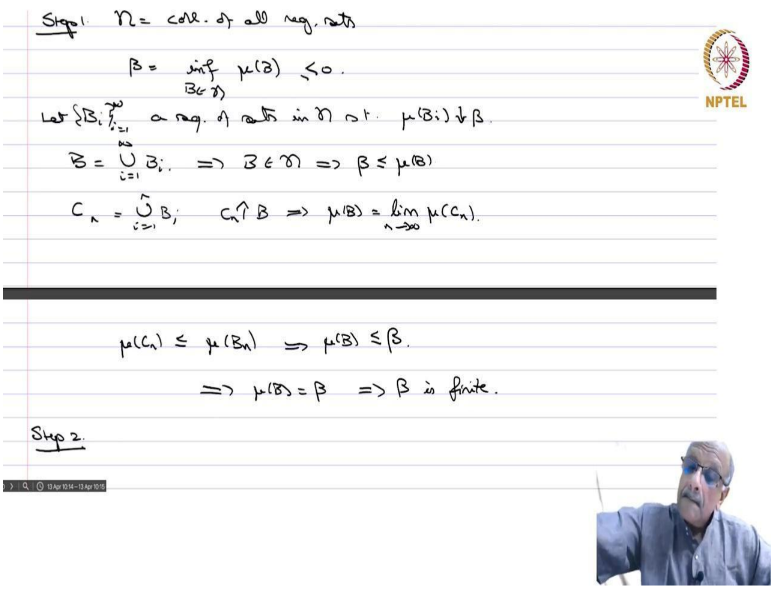

Hahn Decomposition:

So, we defined, we were looking at signed measures. so signed measure is a countably additive function which is 0 on the empty set and takes at most plus 1, plus infinity or minus.

It cannot take both of them. And then given a signed measure, we proved some standard properties and then we introduced the notion of a positive set and the negative set.

A positive set is a set whose measure is less non-negative, and not only that, every subset of it is also of non-negative measures. Similarly, a negative set has a non-positive measure and the same is true for every subset of it. And then, so now, we have following very important theorem.

So, this is called the

Theorem (Hahn Decomposition):. So (X, S) measurable space signed μ signed measure, then there exist two disjoint sets A and B such that $X = A \cup B$ so you, that is why we call it a decomposition, and A is a positive set, and B is a negative set. So, we are going to prove this in several stages.

Proof: So, without loss of generality, assume $- \infty < \mu(E) \leq + \infty$ for all $E \in S$. So we are saying that it takes the value plus infinity. If you are going to, if you took the value only minus infinity then you work with minus mu in this proof, and everything will go through.

Step 1: so N, collection of all negative sets. And you define $\beta = inf\ \mu(B) \leq 0$, B in N. So of course, this will be less than or equal to 0. So then let B i be a minimizing sequence, i equals 1 to infinity, a sequence of sets in n such that $\mu(B_i)$ decreases to B. You can always find the minimizing sequence.

So $B = \bigcup_{n=1}^{\infty} B_i$. Now, we have already seen that a countable union of negative sets is negative, therefore B belongs to N. And that also implies that beta is therefore less than or equal to $\mu(B)$. On the other hand, you said $C_n = \bigcup_{i=1}^{n} B_i$.

Then C_n increases to B and therefore, you have $\mu(B)$ is the limit as n tends to infinity, $\mu(C_n)$. Now, C_n we have seen in the proposition just at the end of the previous session that $\mu(C_n)$ is less than or equal to the minimum. So in particular, this is less than equal to $\mu(B_n)$.

In fact, it is, C_n is less than the minimum of B_1 to B_n. So it is less than or equal to B_n and therefore, this implies that $\mu(B\ )$, which is the limit of the C_n is also less than or equal limit of the $\mu(B_n)$ and that is equal to beta. So this implies that $\mu(B\ )$ equal to beta. And in particular, in fact the beta is finite. And all negative sets have finite measure.

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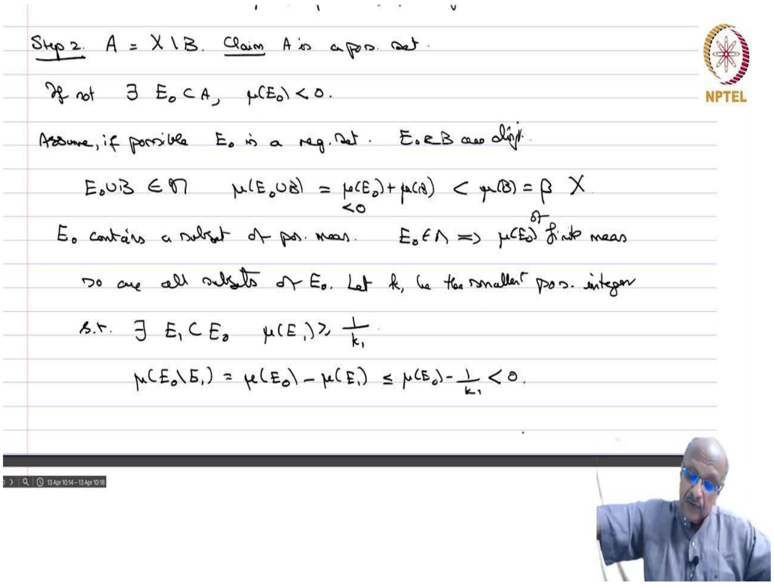

Step 2. So we have got a set of negative measure and its lowest possible value of that measure. so we expect that to be the set which we want to and so we take its complement

$A = X\backslash B$

So claim A is a positive set. Then our theorem is proved. So if not there exists a $E_0 \subset B$ and $\mu(E_0) < 0$. So that is, that is why, how a positivity will fail for the set.

So assume, if possible, E_0 is a negative set. But then E_0 *and* B are disjoint. And therefore, E naught union B is also a negative set and $\mu(E_0 \cup B)$ by additivity is $\mu(E_0) + \mu(B\ )$. This is strictly less than 0, so this is strictly less than mu of B equal to beta. But that is a contradiction, because beta is the infimum of all that.

You cannot have a negative set which has measure less than beta. And therefore, E_0 contains a subset of positive measure, but E_0 is in N. And this implies that mu of E naught is finite. So are all subsets of E_0. Finite means finite measure. E_0 of finite measure and so, and all the subsets of E_0, because a subset of set of finite measure is also finite.

Step-3, Let k_1 be the smallest positive integer such that there exists a $E_1 \subset E_0$ and $\mu(E_1)$ is bigger than or equal to 1 over k_1. That means, we are looking for the largest positive, set with the largest possible, positive measure contained in E_0. So we are looking because k_1 is the smallest positive integer and you are looking at $1/k_1$ here..So we are trying to exhaust as fast as possible.so we are looking for the largest possible measured set inside E naught.

So, now, you look at mu of $\mu(E_0\setminus E_1)$ is equal to, everything is finite. so subtractive property holds, and therefore, $\mu(E_0\setminus E_1)$ which is less than equal to $\mu(E_0)$ minus $1/k_1$, and that is of course, strictly less than 0. And of course, it is contained in the E_0.

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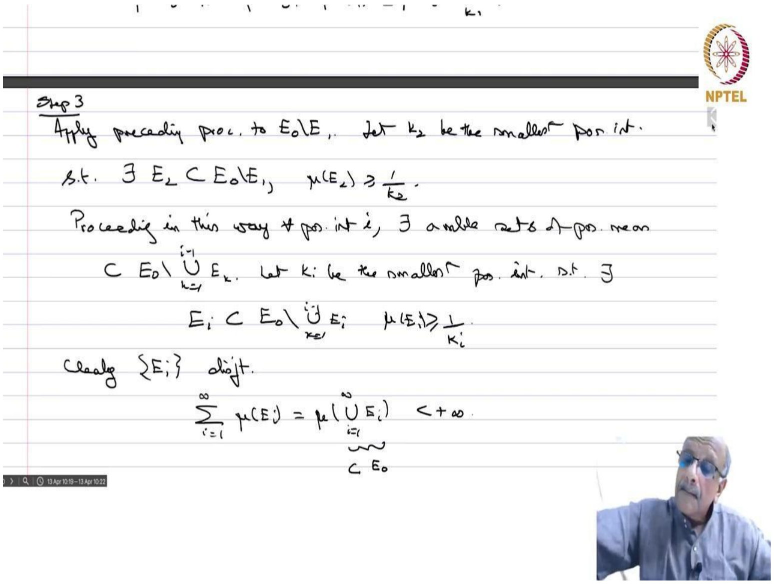

And therefore, we have, apply proceeding procedure to $E_0\setminus E_1$ because, this again, that means, this cannot be a negative set so it will contain a set of positive measure..So let k_2 be the smallest positive integer such that there exists E_2 contained in $E_0\setminus E_1$ and $\mu(E_2) \geq 1/k_2$.

And now we proceed inductively. Just now, you take $E_0\setminus E_1 \cup E_2$ and so on. So proceeding like this, proceeding in this way, for every positive integer i, there exists a measurable set E_i, $\mu(E_i) \geq 1/k_i$, sorry, of positive measure contained in E_0 minus union k equals 1 to i minus 1 E_k. And let k i be the smallest positive integer such that there exists E_i contained in E_0 minus union k equals 1 to i minus 1 E_i and mu of E_i greater than or equal to 1 by k_i.

So, clearly E_i are all disjoint because each time you are looking at something in the compliment, and therefore, E is are all clearly disjoint and you have sigma i equals 1 to infinity, mu of E_i is equal to mu of union i equals 1 to infinity E_i. And this is of course, contained in E_0, which has finite measure and therefore, you have that this is less than plus infinity. So in particular this implies at $\mu(E_i)$ goes to 0, that is k_i tends to infinity. So this is step 3.

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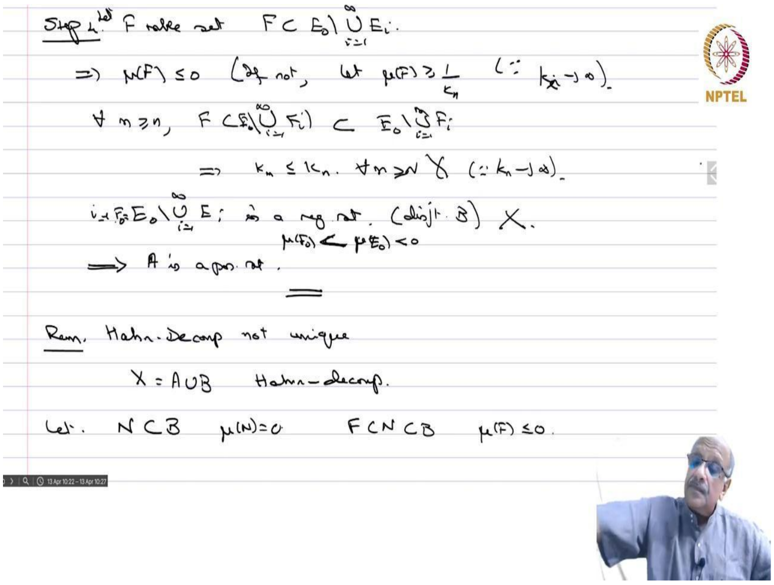

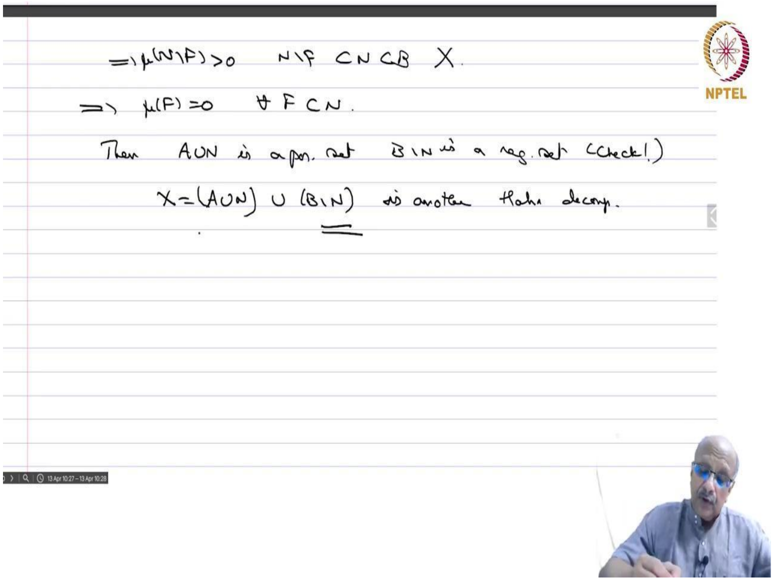

Step 4. F measurements set, let $F \subset E_0 \setminus \bigcup_{i=1}^{\infty} E_i$. So then this implies it $\mu(F) \leq 0$. If not, let $\mu(F) \geq 1/k_n$. Since, k_i tends to infinity, you can always find some, how will be positivity. so you can always find $1/k_n$ goes to 0, so however small this is, you can always find a k_n like this.

Now, for all m greater or equal to n, you have F is contained in union E_0 minus union i equals 1 to infinity E I, which is contained in E_0 minus union k equals, i equals to 1 to m F i. And then, it has positive measures and therefore, this implies that k m is less than equal to k_n

because $\mu(F_n)$ is greater than $1/k_n$. But then, by definition it is like this. How we have chosen, or chose what, , $1/k_m$ is the smallest positive integer set such that there exists a subset of positive measure greater than $1/k_m$.

So, here this implies, but this is for all m greater than equal to n contradictions since k n again tends to infinity. So this is not possible. And therefore, $\mu(F\ )$ is less than equal to 0, that is in E naught minus union i equals 1 to infinity E i is a negative set. And it is disjoint from B, and that again, is the beginning of step 2, is a contradiction because you cannot have a set like this.

And consequently. so this implies that A is a positive set. If you call this F_0, you have $\mu(F_0)$ is contained, is a subset, is less than $\mu(E_0)$ every time we saw that and therefore, this is, which is strictly less than 0 and is joined from B and it is not possible. So that is the conclusion. So this we have, this completes the proof.

Remark, Hahn Decomposition not unique. Let us see why it cannot be unique. So let us take $X = A \cup B$. So this is called Hahn Decomposition, breaking it up into a positive set and a negative set, and they are disjoint and that is called a Hahn. So if $X = A \cup B$. is Hahn Decomposition, let N be a subset of B with measure of N equal to 0.

So let F be contained in N. So N is contained in B, F is contained in N, and N is contained will B, so $\mu(F) \leq 0$ is less than or equal to 0. So, if $\mu(F)$ is strictly negative, then mu of 0 equals μ of N, which is equal to $\mu(F^+)$ $\mu(F)$ of N minus F. And that would mean N minus F, $\mu(F)$ of N minus F strictly positive, but N minus F is contained in N contained in B and that is a contradiction, so it is not possible.

Therefore, $\mu(F) = 0$ for all F contained in N. Then A union N is a positive set and B minus N is a negative set. And check. And $X = A \cup N \cup B\backslash N$, and these are disjoint and therefore, is another Hahn Decomposition. So you can take sets of measure 0 in these, in a Hahn Decomposition and then put it either place and then produce new Hahn Decomposition. And this is the only way you can do it, and that comes from the following proposition.

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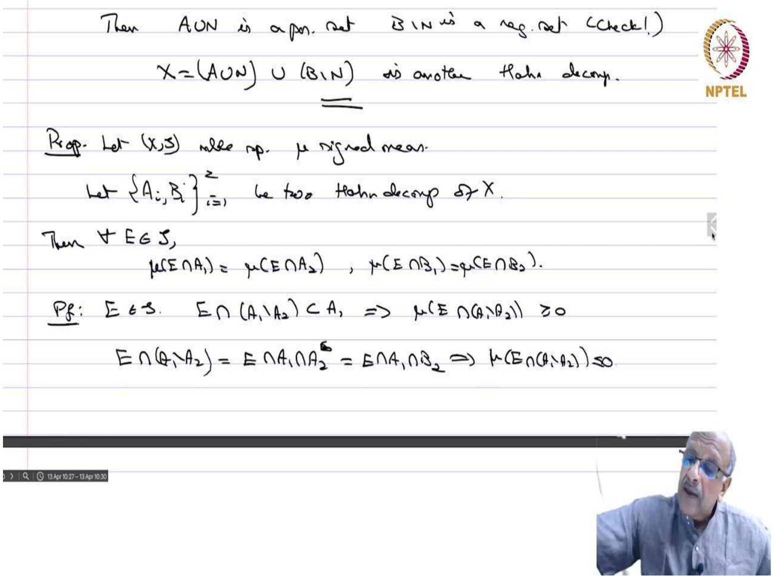

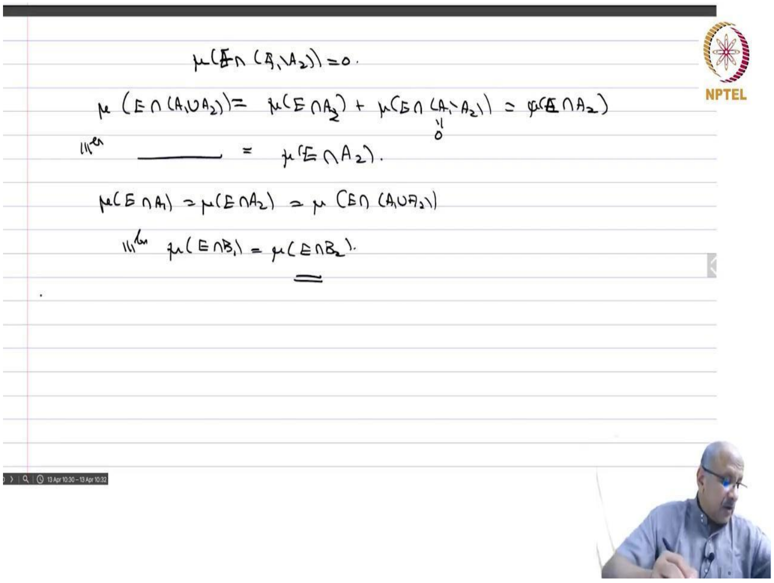

Proposition: *Let* (X, S), measurable space and μ, signed measure. Let $\{A_i, B_i\}_{i=1}2$ be two Hahn Decompositions. Then, for every $E \in S$, you have $\mu(E \cap A_1) = \mu(E \cap A_2)$. And $\mu(E \cap B_1) = \mu(E \cap B_2)$. So however you do it, it does not matter.

Proof, Let $E \in S$ and we have $E \cap A_1 \setminus A_2 \subset A_1$. And therefore, this implies that $\mu(E \cap A_1 \setminus A_2) \geq 0$. But $E \cap A_1 \setminus A_2 = E \cap A_1 \cap A_2^c = E \cap A_1 \cap B_2$. And that is a negative set, and this implies that $\mu(E \cap A_1 \setminus A_2) \leq 0$.

So, $\mu(E \cap A_1 \backslash A_2) = 0$. So

$$\mu(E \cap (A_1 \cup A_2)) = \mu(E \cap A_2) + \mu(E \cap (A_1 \backslash A_2)) = \mu(E \cap A_2).$$

Similarly this = $\mu(E \cap A_2)$

And therefore, they two are the same. So

$$\mu(E \cap A_1) = \mu(E \cap A_2) = \mu(E \cap (A_1 \cup A_2))$$

Similarly, $\mu(E \cap B_1) = \mu(E \cap B_2)$. So if you have any Hahn Decomposition ,they will only differ by sets of measure 0.

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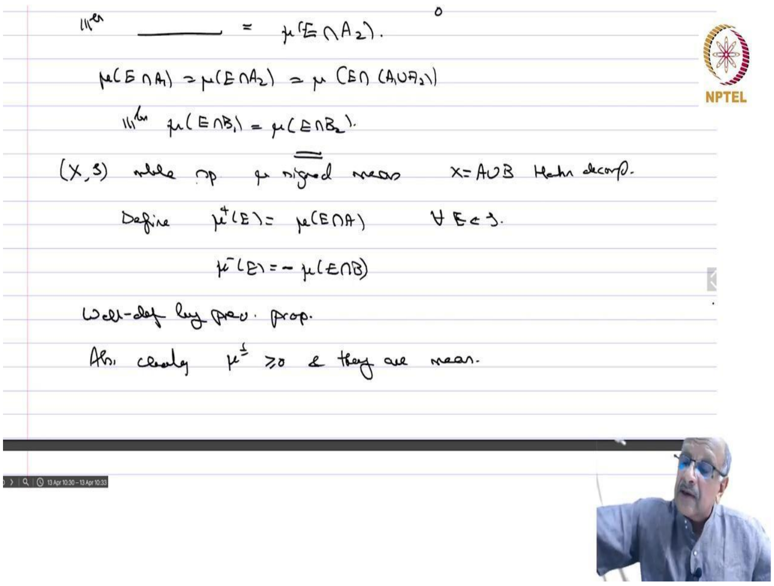

So, now you define, so Let (X, S), measurable space and μ, signed measure, and $X = A \cup B$ is a Hahn Decomposition. Then define $\mu^+(E) = \mu(E \cap A)$ and $\mu^+(E) = -\mu(E \cap B)$ with a minus sign, and for every E in S. So, well defined by previous proposition.

It does not depend on the Hahn Decomposition. Whatever Hahn Decomposition you take, these numbers will be, will not change. Also, clearly, mu plus minus are non-negative and they are measures, because you define it as $E \cap A$ and when you restrict this measure to a particular set by intersection, it is also a new measure.

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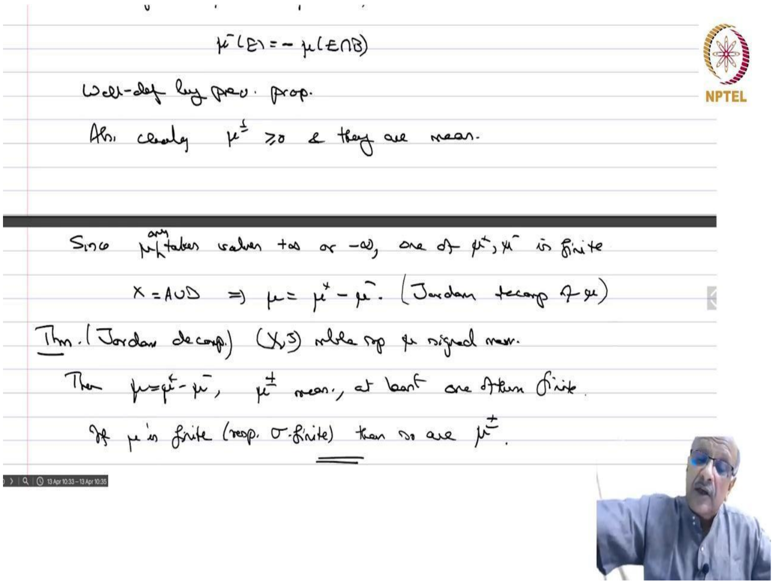

And since μ takes values plus infinity or minus infinity only, μ only takes values plus infinity or minus infinity, one of μ^{+}, μ^{-} is finite. And

$$X = A \cup B \Rightarrow \mu = \mu^{+} - \mu^{-}.$$

Therefore, we have proved the following theorem. This is called the Jordan Decomposition.

Theorem (Jordan Decomposition): So (X, S), measurable space, μ signed measure, then

$$\mu = \mu^{+} - \mu^{-}$$

measures, at least one of them finite. If μ is finite, respectively sigma finite, then so are μ^{+-}. That is easy to see, because if μ is finite, then μ^{+}, μ^{-} must both be finite, and if μ is sigma finite, it can witness a disjoint union of sets which are a finite measure. And the same thing carries over to the mu plus and the mu minus.

So, we will continue with this next time. So this decomposition, $\mu = \mu^{+} - \mu^{-}$ is called the Jordan Decomposition of mu. So you have a Hahn Decomposition. In terms of the Hahn Decomposition, you can define the Jordan Decomposition. And of course, though the Hahn Decomposition is not unique, the Jordan Decomposition becomes unique.