Measure and Integration
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Lecture No-6
Caratheodory's method

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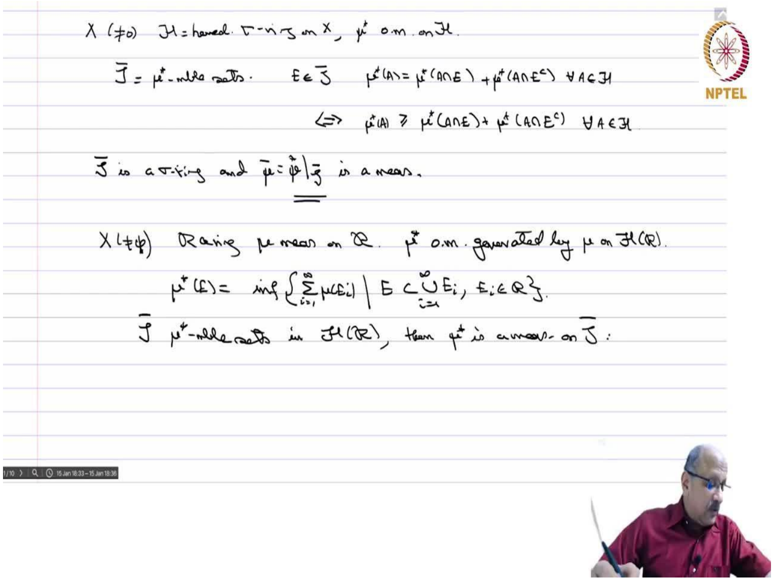

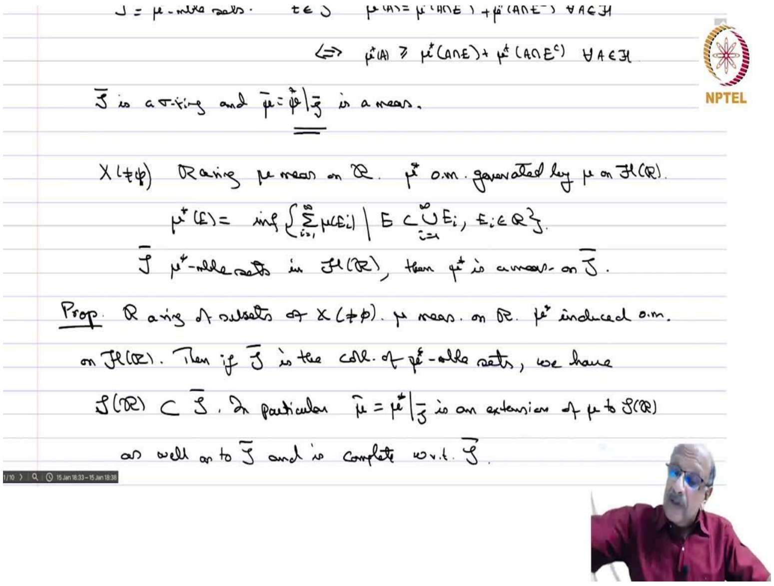

So, last time we were looking at set X (non-empty) and H-hereditary σ-ring on X and μ *-an outer measure on H. Then we define $\bar{S}$ = μ *-measurable sets. So, $E \in \bar{S}$. So, this means

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c), \ \forall A \in H$$

$$\Leftrightarrow \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c), \ \forall A \in H.$$

And then we prove that $\overline{S}$ is a σ-ring, first we prove it is a ring then we proved it is a σ-ring and $\overline{\mu} = \mu^*|_{\overline{S}}$ is a measure.

So, now, given X (non-empty) and R-ring on X and μ-a measure on R. So, we had natural μ^* outer measure generated by μ on $H(R)$, which is the smallest shady training containing R. So, how was this mu star phi defined?

$$\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \mu(E_i) : E \subset \cup_{i=1}^{\infty} E_i,\ E_i \in R\}.$$

So, now if you now $\overline{S}$ -μ^* measurable sets in $H(R)$, then μ^* is a outer measure on $\overline{S}$.

So, we have now got a different set of sigma rings on which you have a measure coming from the original measure. Now, the question is, is this an extension of the original measure or not? So, that is the question which you want to answer and therefore, we have the following proposition.

Proposition: R ring of subsets of X (non-empty) and μ measure on R, μ^* induced outer measure on $H(R)$. Then if $\overline{S}$ is the collection of μ^* measurable sets, we have $S(R) \subset \overline{S}$. Here $S(R)$ is the σ-ring generated by R. So, in particular, $\overline{\mu} = \mu^*|_{\overline{S}}$ is an extension of μ to the σ-ring generated by R as well as to $\overline{S}$ and is complete with respect to $\overline{S}$.

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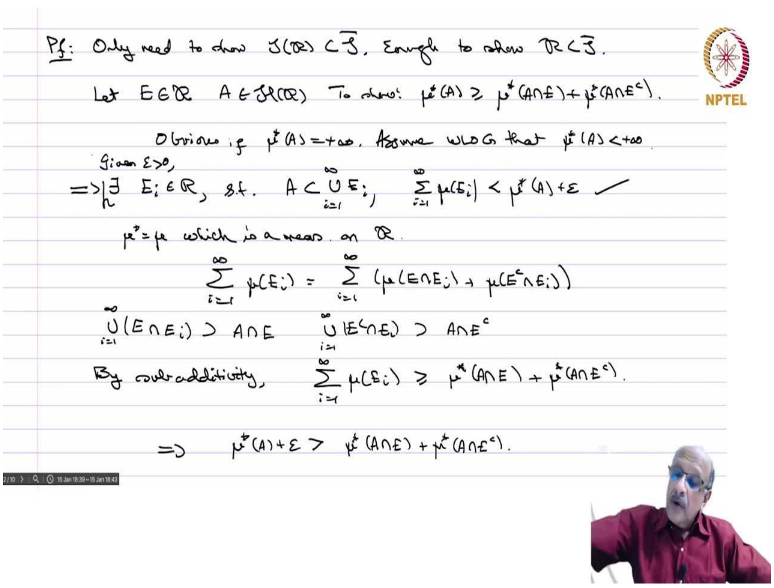

proof: Only need to show $S(R) \subset \overline{S}$. All the other things will follow. So, enough to show $R \subset \overline{S}$, because if $R \subset \overline{S}$, $\overline{S}$ is a sigma ring. So, it should contain the smallest sigma ring containing R and therefore it contains $S(R)$ also, so, this is the… it is enough to show that $R \subset \overline{S}$.

So, let us take so let $E \in R$ and let $A \in H(R)$. So, to show in fact, that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Now, obvious if $\mu^*(A) = \infty$. So, assume without loss of generality, that $\mu^*(A)$ is finite.

So, by the definition of μ^*, for given $\epsilon > 0$, there exists $E_i \in R$ such that $A \subset \cup_{i=1}^{\infty} E_i$ and

$$\sum_{i=1}^{\infty} \mu^*(E_i) < \mu^*(A) + \epsilon$$

Now, we have that $\mu^* = \mu$ on R and therefore, you have

$$\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} (\mu(E \cap E_i) + \mu(E^c \cap E_i))$$

Now $A \cap E \subset \cup_{i=1}^{\infty}(E \cap E_i)$ and $A \cap E^c \subset \cup_{i=1}^{\infty}(E^c \cap E_i)$. Therefore by subadditivity of the measure we have that $\sum_{i=1}^{\infty} \mu(E_i) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

$$\Rightarrow \mu^*(A) + \epsilon > \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

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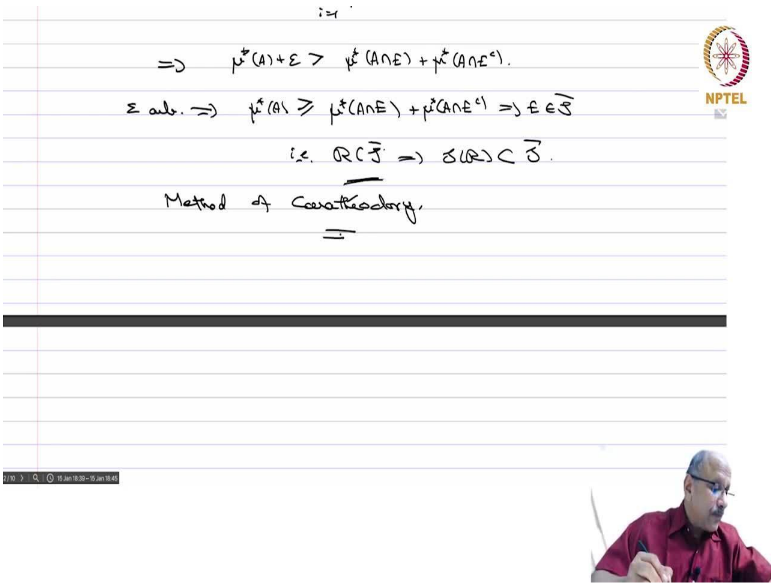

So, you have that epsilon arbitrarily and so

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\Rightarrow E \in \overline{S}.$$

i.e., $R \subset \overline{S}$ and this gives $S(R) \subset \overline{S}$. The theorem is completely proved.

So, this construction extension, construction of a complete extension of a measure so, this method is due to Caratheodory, so method of Caratheodory so, this called Caratheodory extension of measures. So, you get a complete measure on all the measurable sets and it also extends it to the sigma ring containing the measure. So, if you construct a measure on a ring you can always extend it to the sigma ring and also you can extend it to a complete measure on a slightly bigger sigma array.

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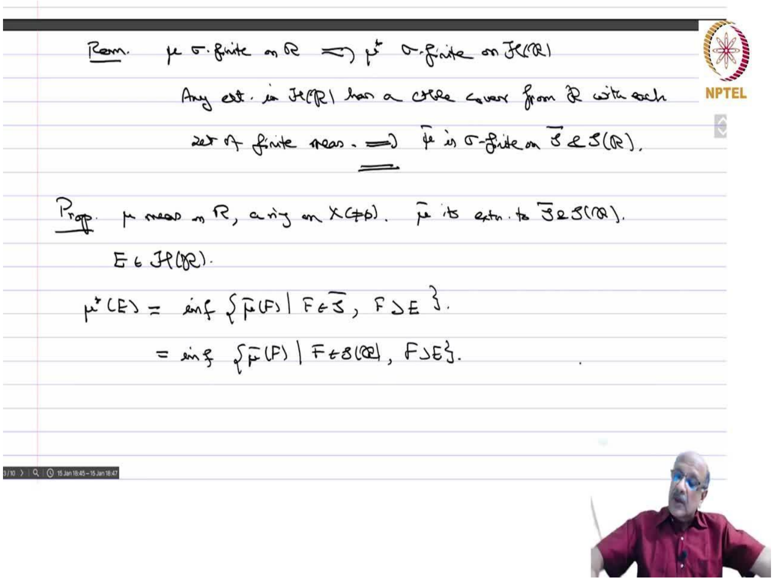

Remark: Suppose μ is σ-finite on R ⇒ μ * σ-finite on $H(R)$. And any element in $H(R)$ has a countable cover from R with each set of finite measure ⇒ $\overline{\mu}$ is σ-finite on $\overline{S}$ and $S(R)$.

Proposition: So, μ measure on R- a ring on X, $\overline{\mu}$ is extension to $\overline{S}$ and $S(R)$ by the above method. So, let $E \in H(R)$. Then we can compute the outer measures of E

$$\mu * (E) = \inf\{\overline{\mu}(F):\ F \in \overline{S},\ E \subset S\} = \inf\{\overline{\mu}(F):\ F \in S(R),\ E \subset S\}.$$

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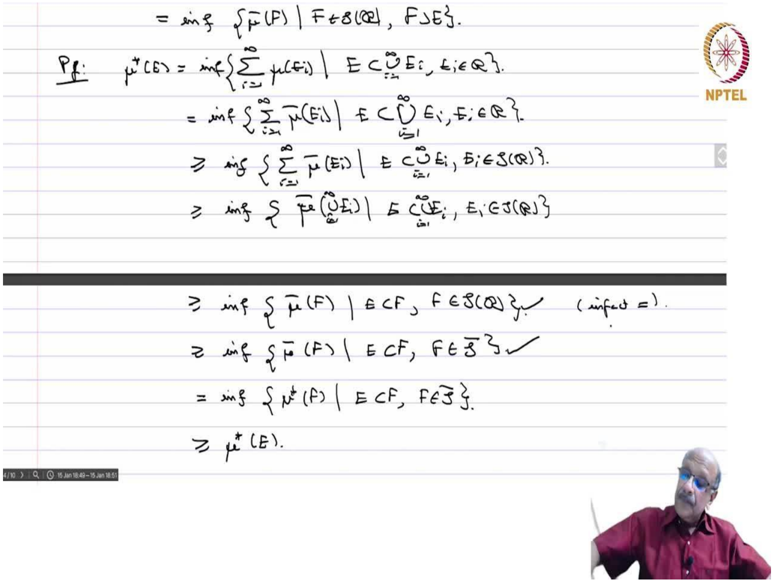

proof: So, $\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \mu(E_i): E \subset \cup_{i=1}^{\infty} E_i, E_i \in R\}$

$$= \inf\{\sum_{i=1}^{\infty} \overline{\mu}(E_i): E \subset \cup_{i=1}^{\infty} E_i, E_i \in R\}$$

$$\geq \inf\{\sum_{i=1}^{\infty} \overline{\mu}(E_i): E \subset \cup_{i=1}^{\infty} E_i, E_i \in S(R)\}$$

$$\geq \inf\{\overline{\mu}(\cup_{i=1}^{\infty} E_i): E \subset \cup_{i=1}^{\infty} E_i, E_i \in S(R)\}$$

$$\geq \inf\{\overline{\mu}(F): E \subset F, F \in S(R)\}$$

$$\geq \inf\{\overline{\mu}(F): E \subset F, F \in \overline{S}\}$$

$$= \inf\{\mu^*(F): E \subset F, F \in \overline{S}\} \geq \mu^*(E).$$

So, everything is equal now, and this proves the theorem.

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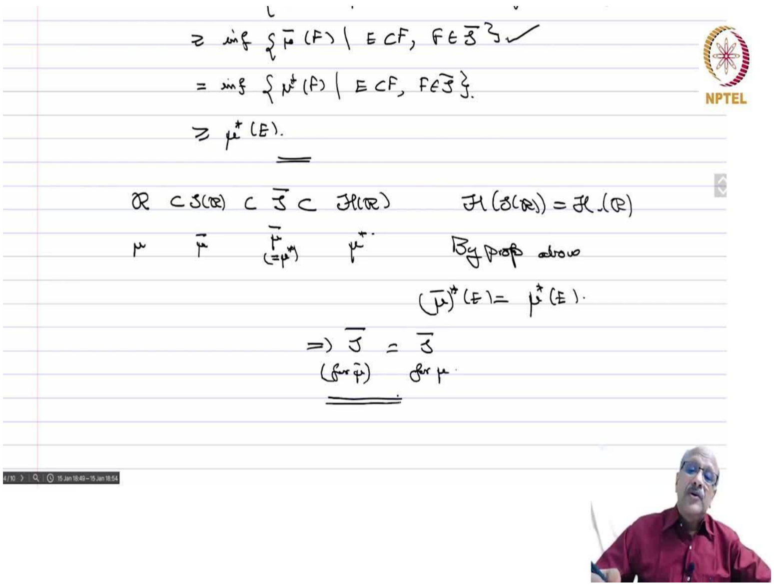

Now, why did we use this result? So, we now have a ring up and we had a measure mu here then we constructed the hereditary out σ ring H R with mu star, then we had mu star measurable sets and here we have mu bar which is equal to mu star then we showed that in fact this contains s of R and so, again here you have $\bar{\mu}$ which is a measure. So, this is how the Caratheodory scheme works. So, from R you go to H of R you go to a S bar then you come to S of R, so, now I can try to play this game again.

So, I can try to I have a sigma ring which is a ring and a measure on it and I can try to find the hereditary sigma ring which contains this but that will be just H of R because H of R already contains it and it is the smallest which contains R, so it will also be the smallest which contains S of R it cannot be anything smaller than that. So, H of S of R is the same as H of R. Now, by the proposition about if you go look at the chain of inequalities here, you get that the mu bar star of E is the same as mu star of E.

So, the thing is if you go see construct the definition you have $\bar{\mu}$ is a measure, if you construct the corresponding outer measure that is precisely the same as we have it here, which is this, and therefore, you have nothing different. So, it says that it is the same. So, in place S bar for mu bar is the same as the S bar for mu. And therefore, the measure is still the same.

So, this construction method is finished once and for all, you cannot try to get something more out of this anymore, because of the previous proposition, it completes the entire process, you cannot play this game again and again, because if you try to do it, you will end up with the same results. So, this tells you that that is an advantage. So now, we will use this method to construct the lebesgue measure.

So, we have this semi closed open intervals, semi open intervals, for which we know you have a notion of a length. Now, we have the ring containing all such intervals. So, we have to show that that generates a measure on the train that is only a job. Once you have that measure. Now, we apply the Caratheodory construction and immediately go to the sigma ring and go to the mu star measurable sets, you get a complete measure, which will be the lebesgue measure. So, this is the idea. And so now all the work has been done. So, at once we construct the measure, which is nothing but the generalization of length, but to finite unions of intervals. That is all we have to do.

Once we do that, then automatically the Caratheodory method will tell you how to do the lebesgue measures. And we will also see that you can simultaneously do the same thing. You do not have to work again and again, in any space dimension $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$ and so on so it is really a very powerful method, and which will give you the lebesgue measure immediately. So, that will be the next thing we do. Of course, before that we will do some exercises based on what we have learned up to now. And after that we will construct the lebesgue measure.