

Measure and Integration
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Lecture 59
Signed measures

(Refer Slide Time: 00:17)

SIGNED MEASURES

(X, \mathcal{S}) mds sp. μ_1, μ_2 meas on \mathcal{S} . $\alpha_1, \alpha_2 \geq 0$.

$$(\alpha_1 \mu_1 + \alpha_2 \mu_2)(E) = \alpha_1 \mu_1(E) + \alpha_2 \mu_2(E) \quad \forall E \in \mathcal{S}$$

$\alpha_1 \mu_1 + \alpha_2 \mu_2$ is a meas.



What if $\alpha_1, \alpha_2 \in \mathbb{R}$? $\mu_1 - \mu_2$ $\mu_1(E)$ & $\mu_2(E)$ are inf.

the $(\mu_1 - \mu_2)(E)$ is not meaningful.

Def (X, \mathcal{S}) mds sp. μ extended real-val. rat. fn. on \mathcal{S} .

μ is called a signed measure if

(i) $\mu(\emptyset) = 0$



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(ii) μ takes at most one of the val. $+\infty$ or $-\infty$.

(iii) μ countably additive.

Ex. μ_1, μ_2 meas. one of them finite.

We will now start a new chapter. This is called Signed Measures. So, let (X, \mathcal{S}) be a measurable space, and let us have μ_1, μ_2 measures and α_1 and α_2 non-negative real numbers.

Then you can define

$$(\alpha_1 \mu_1 + \alpha_2 \mu_2)(E) = \alpha_1 \mu_1(E) + \alpha_2 \mu_2(E), \quad \text{for every } E \in \mathcal{S}.$$

And then $\alpha_1\mu_1 + \alpha_2\mu_2$ is a measure. This is very, we have seen it and it is easy to prove.

So, now, what if $\alpha_1, \alpha_2 \in \mathbb{R}$. That means, they are not just non-negative numbers. Now, so, for instance if you have $\mu_1 - \mu_2$, then the difficulty is, if you, if both $\mu_1(E)$ and $\mu_2(E)$ are infinite, then $\mu_1 - \mu_2(E)$ is not meaningful. So, at least 1 of them has to be finite. And this kind of situation we already encountered in defining integrable functions. We wanted the integral f plus or the integral f minus to be finite.

Of course, we, when we define integrability, we said both of them are finite, but you can define an integral if at least one of them is finite. In the same way, if you want a measure, at least one of them should be finite. Of course, now the possibility is that at least, I mean some sets may have negative values for the measure. So, the measure will now, that is why, we call them signed measure. So based on this, we make the following definition.

So, (X, S) measurable space, and μ extended real valued set function on S . μ is called a signed measure if one,

$$(i) \mu(\Phi) = 0,$$

$$(ii) \mu \text{ takes at most one of the values, } +\infty \text{ or } -\infty.$$

That means, if at all the set has infinite measure, it will always be plus infinity or it will be always minus infinity. You cannot have some sets plus infinity some sets minus infinity because, to avoid the kind of confusion which we mentioned before.

$$(iii) \mu \text{ is countably additive.}$$

So if we have these three properties, then you say that μ is a signed measure.

(Refer Slide Time: 04:20)

(iii) μ countably additive.

Ex. μ_1, μ_2 meas. one of them finite. $\mu_1 - \mu_2$ is a signed meas.

Ex. (X, \mathcal{S}, μ) meas sp. f integrable real-val. fn.

$$v(E) = \int_E f d\mu \quad E \in \mathcal{S}$$

defines a signed meas on \mathcal{S} .

Def. A signed meas is called finite if $|\mu(E)| < +\infty \quad \forall E \in \mathcal{S}$

\mathcal{X} is called σ -fn. if $X = \bigcup_{n=1}^{\infty} E_n \quad |\mu(E_n)| < +\infty \quad \forall n$.



Example: μ_1 and μ_2 are measures, one of them finite. Then you have $\mu_1 - \mu_2$ is a signed measure.

Example: (X, S, μ) measure space, f integrable real valued function, then

$$v(E) = \int_E f d\mu, \quad E \in S,$$

defines a signed measure. So definition,

Definition: A signed measure is called finite if $\mu(E) < +\infty$ for all $E \in S$. It is called sigma finite if $X = \bigcup_{n=1}^{\infty} E_n, \mu(E_n) < +\infty$. So, then this is called a signed measure.

(Refer Slide Time: 06:28)

$\int_{\mathcal{G}} (X, \mathcal{G}, \mu)$ mean sp. f. integrable real-val. m.

$$\nu(E) = \int_E f d\mu \quad E \in \mathcal{G}$$


defines a signed measure on \mathcal{G}

Def. A signed measure is called finite if $|\mu(E)| < +\infty \quad \forall E \in \mathcal{G}$

ν is called σ -finite if $X = \bigcup_{n=1}^{\infty} E_n \quad |\mu(E_n)| < +\infty$

Rem. Signed measure is clearly finitely additive

$$|\mu(E)| < +\infty, \quad E \subset F \quad \mu(F \setminus E) = \mu(F) - \mu(E)$$



So, remarks.

Remark: So signed measure is clearly finitely additive, because as usual $\mu(\Phi) = 0$ so, you can fill in the this for the to make it accountable thing, and by countable additivity, it becomes finitely additive. And if $\mu(E)$ is finite, then this implies that if E and E is a subset of F then $\mu(F \setminus E) = \mu(F) - \mu(E)$. The same proof as we had for measures because F can witness the disjoint union, F minus E union E and by finite additivity, μF will be μ of F minus E plus $\mu(E)$, and because μE is finite, you are allowed to subtract. So, we have this.

(Refer Slide Time: 07:39)

Rem. Signed meas. is closely finitely additive

$$|\mu(E)| < +\infty, \quad E \subset F \quad \mu(F \setminus E) = \mu(F) - \mu(E)$$



Fig. (X, \mathcal{S}) mte sp., μ a signed meas. on \mathcal{S} , Let $E, F \in \mathcal{S}, E \subset F$.

$\mu(F)$ finite $\Rightarrow \mu(E)$ finite

Pf. $\mu(F) = \mu(F \setminus E) + \mu(E)$

Even if one term on RHS is inf. then $\mu(F)$ inf. \times

$\Rightarrow \mu(E)$ finite.



Proposition, (X, \mathcal{S}) measurable space μ , a signed measure on \mathcal{S} . Let $E, F \in \mathcal{S}, E \subset F$. $\mu(F)$ finite implies $\mu(E)$ is also finite.

Proof: So, we have finite sets, then say, subsets will also have finite measure because

$$\mu(F) = \mu(F \setminus E) + \mu(E)$$



Now, even if one of them is infinity, then since you always have only plus infinity or minus infinity, there is no possibility of cancellations et cetera, so, even if one of them is infinite, $\mu(F)$ will be infinite. So, even if one term on RHS is infinite then $\mu(F)$ infinite. And that is a contradiction. And therefore, this implies that $\mu(E)$ finite. So is $\mu(F \setminus E)$, of course.

(Refer Slide Time: 09:32)

Even if one term on RHS is inf then $\mu(F)$ inf X
 $\Rightarrow \mu(E)$ finite.

Prop. (X, S) measurable sp. μ signed meas on S. $\{E_n\}_{n=1}^{\infty}$ disjoint meas set
 Assume $|\mu(\bigcup_{n=1}^{\infty} E_n)| < +\infty$. Then $\sum_{n=1}^{\infty} \mu(E_n)$ is abs. convt.

Prf: $E_n^+ = \begin{cases} E_n & \mu(E_n) \geq 0 \\ \emptyset & \mu(E_n) < 0 \end{cases}$
 $E_n^- = \begin{cases} E_n & \mu(E_n) \leq 0 \\ \emptyset & \mu(E_n) > 0 \end{cases}$

Proposition, (X, S) measurable space, μ signed measure on S. $\{E_n\}$ disjoint measurable sets. Assume $|\mu(\bigcup_{n=1}^{\infty} E_n)| < +\infty$. Then $\sum_{n=1}^{\infty} \mu(E_n)$ is absolutely convergent.

Proof: We define

$$E_n^+ = E_n, \text{ if } \mu(E_n) \geq 0, \text{ empty set if } \mu(E_n) < 0.$$

$$E_n^- = E_n \text{ if } \mu(E_n) \leq 0, \text{ and empty set if } \mu(E_n) > 0.$$



(Refer Slide Time: 11:17)

Then $\mu(\bigcup_{n=1}^{\infty} E_n^+) = \sum_{n=1}^{\infty} \mu(E_n^+)$
 $\mu(\bigcup_{n=1}^{\infty} E_n^-) = \sum_{n=1}^{\infty} \mu(E_n^-)$

The sum of the two series is $\mu(\bigcup_{n=1}^{\infty} E_n)$ which is fin.

\Rightarrow Both series are finite. But terms are the sum of the positive & sum of -ve terms of a convt. series $\sum \mu(E_n)$.
 $\Rightarrow \sum \mu(E_n)$ is abs. convt.

Prop. (X, S) measurable sp. μ signed meas. $\{E_n\}_{n=1}^{\infty}$ in c. seq. of disjoint
 then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \mu(E_n)$

Then, μ of union n equals 1 to infinity E_n^+ will be equal to $\sum_{n=1}^{\infty} \mu(E_n^+)$. And similarly, μ of union n equals 1 to infinity E_n^- because this is the other, same E_n^- which you have defined and they are all disjoint and so, by countable additivity, you have this is equal to $\sum_{n=1}^{\infty} \mu(E_n^-)$. Sum of these will be 0 in the, in between.

And the sum of the two series is μ of union n equals 1 to infinity E_n , which is finite. So, this implies both series are finite. But these are the sum of the positive terms and sum of negative terms of a convergent series $\sum \mu(E_n)$. This implies $\sum \mu(E_n)$ is absolutely convergent because the series is absolutely convergent if the sum of the positive terms and sum of the negative terms, independently, are convergent.

Next proposition,

Proposition: (X, \mathcal{S}) measurable space, μ signed measure, $\{E_n\}$ increasing sequence of measurable sets and $\mu(E_n) < \infty$ for some n , then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$$

just as in the case of measures.

(Refer Slide Time: 14:19)

$\Rightarrow \mu(E_n)$ is conv. \uparrow

Prop. (X, \mathcal{S}) mds sp. μ signed meas. $\{E_n\}_{n=1}^{\infty}$ inc. seq. of sets
 then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \mu(E_n)$
 If $\{E_n\}_{n=1}^{\infty}$ dec. seq. & $\mu(E_1)$ finite for some n , then
 $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \mu(E_n)$

Pf. Same as in the case of meas. since subsets of sets of fin. meas are also finite and subtractive property holds.



Similarly, if $\{E_n\}$ increasing sequence of measurable sets and $\mu(E_n) < \infty$ for some n , then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$$

Proof, same as in the case of measures. Since subsets of sets of finite measure are also finite and subtractive property holds. So you can just copy that proof, everything is true, everything is justified and therefore, we need not go ahead for this.

(Refer Slide Time: 15:54)

$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$

Pf. Same as in the case of meas. since subsets of sets of fin. meas are also finite and subtractive property holds.

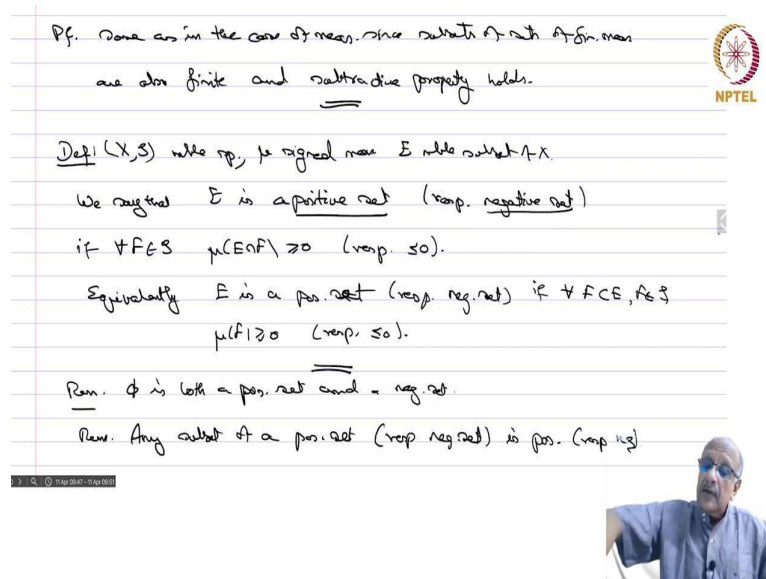
Def (X, \mathcal{S}) mds sp. μ signed meas. E mds subset $A \subset X$
 We say that E is a positive set (resp. negative set)
 if $\forall F \in \mathcal{S}$ $\mu(E \cap F) \geq 0$ (resp. ≤ 0).
 Equivalently E is a pos. set (resp. neg. set) if $\forall F \in \mathcal{S}$
 $\mu(F) \geq 0$ (resp. ≤ 0).



Now, we make an important definition.

Definition: Let (X, S) measurable space, μ signed measure, E measurable set, subset of X . So we say that E is a positive set, (respectively negative set) if for every $F \in S$, $\mu(E \cap F) \geq 0$, respectively, less than equal to 0. Equivalently, E is a positive set, respectively negative set, if for every $E \subset F$, F measurable, $\mu(F) > 0$, respectively less than or equal to 0.

(Refer Slide Time: 17:39)



Pf. Same as in the case of meas. since subsets of each of S_+ and S_- are also finite and additive property holds.

Def: (X, S) mds sp, μ signed mea. E mds subset of X .

We say that E is a positive set (resp. negative set) if $\forall F \in S$ $\mu(E \cap F) \geq 0$ (resp. ≤ 0).

Equivalently E is a pos. set (resp. neg. set) if $\forall F \in S, A \in S$ $\mu(F) > 0$ (resp. < 0).

Rem. \emptyset is both a pos. set and a neg. set.

Rem. Any subset of a pos. set (resp. neg. set) is pos. (resp. neg.)

So, a positive set means a set of positive measure, and all its subsets have also, have positive measures. And similarly, then for negative, so remark, empty set is both a positive set and a negative set.



Remark, any subset of a positive set, respectively negative set is positive, respectively negative because further subsets are all going to have the measures of the same sign.

(Refer Slide Time: 18:36)

Lem. If $\{A_i\}$ is countable disjoint union of pos. sets and $\{B_i\}$ is neg. sets.
 Then Any subset of a pos. set (resp. neg. set) is pos. (resp. neg.)

Any finite or countable disjoint union of pos. sets (resp. neg. sets)
 is pos. (resp. neg.)

If $A_i, i=1,2$ are pos. (resp. neg.) then so is $A_1 \setminus A_2$.
 \Rightarrow Every countable union of pos. (resp. neg. sets) is pos. (resp. neg.)

Any finite or countable disjoint union of positive sets, respectively negative sets is positive, respectively negative because if you take any subset of it, of each one of them, each intersection is going to have positive measure or negative measure depending on what the set is and therefore, it is it will be positive or negative since the all the terms of the series are of the same sign.

Now if A_i, i equals 1, 2 are positive, respectively negative then so is $A_1 - A_2$ because it is a subset, and therefore, so this implies that every countable union of positive, respectively negative sets, every countable union can witness as a countable disjoint union using differences. And therefore, every countable union of positive sets is a countable disjoint union of positive sets and negative sets similarly, and therefore, every countable union of positive, respectively negative sets is positive, respectively negative.

(Refer Slide Time: 20:34)

Prop. Let (X, S) be a measurable sp., μ signed measure. $\{B_i\}_{i=1}^n$ a finite collection of neg sets. $B = \bigcup_{i=1}^n B_i$. Then

$$\mu(B) \leq \min_{1 \leq i \leq n} \mu(B_i) \quad (*)$$

Pf. $B_1 \cup B_2 = B_1 \cup (B_2 \setminus B_1)$. disjoint.

$$\mu(B_1 \cup B_2) = \mu(B_1) + \underbrace{\mu(B_2 \setminus B_1)}_{\leq 0} \leq \mu(B_1)$$

Similarly $\mu(B_1 \cup B_2) \leq \mu(B_2)$.

(*) True for $n=2$. gen case follows by induction.



Proposition, Let (X, S) measurable space, μ signed measure. $\{B_i\}_{i=1}^n$, a finite collection of negative sets. $B = \bigcup_{i=1}^n B_i$. Then

$$\mu(B) \leq \min_{\{1 \leq i \leq n\}} \mu(B_i)$$

Proof. So, $B_1 \cup B_2 = B_1 \cup (B_2 \setminus B_1)$. And this is a disjoint union.

$$\text{So, } \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2 \setminus B_1) \leq \mu(B_1).$$

So, by symmetry, similarly, $\mu(B_1 \cup B_2) \leq \mu(B_2)$. So, it is called the star. So, star, true for $n=2$, general case follows by induction.

And that is proved. So, this, we will use this notion of positive sets and negative sets to show that, so one of the aims which we want to do is to say that every signed measure can witness a difference of two measures, one of them finite, at least one of them finite. And for that, we need this decomposition into positive and negative sets.

So that measure will be of uniform sign on that particular part. And then we will show that the original measures can be written as the difference of two original signed measure. It will be the difference of two measures, one, at least one of them which will be finite. So, for that we need this notion of positive and negative sets, and we will look at it next time.