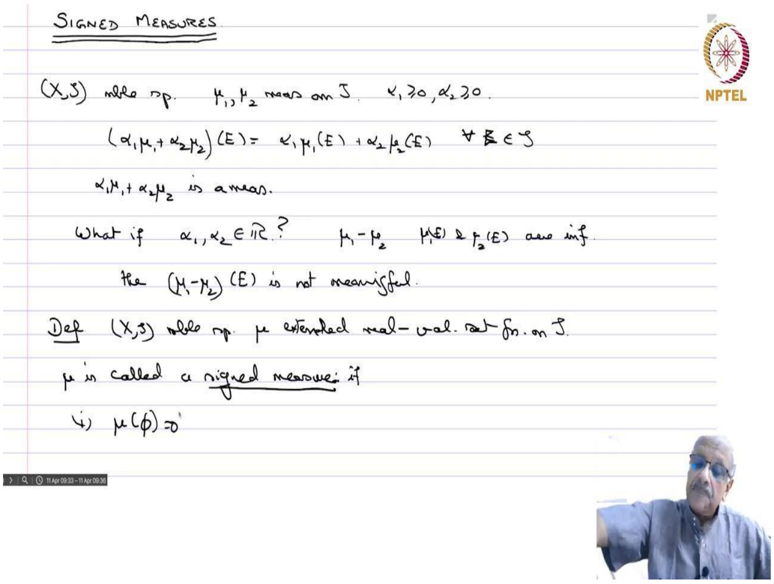
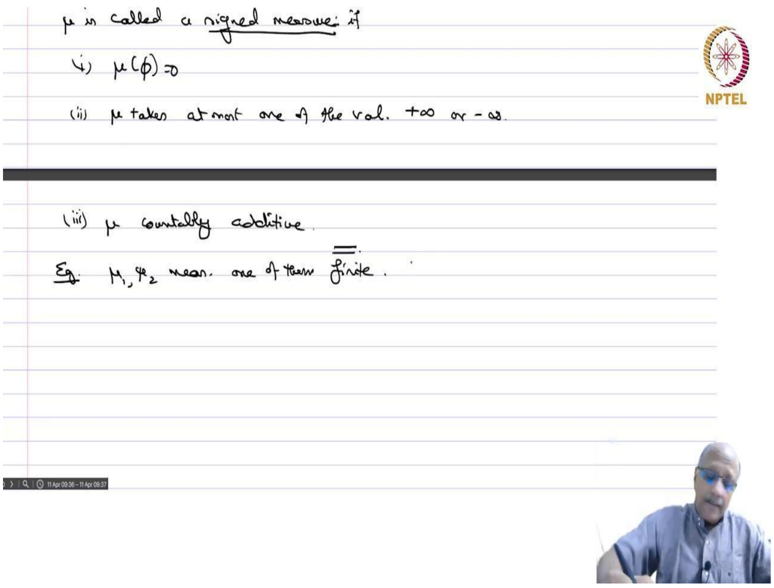
**Measure and Integration**
**Professor S. Kesavan**
**Department of Mathematics**
**The Institute of Mathematical Sciences, Chennai**
**Lecture 59**
**Signed measures**

(Refer Slide Time: 00:17)




We will now start a new chapter. This is called Signed Measures. So, let $(X, S)$ be a measurable space, and let us have $\mu_1$, $\mu_2$ measures and $\alpha_1$ and $\alpha_2$ non-negative real numbers. Then you can define

$$(\alpha_1\mu_1 + \alpha_2\mu_2)(E) = \alpha_1\mu_1(E) + \alpha_2\mu_2(E), \quad for\ every\ E \in S.$$

And then $\alpha_1\mu_1 + \alpha_2\mu_2$ is a measure. This is very, we have seen itm and it is easy to prove.

So, now, what if $\alpha_1, \alpha_2 \in \mathbb{R}$. That means, they are not just non-negative numbers. Now, so, for instance if you have $\mu_1 - \mu_2$, then the difficulty is, if you, if both $\mu_1(E)$ $and$ $\mu_2(E)$ are infinite, then$\mu_1 - \mu_2(E)$ is not meaningful. So, at least 1 of them has to be finite. And this kind of situation we already encountered in defining integrable functions. We wanted the integral f plus or the integral f minus to be finite.

Of course, we, when we define integrability, we said both of them are finite, but you can define an integral if at least one of them is finite. In the same way, if you want a measure, at least one of them should be finite. Of course, now the possibility is that at least, I mean some sets may have negative values for the measure. So, the measure will now, that is why, we call them sin measure. So based on this, we make the following definition.

So, $(X, S)$ measurable space, and $\mu$ extended real valued set function on $S$. $\mu$ is called a signed measure if one,

$(i)$ $\mu\ (\Phi) = 0,$

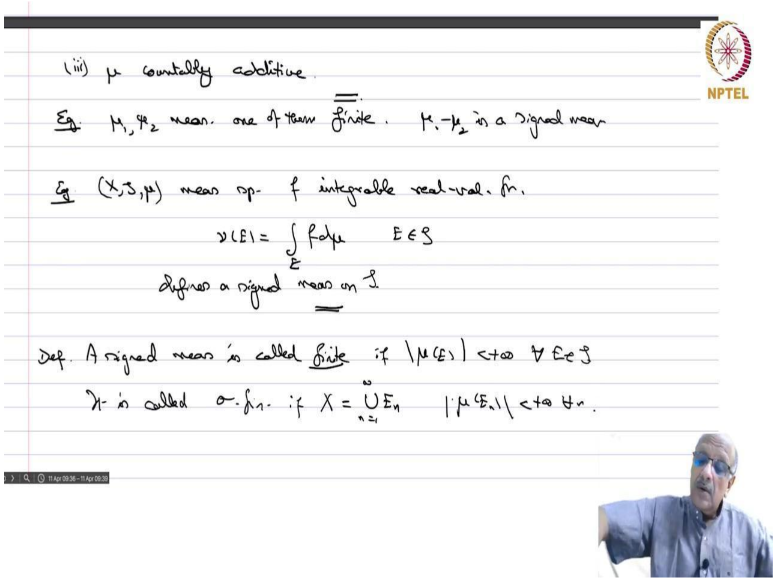
$(ii)$ $\mu$ takes at most one of the values, $+\ \infty\ or\ -\ \infty$.

That means, if at all the set has infinite measure, it will always be plus infinity or it will be always minus infinity. You cannot have some sets plus infinity some sets minus infinity because, to avoid the kind of confusion which we menti1d before.

$(iii)$ $\mu$ is countably additive.

So if we have these three properties, then you say that $\mu$ is a signed measure.

(Refer Slide Time: 04:20)




**Example:** $\mu_1$ *and* $\mu_2$ *are* measures, one of them finite. Then you have $\mu_1 - \mu_2$ is a signed measure.

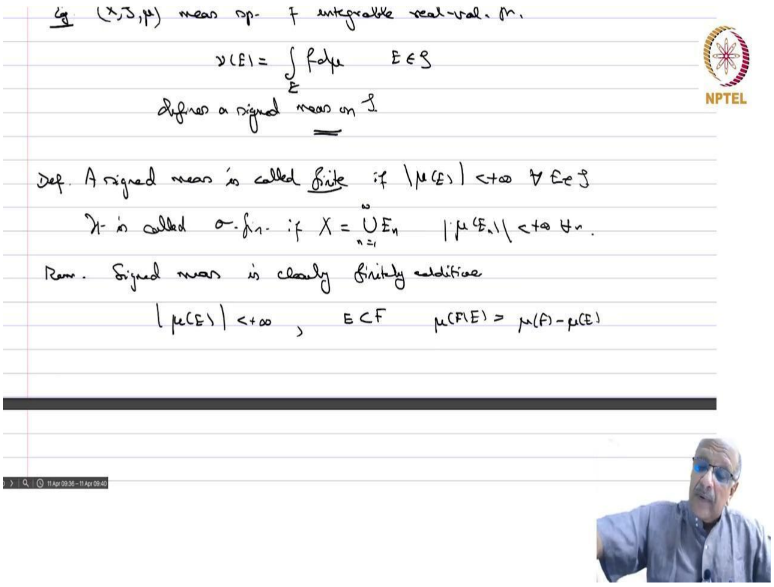
**Example:** $(X, S, \mu)$ measure space, $f$ integrable real valued function, then

$$\nu(E) = \int_E f d\mu, \quad E \in S,$$

defines a signed measure. So definition,

**Definition:** A signed measure is called finite if $\mu(E) < +\infty$ for all $E \in S$. It is called sigma finite if $X = \bigcup_{n=1}^{\infty} E_n$, $\mu(E_n) < +\infty$. So, then this is called a signed measure.

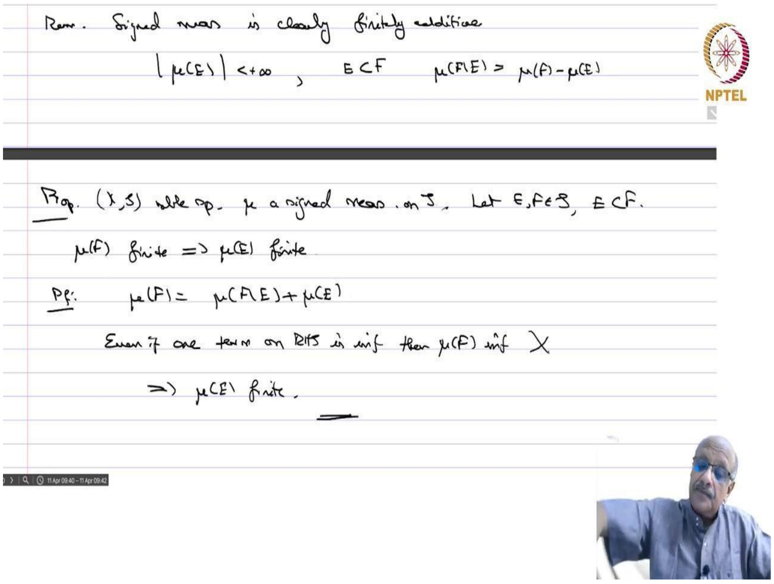
(Refer Slide Time: 06:28)




So, remarks.

**Remark:** So signed measure is clearly finitely additive, because as usual $\mu(\Phi) = 0$ so, you can fill in the phis for the to make it accountable thing, and by countable additivity, it becomes finitely additive. And if $\mu(E)$ is finite, then this implies that if E and E is a subset of F then $\mu(F\backslash E) = \mu(F) - \mu(E)$. The same proof as we had for measures because F can witness the disjoint union, F minus E union E and by finite additivity, mu F will be μ of F minus E plus $\mu(E)$ , and because μ E is finite, you are allowed to subtract. So, we have this.

(Refer Slide Time: 07:39)



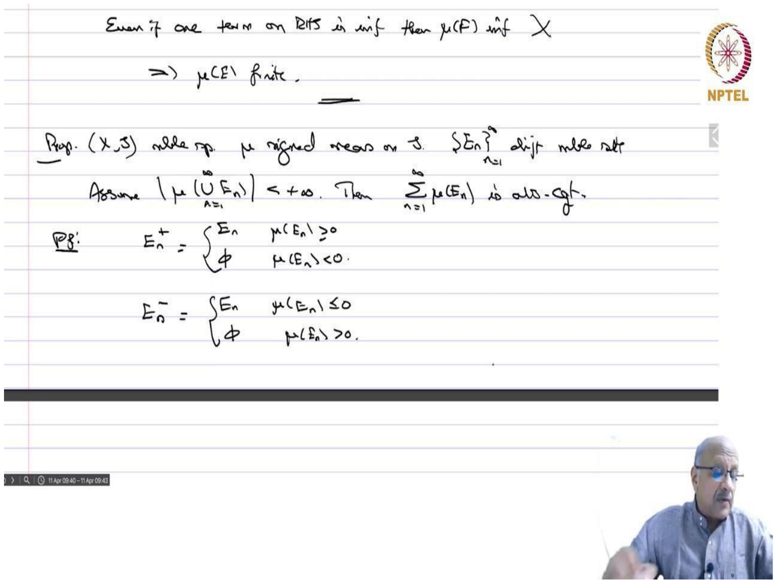

**Proposition,** $(X, S)$ measurable space μ, a signed measure on S. Let $E, F \in S,\ E \subset F$. $\mu(F)$ finite implies $\mu(E)$ is also finite.

**Proof:** So, we have finite sets, then say, subsets will also have finite measure because

$\mu(F) = \mu(F\backslash E) + \mu(E)$

Now, even if one of them is infinity, then since you always have only plus infinity or minus infinity, there is no possibility of cancellations et cetera, so, even if one of them is infinite, $\mu(F)$ will be infinite. So, even if one term on RHS is infinite then $\mu(F)$ infinite. And that is a contradiction. And therefore, this implies that $\mu(E)$ finite. So is $\mu(F\backslash E)$, of course.

(Refer Slide Time: 09:32)



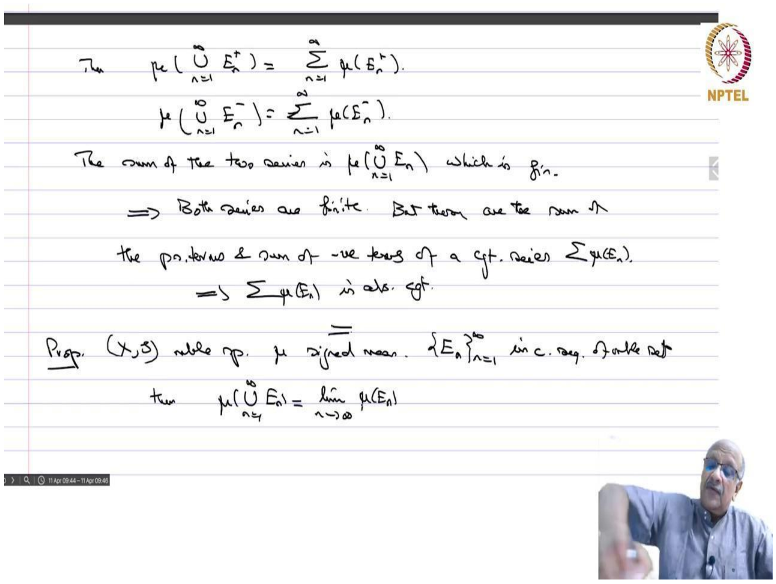

**Proposition**, $(X, S)$ measurable space, μ signed measure on S. $\{E_n\}$ disjoint measurable sets. Assume $|\mu(\bigcup_{n=1}^{\infty} E_n) <+ \infty$. Then $\sum_{n=1}^{\infty} \mu(E_n)$ is absolutely convergent.

**Proof:** We define

$E_n^{\ +} = E_n$, if $\mu(E_n) \geq 0$, empty set if $\mu(E_n) < 0$.

$E_n^{\ -} = E_n$ if $\mu(E_n) \leq 0$, and empty set if $\mu(E_n) > 0$.

(Refer Slide Time: 11:17)



Then, μ of union n equals 1 to infinity $E_n^+$ will be equal to sigma n equals 1 to infinity μ of $E_n^+$. And similarly, μ of union n equals 1 to infinity $E_n^-$ because this is the other, same $E_n$ which you have defined and they are all disjoint and so, by countable additivity, you have this is equal to sigma n equals 1 to infinity, μ of $E_n^-$. Sum of these will be 0 in the, in between.

And the sum of the two series is μ of union n equals 1 to infinity $E_n$ , which is finite. So, this implies both series are finite. But these are the sum of the positive terms and sum of negative terms of a convergent series sigma $\mu(E_n)$. This implies sigma $\mu(E_n)$ is absolutely convergent because the series is absolutely convergent if the sum of the positive terms and sum of the negative terms, independently, are convergent.

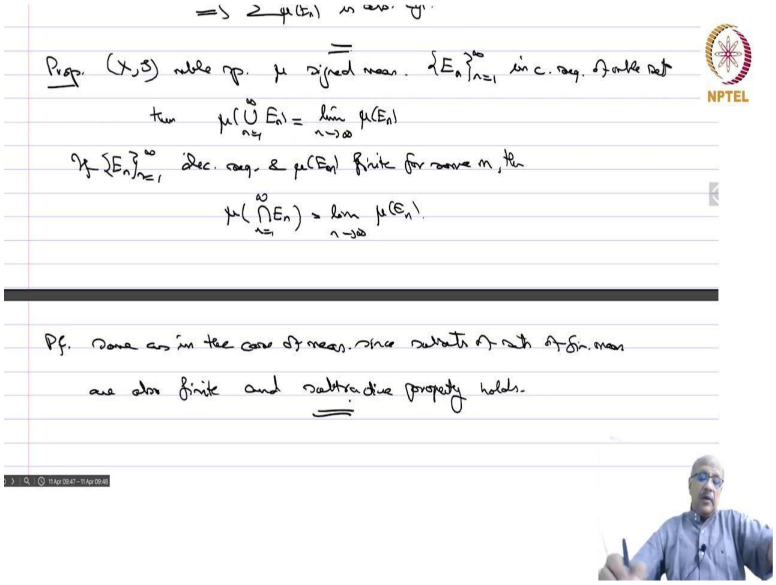
Next proposition,

**Proposition:** $(X, S)$ measurable space, μ signed measure, $\{E_n\}$ increasing sequence of measurable sets and $\mu(E_n) < \infty$ for some n , then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$$

just as in the case of measures.

(Refer Slide Time: 14:19)

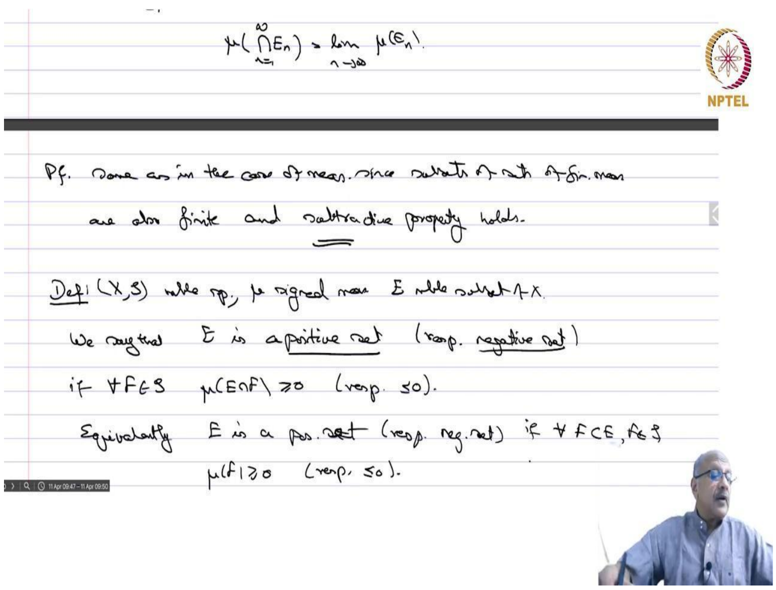



Similarly, if $\{E_n\}$ increasing sequence of measurable sets and $\mu(E_n) < \infty$ for some n, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n)$$

**Proof**, same as in the case of measures. Since subsets of sets of finite measure are also finite and subtractive property holds. So you can just copy that proof, everything is true, everything is justified and therefore, we need not go ahead for this.

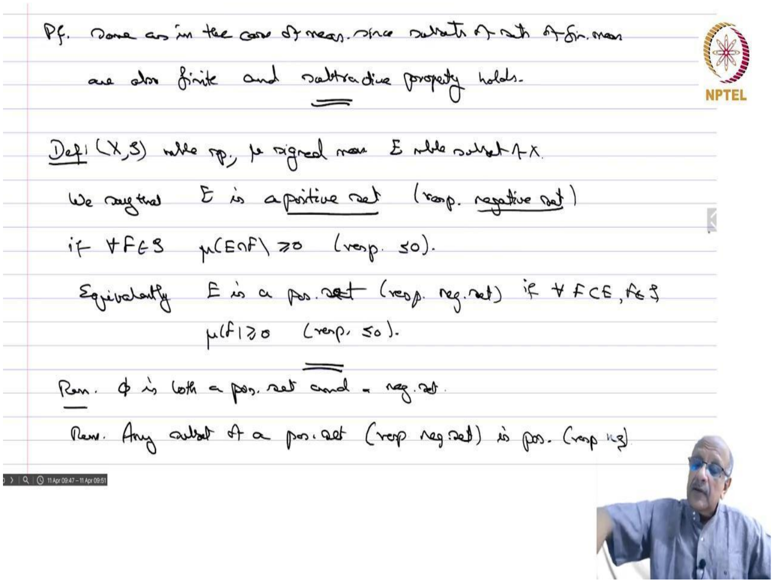
(Refer Slide Time: 15:54)




Now, we make an important definition.

**Definition:** Let $(X, S)$ measurable space, μ signed measure , E measurable set, subset of X. So we say that $E$ is a positive set, (respectively negative set) if for every $F \in S$, $\mu(E \cap F) \geq 0$, respectively, less than equal to 0. Equivalently, $E$ is a positive set, respectively negative set, if for every $E \subset F$, $F$ measurable, $\mu(F) > 0$, respectively less than or equal to 0.

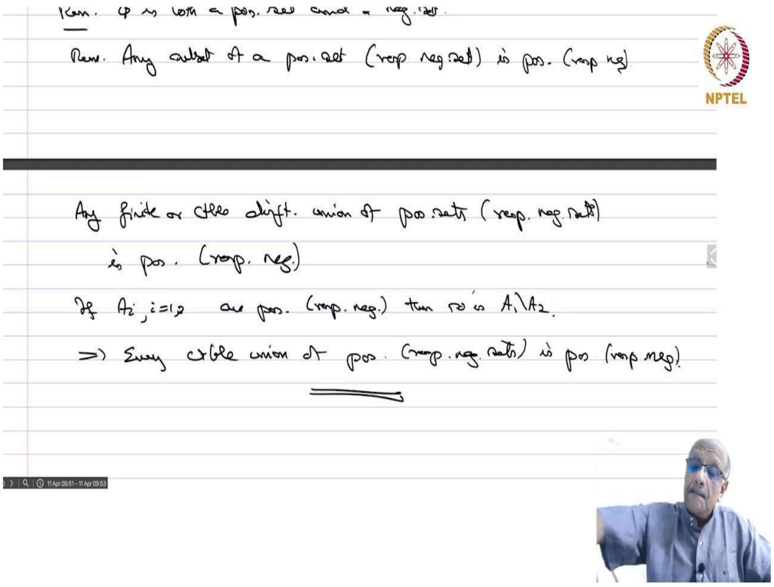
(Refer Slide Time: 17:39)




So, a positive set means a set of positive measure, and all its subsets have also, have positive measures. And similarly, then for negative, so remark, empty set is both a positive set and a negative set.

**Remark,** any subset of a positive set, respectively negative set is positive, respectively negative because further subsets are all going to have the measures of the same sign.

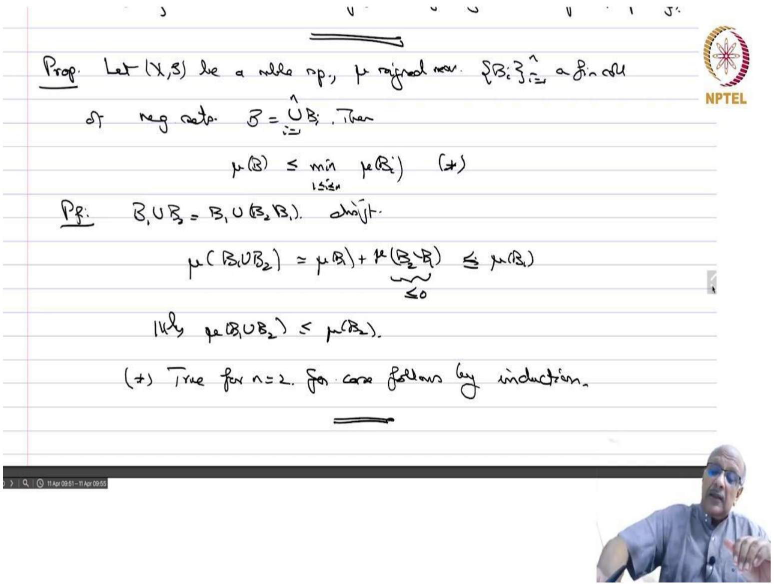
(Refer Slide Time: 18:36)




Any finite or countable disjoint union of positive sets, respectively negative sets is positive, respectively negative because if you take any subset of it, of each one of them, each intersection is going to have positive measure or negative measure depending on what the set is and therefore, it is it will be positive or negative since the all the terms of the series are of the same sign.

Now if $A_i$, i equals 1, 2 are positive, respectively negative then so is $A_1 - A_2$ because it is a subset, and therefore, so this implies that every countable union of positive, respectively negative sets, every countable union can witness as a countable disjoint union using differences. And therefore, every countable union of positive sets is a countable disjoint union of positive sets and negative sets similarly, and therefore, every countable union of positive, respectively negative sets is positive, respectively negative.

(Refer Slide Time: 20:34)



**Proposition**, Let $(X, S)$ measurable space, μ signed measure. $\{B_i\}_{i=1}^{n}$, a finite collection of negative sets. $B = \bigcup_{i=1}^{n} B_i$, Then

$\mu(B) \leq min_{\{1\leq i\leq n\}}\mu(B_i)$

**Proof.** So, $B_1 \cup B_2 = B_1 \cup (B_2 \backslash B_1)$. And this is a disjoint union.

So, $\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2 \backslash B_1) \leq \mu(B_1)$.

So, by symmetry, similarly, $\mu(B_1 \cup B_2) \leq \mu(B_2)$. So, it is called the star. So, star, true for n =2, general case follows by induction.

And that is proved. So, this, we will use this notion of positive sets and negative sets to show that, so one of the aims which we want to do is to say that every signed measure can witness a difference of two measures, one of them finite, at least one of them finite. And for that, we need this decomposition into positive and negative sets.

So that measure will be of uniform sign on that particular part. And then we will show that the original measures can be written as the difference of two original signed measure. It will be the difference of two measures, one, at least one of them which will be finite. So, for that we need this notion of positive and negative sets, and we will look at it next time.