Measure and Integration
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Lecture 58
Exercises

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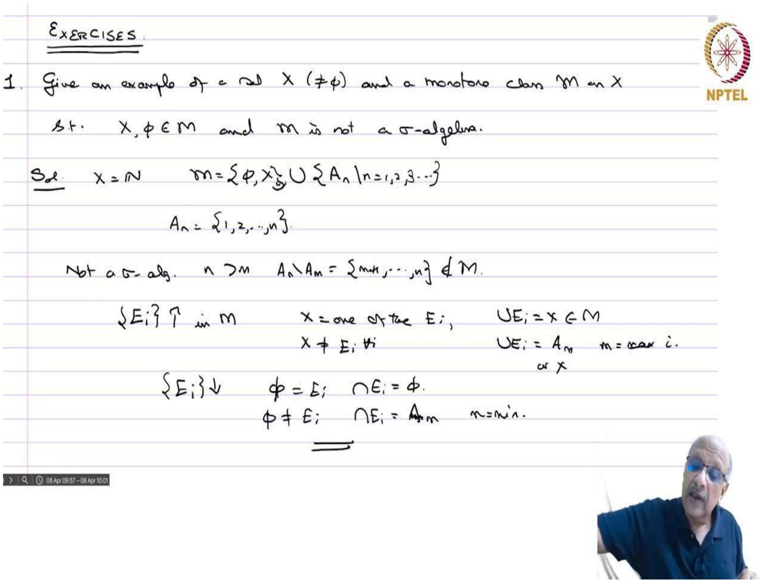

Exercises:

So now, time for some exercises.

(1): give an example of a set $X \neq \Phi$ and a monotone class $\mathfrak{M}$ on X such that X empty set belongs to $\mathfrak{M}$ and $\mathfrak{M}$ is not a σ algebra. So, we saw in many cases, if you have a monotone class with the often it, which is in algebra then generated by an algebra then it is the same as the σ algebra. So, we wanting a monotone class which is not a σ algebra and so, solution. So, let us take $X = \mathbb{N}$ and , $\mathfrak{M} = \{\Phi, X\} \cup \{A_n | n = 1, 2, 3 ...\}$ etc where $A_n = \{1, 2, 3, ..., n\}$..

So, this is not a σ algebra, since, if n is bigger than m, we have A_n minus A_m is equal to m plus 1 up to n which does not belong to $\mathfrak{M}$ and therefore, it is not closed under differences so, this is not a σ algebra. And it is a monotone class because if you take $\{E_i\}$ increasing in $\mathfrak{M}$, then if $X = E_i$, we already have that union E_i equal to X belongs to $\mathfrak{M}$. If X is not in equal to any E_i for all i, then you have union E_i is A_m where m is the max of the is because, yeah.

Similarly, if you have intersection E_i decreasing then the $\Phi = E_i$ then the intersection E_i equal to Φ otherwise if $\Phi \notin E_i$ then intersection E_i will be the minimum, will be A_m, m equals minimum, there will be a minimum in this case. You cannot have a super inf because in this case, you have this. So, here we put, or X, it could be X also. Then m is the minimum. So this is a monotone class and consequently, you have, it is not a σ algebra.

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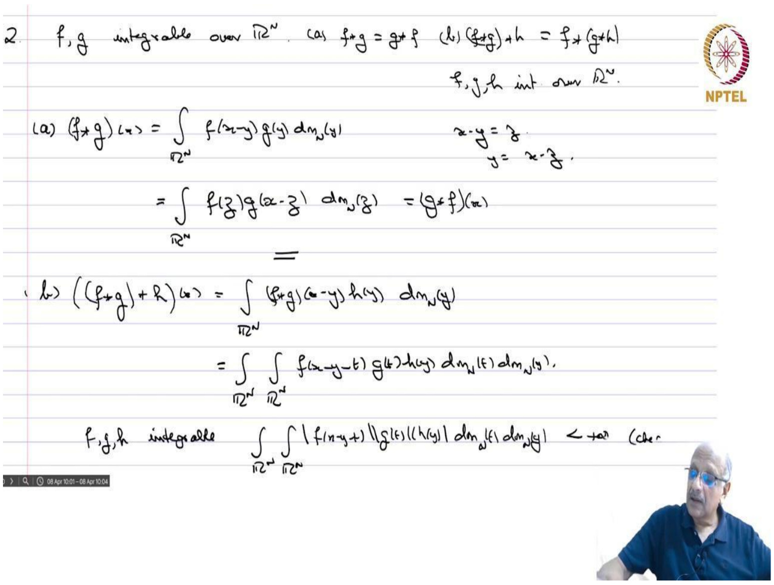

(2), Let f *and* g integrable over $\mathbb{R}^N$ and then show that

(a): $(f * g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y)dm_N(y)$

Now, in this case f star g is integrable if f and g is integrable and h is integrable, so, the convolution is defined. Similarly $g * h$ is integrable and therefore, $f * g * h$ is also integrable. So everything is well defined. So, this is just a equation of Fubini's theorem and translation invariance. So $(f * g)(x) = \int_{\mathbb{R}^N} f(\xi)g(x - \xi)dm_N(\xi) = (g * f)(x)$

And then we have seen this x already when you make a translation in $\mathbb{R}^N$, then the integral just behaves like the usual substitution law and therefore, this becomes f of z and g of x minus z d m N z. And that is equal to $(g * f)(x)$. We have seen that if you translate a function and then integrate it just integrates over the translated domain. But here, we are integrating over $\mathbb{R}^N$, and therefore the translated domain is the same.

(b). $((f * g) * h)(x) = \int\limits_{\mathbb{R}^N} (f * g)(x - y)h(y)dm_N(y)$

$$= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} f(x - y - t)g(t)h(y)dm_N(t)dm_N(y)$$

Now $\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |f(x - y - t)||g(t)||h(y)|dm_N(t)dm_N(y) < \infty$. We have checked it many times, translation invariance matters, and therefore Fubini applies.

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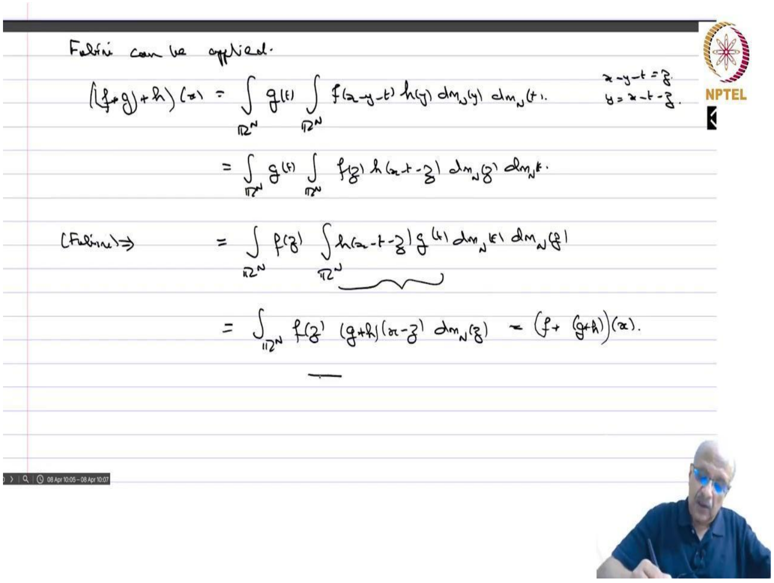

Fubini can be applied. So $((f\ *\ g)\ *\ h)(x)$ is equal to integral $\mathbb{R}^N$. Now, I will bring out g of t because it does not seem to, so I am going to integrate first with respect to y. So, you have f(x-y) integral $\mathbb{R}^N$ f(x-y-t)h(y) d m N y d m N t. Now, you put x minus y minus t equal to z. And then y equals x minus t minus z. And therefore, this becomes integral $\mathbb{R}^N$ g of t integral over $\mathbb{R}^N$, same as we did for the commutativity.

So, x minus y minus t is f of z and then h of x minus t minus z d m N z d m N t. Once again we apply Fubini, so, again Fubini. Integral $\mathbb{R}^N$ f of z, I bring out, and then the integral over $\mathbb{R}^N$ h of x minus t minus z g of t, and then d m N t d m N z. Now, this integral here, so that is equal integral over $\mathbb{R}^N$ f of z g star h at x minus z d m N z. And that is now equal to $((f\ *\ g)\ *\ h)(x)$, because whether it is $g\ *\ h\ *\ f$ or $f\ *\ g\ *\ h$, it does not matter because of the commutativity, we already have. So, that proves.

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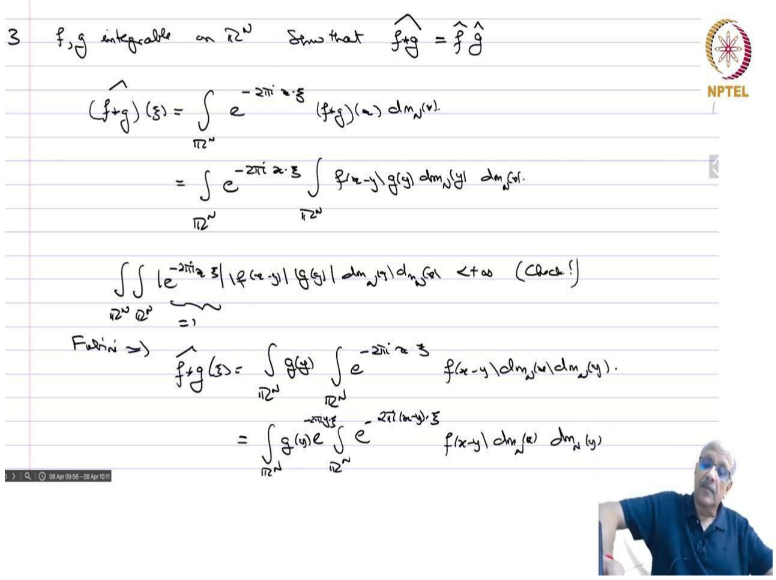

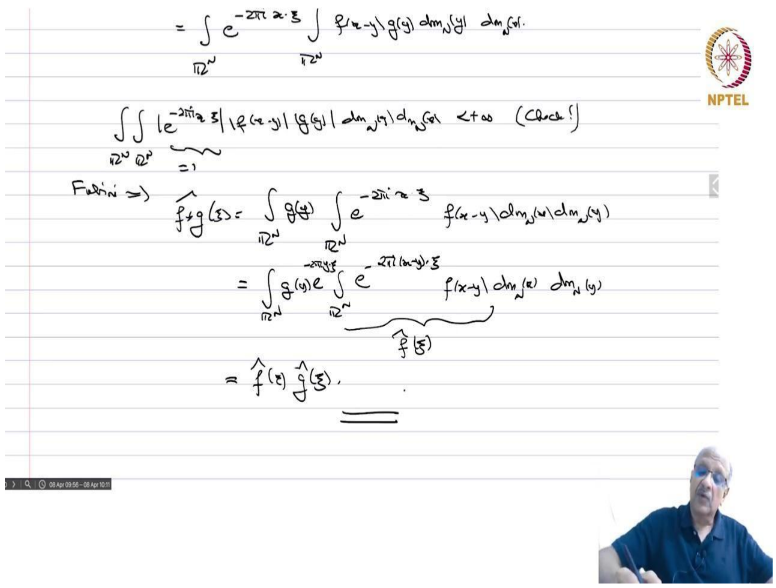

(3):, the f, g integrable on $\mathbb{R}^N$. Show that

$$\widehat{(f * g)} = \hat{f}\hat{g}.$$

This is one of the most important properties of the convolution and Fourier transform. These are used very often in the 3D analysis, partial differential equations. So, you take the convolution product and take the Fourier transform. Then it becomes the usual multiplication.

And so,

$$(f \hat{*} g)(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}(f * g)(x)\, dm_N(x)$$

$$= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} \int_{\mathbb{R}^N} f(x - y)g(y)dm_N(y)\, dm_N(x)$$

Again, $\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}$. This equal to 1 mod f of x minus y mod g y d m N y d m N x is finite.

Again, this is usual, you have done it many times. And therefore, Fubini, so, Fubini implies f star g hat psi equal to integral $\mathbb{R}^N$, g y comes out and then you have $\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} f(x - y)dm_N(x)dm_N(y)$. And that is equal the integral $\mathbb{R}^N$ g of y integral $\mathbb{R}^N$ $e^{-2\pi i(x-y)\cdot\xi}$ f of x minus y d m N x.

And then you put e power minus 2 $2\pi i y \cdot \xi$ d m N y. Now, again by translation in radians, x minus y you can change it to some z if you like, and therefore, this will become nothing but f star f hat ξ. So, you can pull it out. So, that is equal to f hat $\hat{f}(\xi)$. And what is inside is nothing but $\hat{g}(\xi)$. And therefore, you have this nice problem.

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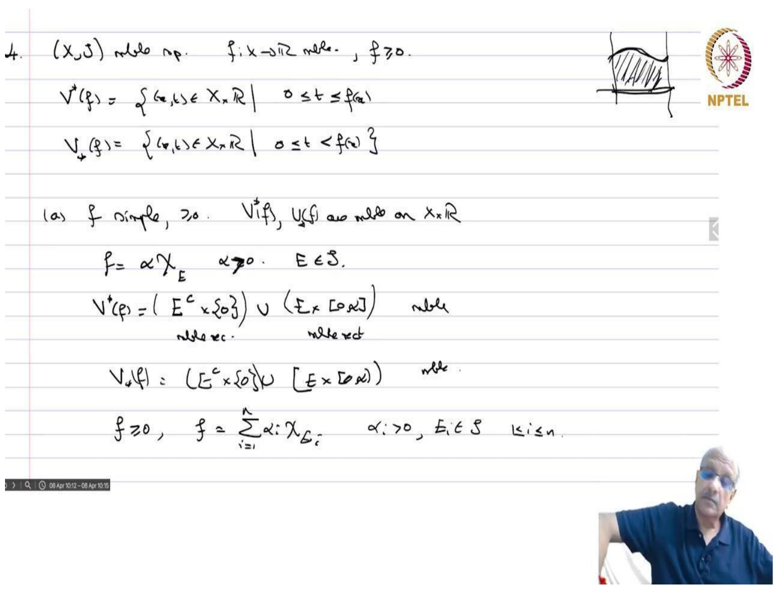

So, now, let us, the next exercise is a generalization of the idea. When we do integration, especially if you have a non-negative function. So, if I have a function say on 0 1 like this, f is some function like this then integral f over this interval is nothing but the area under this curve. So, it is the area of this set. So, we are going to generalize this notion in next exercise.

(4): Let (X, S) measurable space, $f: X \to \mathbb{R}$ measurable, and non negative. You define these

$V^*(f) = \{(x, t) \in X \times \mathbb{R},\ 0 \leq t \leq f(x)\}$

So, it is, we are precisely defining this area, and we are including these two portions. And $V_*(f) = \{(x, t) \in X \times \mathbb{R},\ 0 \leq t < f(x)\}$ So, we are excluding the boundary.

(a):, f simple non negative then $V^*(f)$, $V_*(f)$ are measurable on $X \times \mathbb{R}$. We give the usual product topology. So, here you have the Lebesgue, I mean the s and here you have the Lebesgue measures. So, let us take f equals alpha times chi of E when alpha is greater or equal to 0. Alpha is strictly positive because with 0, it is a 0 function and there is nothing for you to do, and E belongs to S.

So, then $V^*(f)$ is nothing but, so, you can take, so, E complement singleton 0 union, because if you take E complement, f is 0 and therefore, t will also be have to be forcibly 0, so, therefore, you have this cross E cross 0 alpha. And then this is measurable rectangles, and therefore, this is measurable.

So, this is a measurable rectangle, this is also measurable rectangle. And $V_*(f)$ is nothing but E complement cross 0 union E cross 0 open alpha. And again, this is measurable. So, now, let f non-zero f equals σ alpha i chi of E_i, i equals 1 to n, alpha i positive and E_i in S, 1 less than to i less than equal to n.

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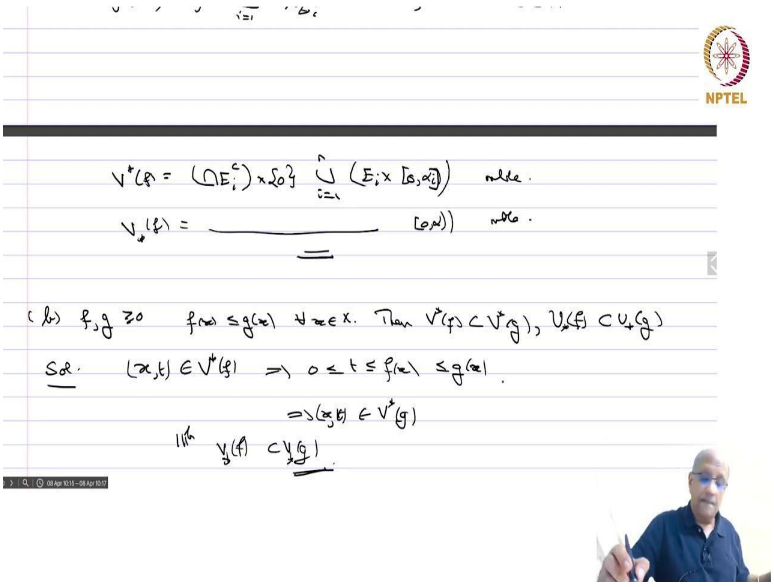

Then $V^*(f)$ equal to intersection E_i complement cross 0. This is where everything is 0, union i equals 1 to n E_i cross 0 alpha i. And then this is measurable again because it is a union of measurable rectangles. And then $V_*(f)$ is the same thing, except you have zero alpha here. So this is also measurable.

(b), Let f, g *be* non negative, $f(x) \leq g(x)$ for all x, then $V^*(f) \subseteq V^*(g)$ and $V_*(f) \subseteq V_*(g)$.

Solution: So, let us take $(x, t) \in V^*(f)$. So this means $0 \leq t \leq f(x) \leq g(x)$ and therefore, $(x, t) \in V^*(g)$. Similarly, $V_*(f) \subseteq V_*(g)$.

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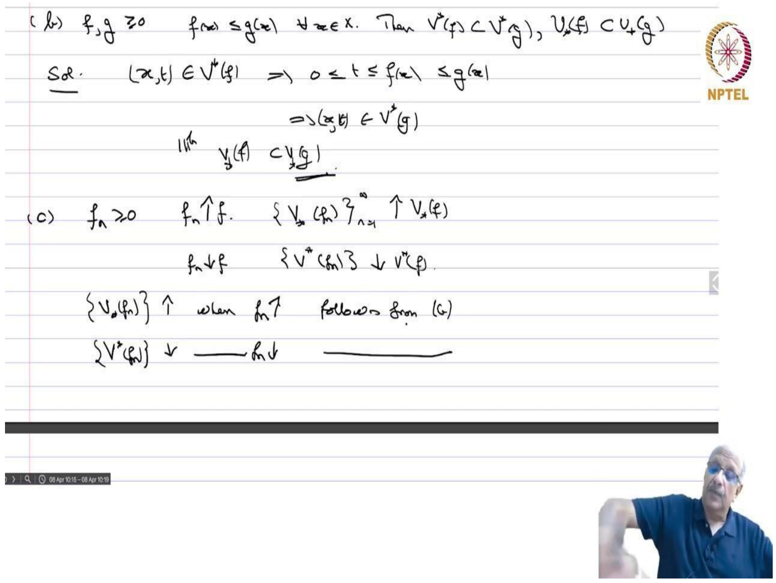

(c): Let $f_n \geq 0$ and $f_n \uparrow f$. Then $\{V_*(f_n)\} \uparrow V_*(f)$, and if $f_n \downarrow f$. Then $\{V^*(f_n)\} \downarrow V^*(f)$

So $V_*(f_n)$ increasing when f_n increasing follows from V. Similarly, $V^*(f_n)$ decreasing when f_n decreases also follows from here.

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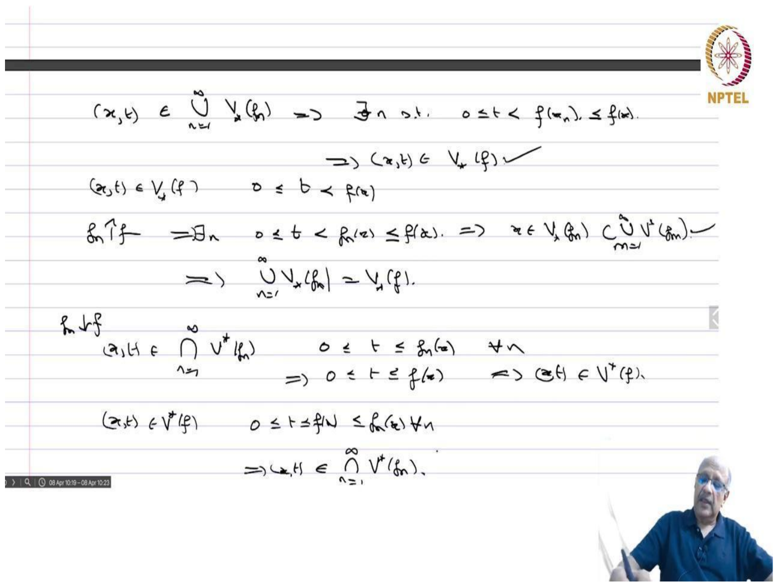

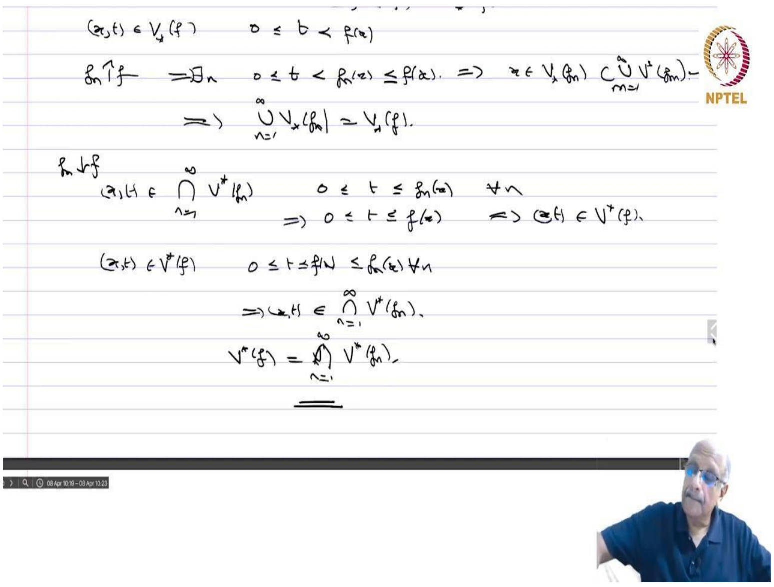

So, let us take $(x, t) \in \cup V_*(f_n)$. This means what? There exists an n such that 0 less than or equal to t less than equal to f of x n. But that goes to f of x. So, this is strictly less. That means $(x, t) \in V^*(f)$. Now if $(x, t) \in V^*(f)$, then you have that 0 less than equal to t strictly less than f of x.

In place, there exists an n since f_n, $f_n \uparrow f$, therefore there exists n such that 0 less than equal to t strictly less than $f_n(x)$, which is less than or equal to f of x. You can insert f_n because $f_n \to f$, and t is strictly less. And therefore, this implies that x belongs to $V^*(f_n)$, which is contained in union n equals 1 to infinity V^* m equals to f_n.

So, we have both in inclusions here, and therefore, you have union $V^*(f_n)$, n equals 1 to infinity equals $V^*(f)$. Similarly, for (x, t), so, $f_n \downarrow f$ if $(x, t) \in \cap V^*(f_n)$, so 0 less than or equal to t less than equal to f_n for all n, and this implies that 0 less than equal to t less than equal to f of x.

So, this implies that x belongs to, $(x, t) \in V^*(f)$. Conversely, $(x, t) \in V^*(f)$. So, this means 0 less than or equal to t less than equal to f x, but $f(x) \leq f_n(x)$ for all n, and this implies. And therefore, those two are equal. So, we have $V^*(f)$ equals V^*, intersection.

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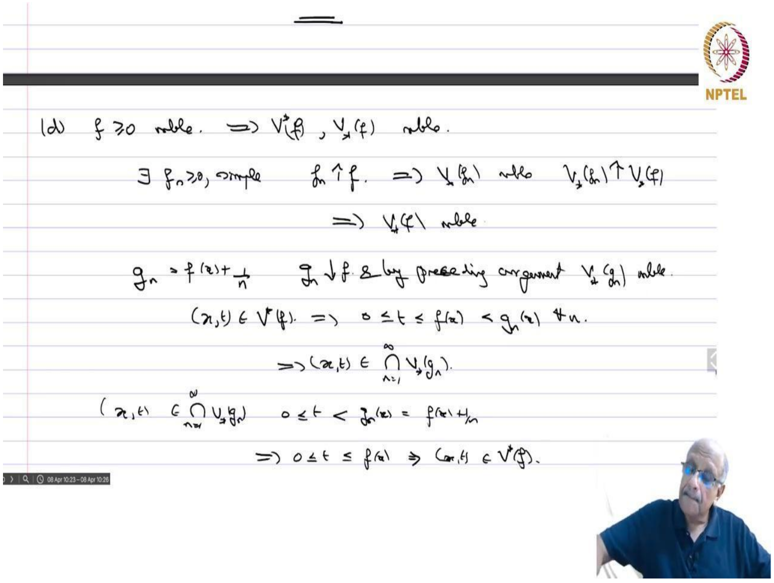

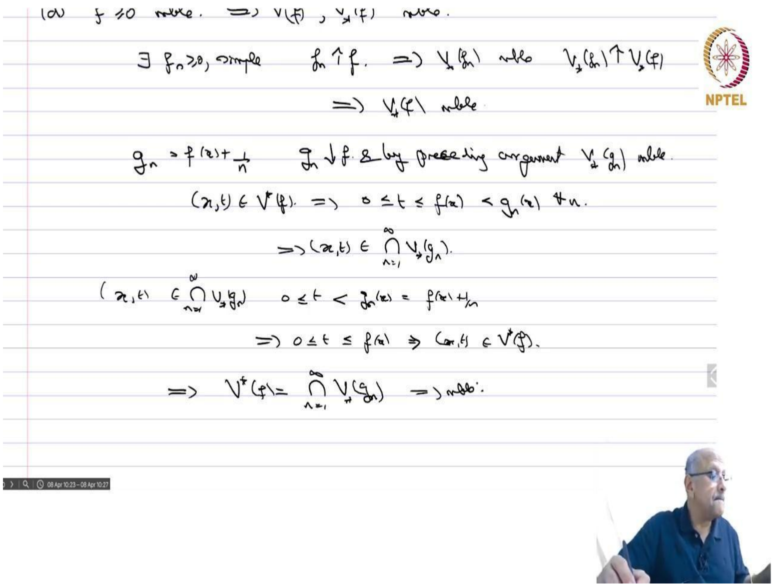

(d), f be non-negative measurable, then $V^{*}(f)$, $V_{*}(f)$ *are* measurable. So up till now, we are dealing with, just doing set theoretic arguments. So now we want to show that these two are measurable. So, f is non-negative measure therefore there exists $f_n \geq 0$ simple and $f_n \downarrow f$.

Then V n, V of, sub star sorry f_n measurable and $\{V_{*}(f_n)\}$ increases to $V^{*}(f)$ and therefore, this implies that $V^{*}(f)$ is measurable. Now, you let g n equals f of x plus 1 by n. Then g n decreases to f. And we know that by preceding, and by preceding argument, $V_{*}(g_n)$ is measurable.

So, let $(x, t) \in V^{*}(f)$. This implies 0 less than or equal to t less than or equal f of x, which is strictly less than g n of x for all n. So this implies $(x, t) \in \cap V_{*}(g_{n})$. And if $(x, t) \in \cap V_{*}(g_{n})$, n equals 1 to infinity, then 0 less than or equal to t less than g n of x which is f of x plus 1 by n.

And this implies 0 less than or equal t less than or equal f of x is equal to V^{*}. So this implies $(x, t) \in V^{*}(f)$. So therefore, we get $V^{*}(f)$ equals intersection n equals 1 to infinity $\cap V_{*}(g_{n})$. And these are all measurable, implies measurable.

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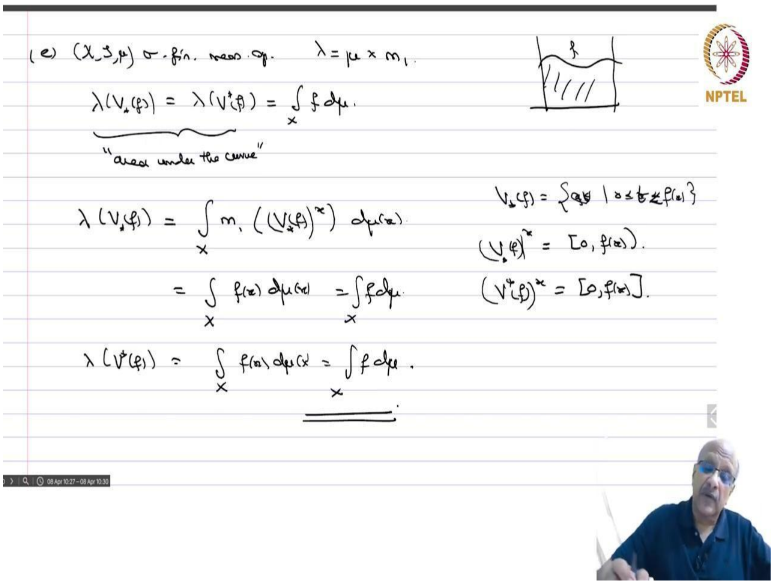

(e): (X, S, μ) σ finite measure space and $\lambda = \mu \times m_{1}$. Then

$$\lambda(V_{*}(f)) = (V^{*}(f))\lambda = \int_{X} f d\mu.$$

So, this is, you can say, area under the curve because you have, we saw that picture. So if, this is f, then this, without the boundary, is V_{*}, with the boundary, it is V^{*}. And that lambda measure is what you want. So, lambda is the product measure. And therefore, that gives you area under the curve.

So, $\lambda(V_{*}(f)) = \int_{X} m_{1}((V^{*}(f))^{x}) d\mu(x) = \int_{X} f(x) d\mu(x) = \int_{X} f d\mu$, that is a measurable set in the product measure, so is equal to, by Fubini's theorem,. What is $V_*(f)$? This equal to set of all (x, t), Since at 0 less than equal to x, t less than equal, strictly less than f of x. And that is equal to, so $V^*(f)^x$, so, for any fixed x, so, this is nothing but the interval [0, f(x)). Similarly, $(V^*(f))^x = [0, f(x)]$.

So, m 1 of this interval, this is equal to integral over x, f of x d mu x. This is equal to integral x f d mu. Similarly, $\lambda(V^*(f))$ is the same thing. So,

$$\lambda(V^*(f)) = \int_X f(x) d\mu(x) = \int_X f d\mu$$

So, you see, we have, the way the measure, product measure is defined, just generalizes the notion of area. So, we will stop with this. So, this chapter has ended. Next time, we will take up a new chapter. These are called signed measures, where measures need not be necessarily positive.