Measure and Integration
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Lecture 57
Measure of the unit ball in N dimensions

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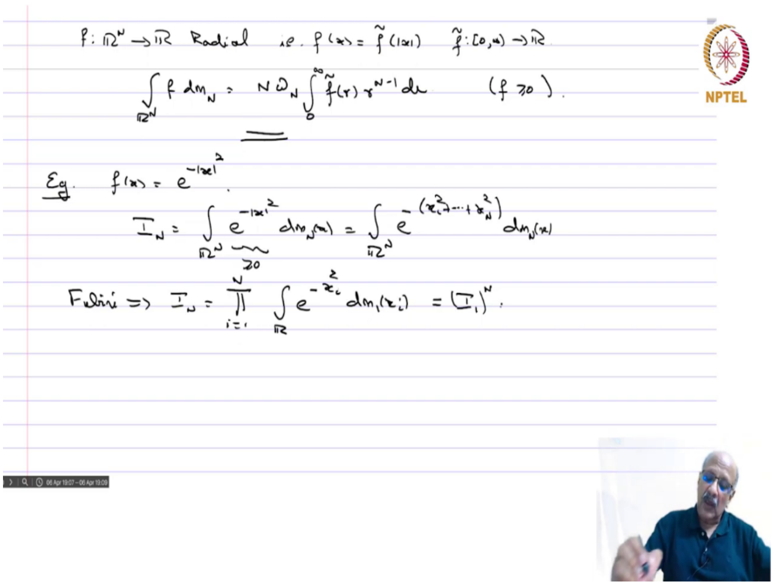

We were looking at the integration of radial functions. So, $f: \mathbb{R}^N \to \mathbb{R}$ radial, $i.e.$, $f(x) = f^{\sim}(|x|)$ some $f^{\sim}: [0, \infty) \to \mathbb{R}$. So, that is such a function, if it is a smaller domain then we will say. Now, how do we say the integral for instance

$$\int_{\mathbb{R}^N} f \, dm_1 = N\omega_N \int_0^{\infty} f^{\sim}(r) \, r^{N-1} dr \quad , \quad f \geq 0$$

So, this was the formula, we said, especially for instance if f is non-negative for instance. Then we have, it can be extended to others. We first proved it in the case of a ball and then we, by limiting arguments to you, like monotone convergence theorem or something, can prove it. For other domains also f is integrable, f is non-negative, in such cases you can extend it to this.

So, now we will see a nice example of this thing.

Example: $f(x) = e^{-|x|^2}$

So, this is the function which we want to look at. So, let us call

$$I_N = \int_{\mathbb{R}^N} e^{-|x|^2} dm_N(x) = \int_{\mathbb{R}^N} e^{-(x_1^2+\ldots+x_N^2)} dm_N(x)$$

Now, this non negative function here and therefore Fubini implies that of $I_N = \prod_{i=1}^{N} e^{-|x_i|^2} dm_1(x_i)$. And this is nothing but $(I_1)^N$. So, this is one way of looking at Fubini's, by application, applying Fubini's theorem, you get this. So, now let us try to compute I 1.

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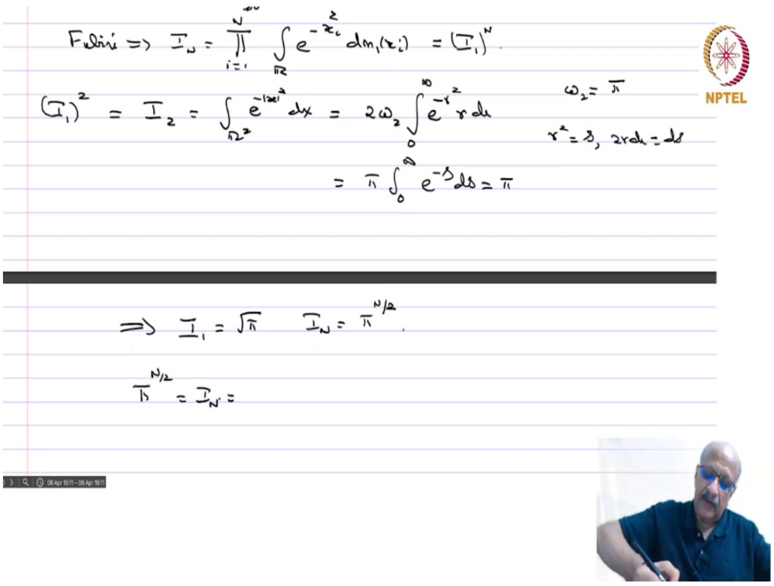

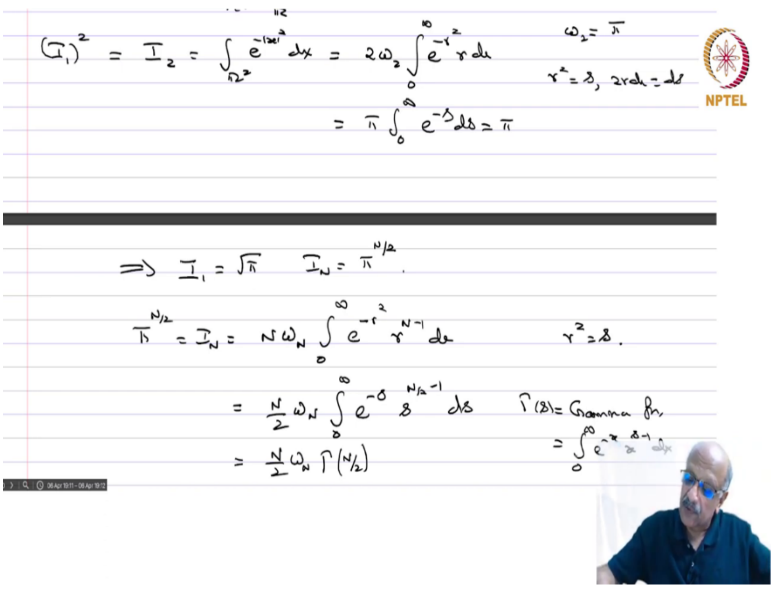

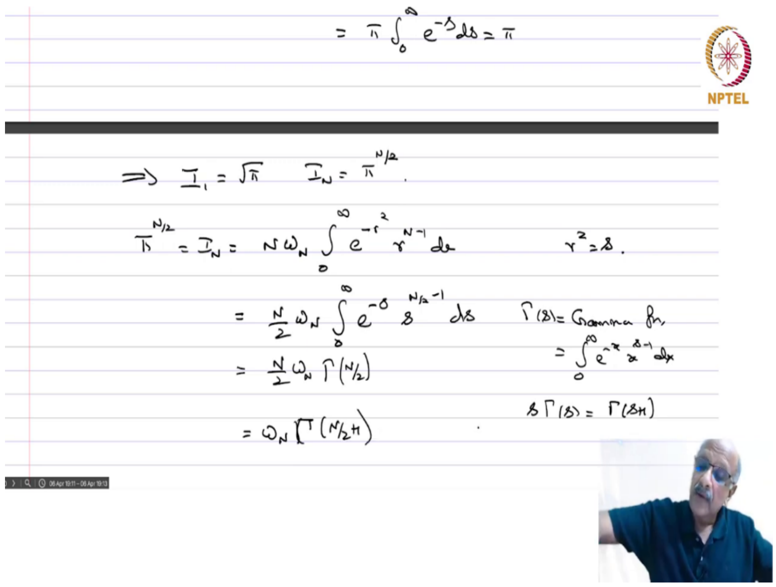

So, we start with I_2 is equal to integral over $\mathbb{R}^2$ e power minus mod x square d x, and now we will, so this is equal to I 1 square. Now, this we will apply the polar coordinate formula integration of a radial function. So, this is equal to 2 ω_2 and ω_2 is volume of the unit ball which is area of the unit circle disc and that is equal to π.

So, 2: ω_2 integral 0 to infinity e power minus r square r d r. So that is the formula which we have. Now, 2 r d r so if I put r square equal to s then 2 r d r equals d s and therefore this becomes ω_2 which isπ integral 0 to infinity e power minus s d s is equal to π. Consequently, so this implies that I 1 equal to root π and I_N, therefore, is equal to $(\pi)^{N/2}$.

Now, having determined this, we will now look at it again. So, $(\pi)^{N/2}$ equals I_N and that is equal to, if I write again e power minus mod x square is nothing but a radial function. So this is equal to N ω_N integral 0 to infinity e power minus r square r power N minus 1 d r.

So once more, I want to write r square equal to s and then this will become N by 2 ω_N integral 0 to infinity e power minus s. This r power N minus 1 d r which if I make r square equal to s transforms to s power N by 2 minus 1 d s. And that is equal to N by 2 ω_N $\Gamma(\frac{N}{2})$ where $\Gamma(s)$ is the Γ function which is equal to integral 0 to infinity e power minus s s minus 1, s power, sorry e power minus x, x power s minus 1 d x.

So this is the Γ function and therefore you have this. Now, this Γ function has some properties. You have $s\Gamma(s) = \Gamma(s + 1)$. These are all properties which you have seen in your calculus course. So this is equal to $\omega_N \Gamma(\frac{N}{2} + 1)$.

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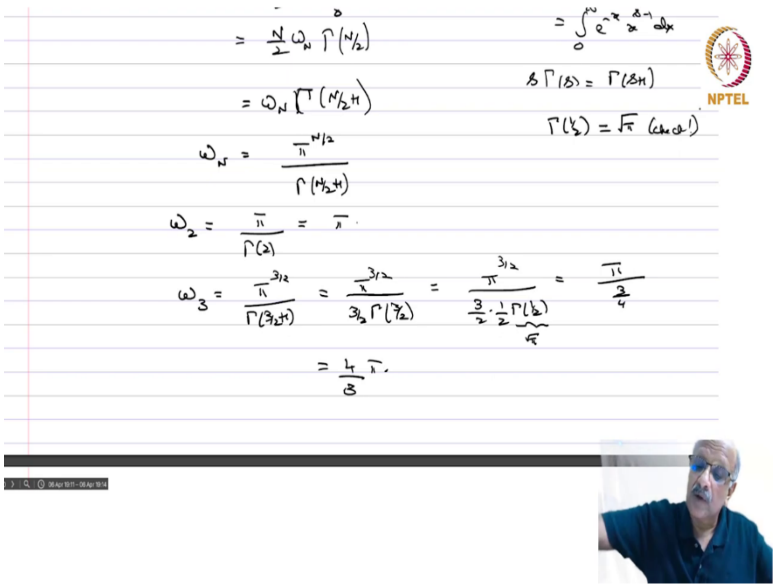

So, therefore from this, you deduce that ω_N, the volume of the unit ball is nothing, the measure of the unit ball is equal $(\pi/2)^N)$, by 2 by $\Gamma(\frac{N}{2} + 1)$. Now, $\Gamma(\frac{1}{2})$ is again the integral which you get when you do I 1. It is same as that integral and therefore you have root π. So check. The same as the integral $\Gamma(\frac{1}{2})$. So, now let us compute ω_2, for instance, from this formula. This equal toπ by $\Gamma(2)$. $\Gamma(2)$ is equal to 1 and therefore this is π. So this is the area of the unit ball.

Let us do ω_3. So this is equal to $\pi^{3/2}$ by $\Gamma(3)$ by 2 plus 1 which is $\pi^{3/2}$ by 3 by 2, $\Gamma(s + 1) = s\Gamma(s)$, Γ 3 by 2. 3 by 2 is 1 plus half and therefore you have this equal to $\pi^{3/2}$ by 3 by 2 into 1 by 2 into $\Gamma(\frac{1}{2})$ is equal to, so this is equal to root π. And therefore that will cancel here, so with one half with one and, so this will just give you $\pi/3$ by 4 is equal to 4 by 3 π which is the formula you know for the volume of the unit ball in r 3.

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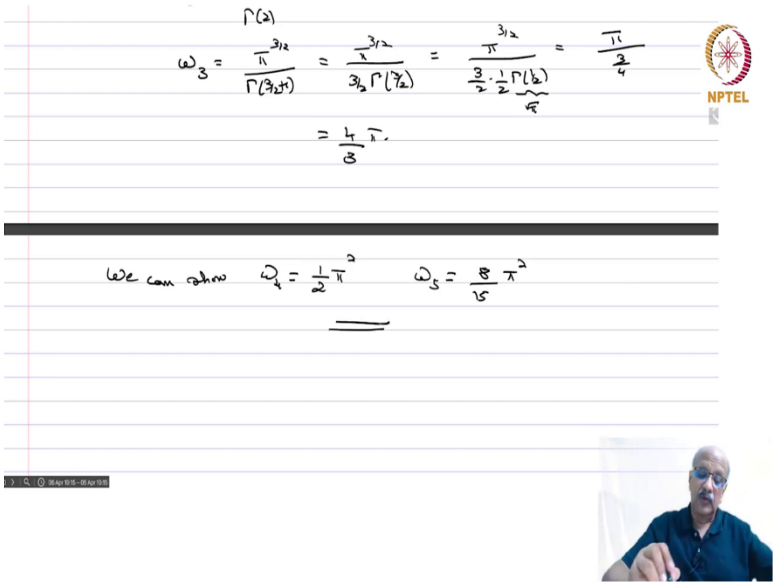

Similarly, we can show ω_4, the measure of the unit ball in four dimensions is one half π square and ω_5 which is the measure of the unit ball in r 5 is 8 by 15. You just apply the formula and use the fact that $\Gamma(s + 1) = s\Gamma(s)$. So, this tells you how to compute the volume of the unit ball in arbitrary space dimensions.