# Measure and Integration

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## Lecture 56

## Integration of radial functions

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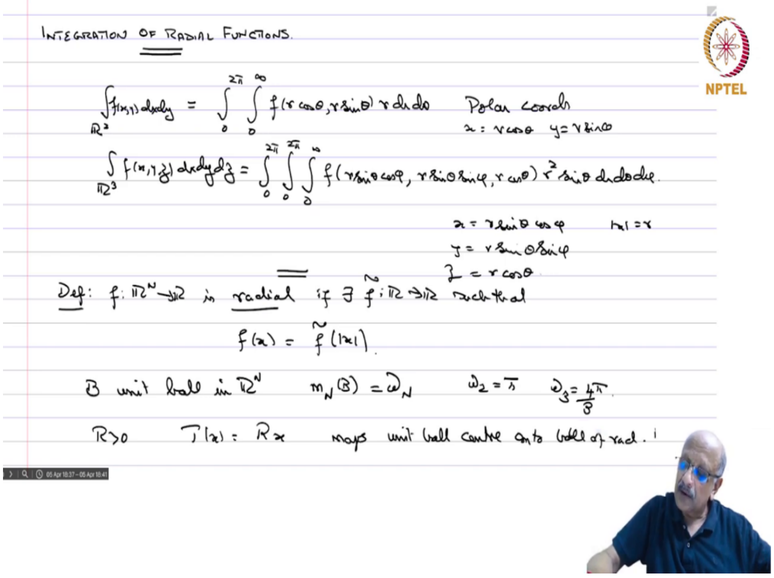


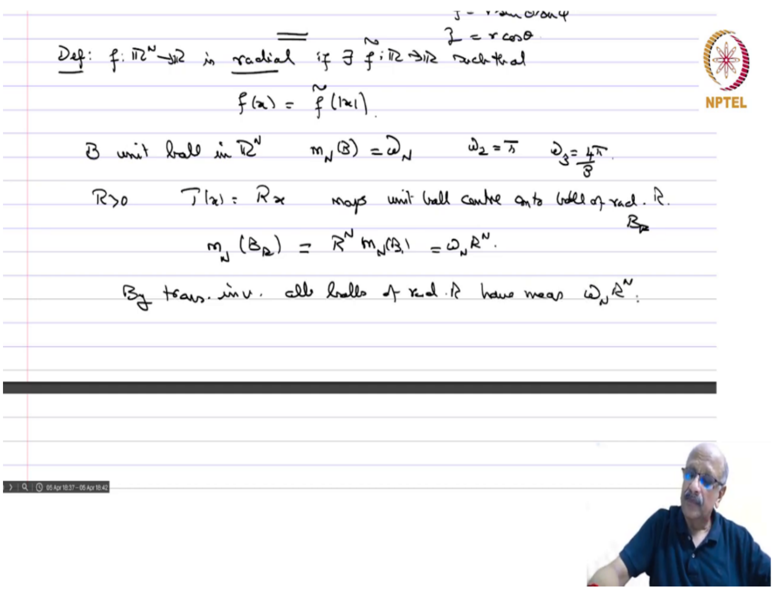


We will now look at **integration of radian functions**. So, when doing Riemann integration, let us say in two dimensions, then you would have written, seen such $\int_{\mathbb{R}^2} f(x, y) dx\, dy = \int_0^{2\pi} \int_0^{\infty} f(r \cos\theta, r \sin\theta) r\, dr\, d\theta$. So this is the change to polar coordinates, so this is polar coordinates. So, $x = r\cos\theta$; $y = r\sin\theta$.

Similarly,

$$\int_{\mathbb{R}^2} f(x, y, z) dx\, dy dz = \int_0^2 \int_0^{2\pi} \int_0^{\infty} f(r \sin \theta \cos \varphi,\ r \sin \theta \sin \varphi,\ r \cos \theta) r^2 \sin \theta\, dr\, d\theta\, d\varphi,$$

So, here you have the polar coordinates $x = r \sin \theta \cos \varphi,\ y = r \sin \theta \sin \varphi,\ z = r \cos \theta.$

So, when you use polar coordinates you can change the integrals like this as you might have seen when doing Riemann integration, double integrals, triple integrals and so on. Now, one can do, justify most of these for Borel measurable functions and the Lebesgue measure associated with the integration with Lebesgue measure. But as you go to n dimensions, these formulae become more and more horrendous and difficult to write down.

Now, what we will see here is a very easy way to integrate radial functions. So, the definition

**Definition:** $f: \mathbb{R}^N \to \mathbb{R}$ is radial if there exists $f^{\sim}: \mathbb{R}^N \to \mathbb{R}$ such that

$$f(x) = f^{\sim}(|x|).$$

So, the function depends only |x|, namely here |x|=r in both the cases, so it depends only on the r variable, the theta and other variables do not matter. So, such a function is called a radial function. And we would like to see how to integrate a radial function over $\mathbb{R}^N$.

So, let $B$ be the unit ball in $\mathbb{R}^N$ and its measure so $m_N(B_R) = R^N m_N(B_1) = \omega_N R^N$. So $\omega_2$ is the area of the unit circle which is pi, $\omega_3$ is the volume of the unit ball in r by 3 in r 3 which is 4 by 3 pi etcetera. General value of $\omega_N$ we will see a little later. So, if R equal, if R is positive then $T(x) = Rx$ maps unit ball centre 0 to, onto a ball of radius R. And this is a linear relationship and you know how.

Therefore $m_N(B_R)$, so $B_R$ is the ball of radius R centre 0, so this is nothing but you must, you might we have seen already is the determinant of the map. Now, the map here is the diagonal map R R R R R, so it is $R^N$ into the measure of the $m_N(B_R) = R^N m_N(B_1) = \omega_N R^N$ . So, by translation, invariance of any ball, all balls of radius R have measure $\omega_N R^N$.

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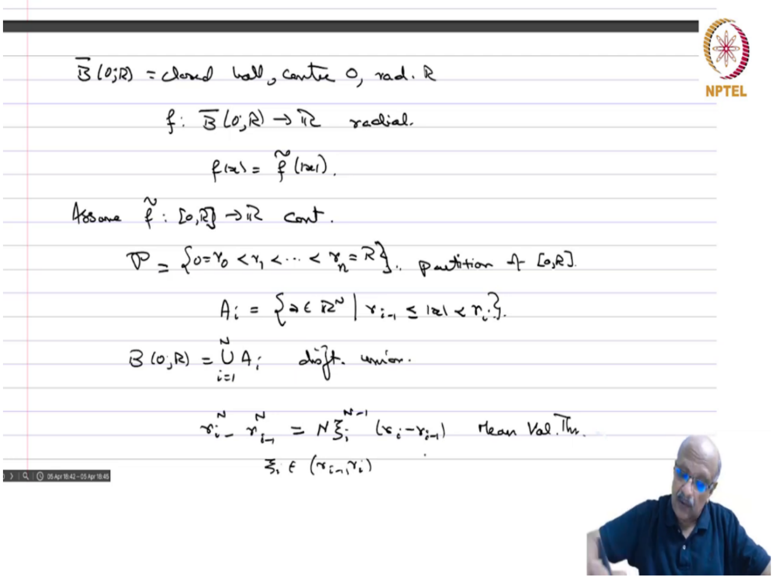


So, now let $\overline{B}(0; R)$ = is the closed ball centre 0, radius R. And $f: \overline{B}(0; R) \to \mathbb{R}$ is radial. So, that means $f(x) = f^{\sim}(x)$ for some $f^{\sim}$. So, now assume $f^{\sim}: [0, R] \to \mathbb{R}$, is continuous, therefore it is also uniformly matrix. Now you take any partition of the interval [0 R].

So, that is equal to 0 equals r naught less than r 1 less than etcetera less than r n equals capital R. So, partition of [0 R]. And you set $A_i = \{x \in \mathbb{R}^N;\ r_{i-1} \leq |x| < r_i\}$. So, there you have that $B(0; R)$ equals union i equals 1 to n $A_i$ disjoint union.

Now, if you take $r_i$ power N minus $r_i$ minus 1 power N, then this is equal to N $\xi_i$ power n minus 1 $r_i$ minus $r_i$ minus 1. This is the mean value theorem. Difference of the value of the function R power N at two points is the value of the derivative at some intermediary point, so you have 0, sorry $r_i$, $\xi_i$ belonging to $r_i$ minus 1 $r_i$. So, this is the mean value theorem.

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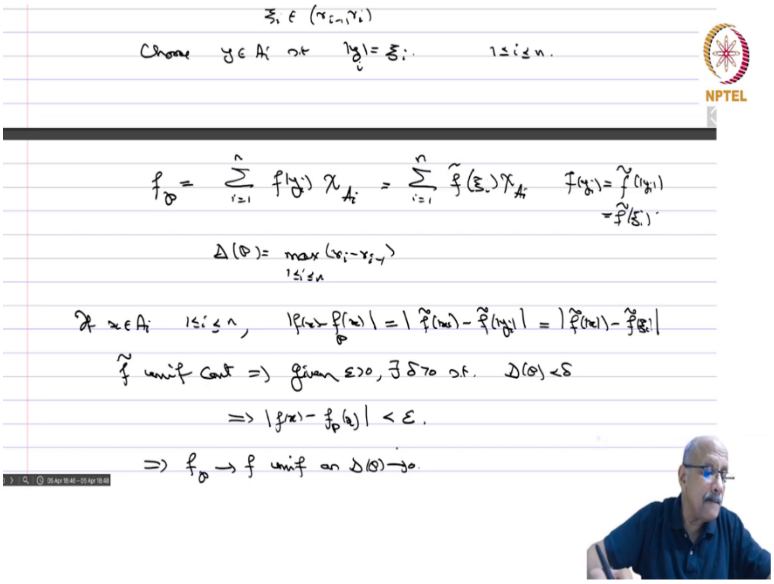


So, choose y in $A_i$ such that mod y i equal to $x_i$ i. So, 1 less than equal to i less than equal to n, we do this for. Now, define the function f of p equal to sigma i equals 1 to n of f of y i chi times $A_i$, that is also equal to sigma i equals 1 to n $f^{\sim}(x_i)$ chi of $A_i$, since f of y i is $f^{\sim}$ of mod y i which is $f^{\sim}(x_i)$ .

So, now you let delta P to be the max of $r_i$ minus $r_i$ minus 1, 1 less equal to i less than equal to n. If x belongs to $A_i$ then what is 1 less equal to i less than equal to n, then you have mod f x minus f of f p of x, this equal to mod of $f^{\sim}$ of mod x minus $f^{\sim}$ of mod y i is equal to mod $f^{\sim}$ of mod x minus $f^{\sim}$ of $x_i$ i.

Now, $f^{\sim}$ is uniformly continuous because it is continuous on $[0, R]$ so this implies given epsilon positive, there exists delta positive such that delta P less than delta implies mod f x minus f P of x which is f of x minus f P of x is nothing but f x minus f of $\xi_i$ mod x minus $\xi_i$ will be less than delta P which is less than small delta and therefore this will be less than epsilon. Therefore, you have f P converges to f uniformly as delta P goes to 0.

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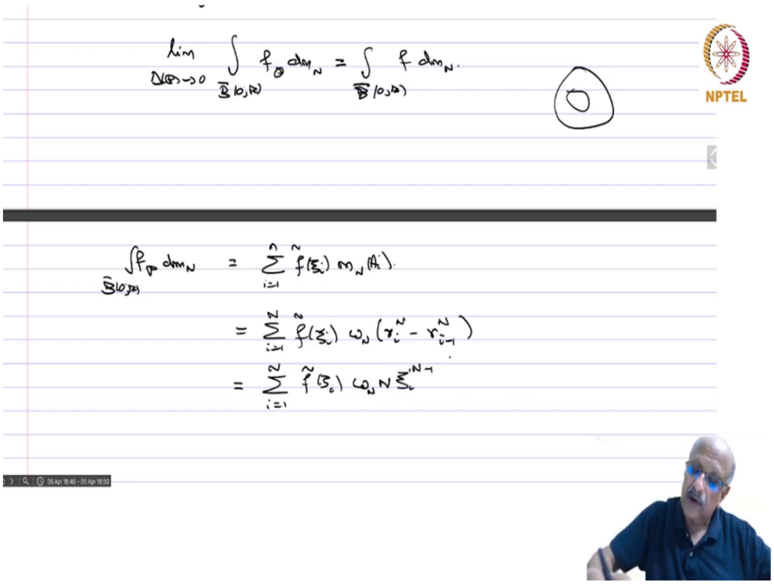


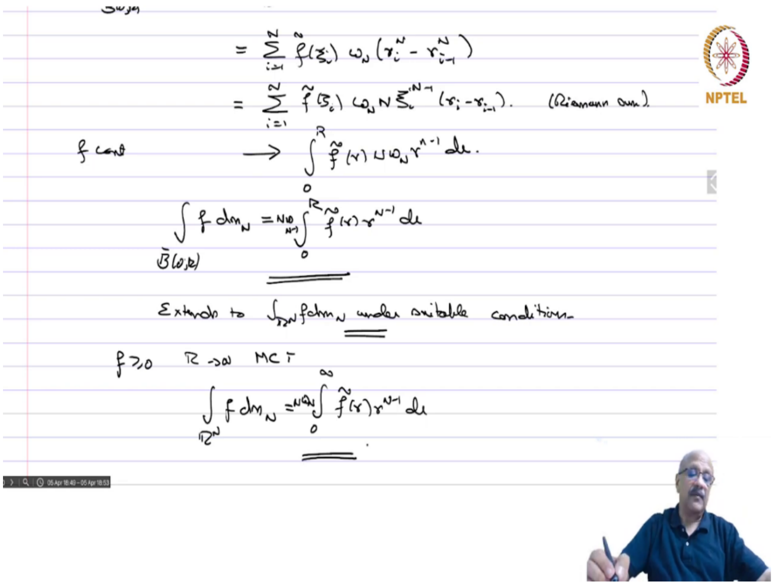

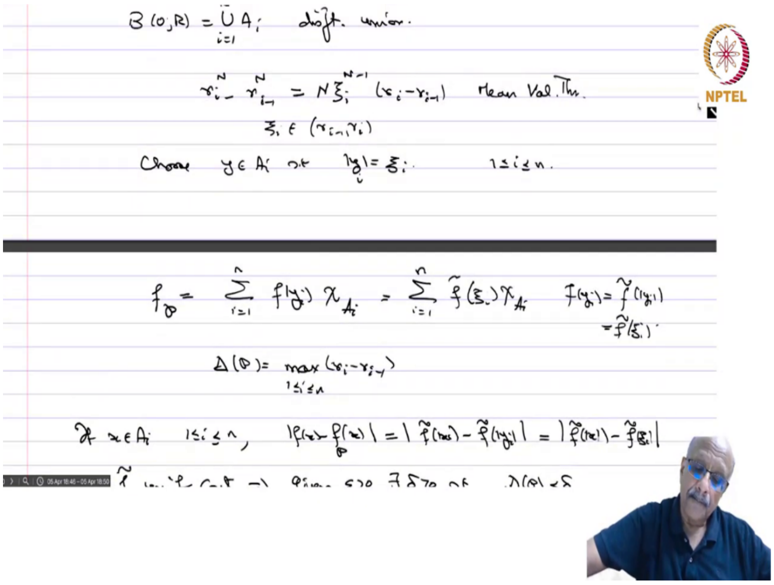

So, now you have if you have a set of finite measure, we have done this exercise or it was the assignment, if you have a set of finite measure then of course if you have uniform convergence then the integral also will converge. So, limit delta P tending to 0 of integral $B(0; R)$ of f P $dm_N$ will be equal to integral $B(0; R)$ of f d$m_N$.

So, let us take what is integral f p d m N. That is, since it is a simple function this is equal to sigma i equals 1 to n f of $\xi_i$ into measure m N of $A_i$ that is equal to sigma i equals 1 to N f of $x_i$ I, what is the measure of $A_i$? $A_i$ is as annular region and therefore that is equal to $\omega_N$ so $r_i$ power N minus $r_i$ minus 1 power N.

And that, we know, is sigma i equals 1 to N by the mean value theorem $f^{\sim}(x_i)$ i, $f^{\sim}(x_i)$ i $\omega_N$ N $\mathbb{R}^N$ minus 1, sorry $\xi_i$ to the N minus 1, this is what we saw earlier, N $\xi_i^{N-1}$ $r_i$ minus 1 minus r. So, f is continuous. That means this converges, this is exactly a Riemann sum so this converges to integral 0 to R of $f^{\sim}(r)$, r N $\omega_N$, r power n, sorry, N $\omega_N$ r power N minus 1, Riemann sum.

So, $\int_{\overline{B}(0,R)} f\, dm_N = \int_0^R f^{\sim}(r)\, r^{N-1}\, dr_N \omega_N$. If f is non negative and integrable, then of course we can extend this by monotone convergence theorem, dominated convergence theorem, et cetera to all of $\mathbb{R}^N$ also. So, extends to integral on $\mathbb{R}^N$ of $f dm_N$ under suitable conditions.

So, for instance $f$ non-negative then you take R tending to infinity then by monotone convergence theorem we

$$\int_{\mathbb{R}^N} f \, dm_N = \int_0^{\infty} f^{\sim}(r)\, r^{N-1}\, dr_N \omega_N.$$

So, we have such formula. We will see a nice application of this.