Measure and Integration
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Lecture 55
The Institute of Mathematical Sciences
8.7 - Examples

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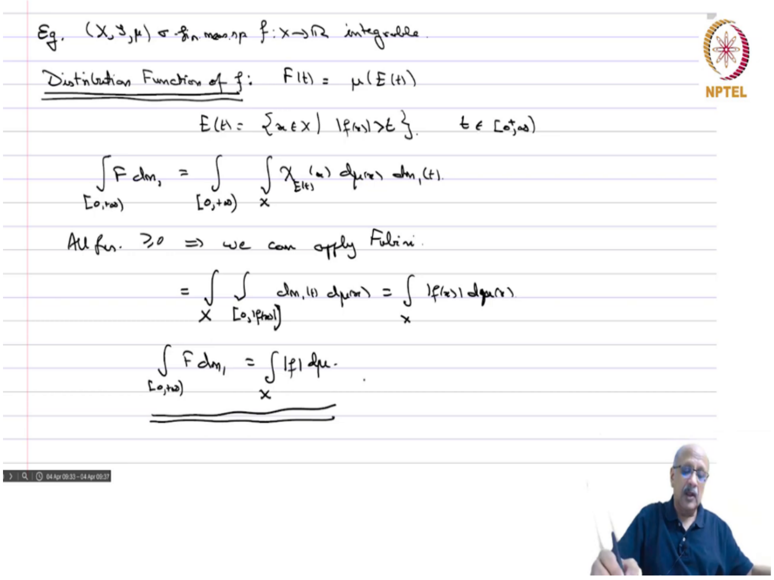

We continue examples of applications of Fubini's theorem. So, example

Example: (X, S, μ) measurable space $f: X \to \mathbb{R}$ integrable. Now, the **distribution function** of f, this is the function

$F(t) = \mu(E(t))$ where $E(t) = \{x \in X \mid |f(x)| > t\},\ t \in [0, \infty)$.

So, these are called the level sets. So, the measure of the **level sets** gives you what is called the distribution function of f.

So, let us look at in, $t \in [0, \infty)$, because we are taking $|f(x)|$. So, now if you look at the

$$\int_0^\infty F dm_1 = \int_0^\infty \int_X \chi_{E(t)} d\mu(x) dm_1(t)$$

So, now we are dealing with all functions that are non negative, which implies we can apply Fubini, so X, S, mu, σ finite measure space. So, we can apply Fubini's theorem. And so what is Fubini's theorem? So, this is

$$\int_0^\infty F dm_1 = \int_X \int_0^{|f(x)|} \chi_{E(t)} dm_1(t) d\mu(x) = \int_X |f(x)|\, d\mu(x)$$

So, we will generalize this later on when we study L^p spaces but for the moment we have these results.

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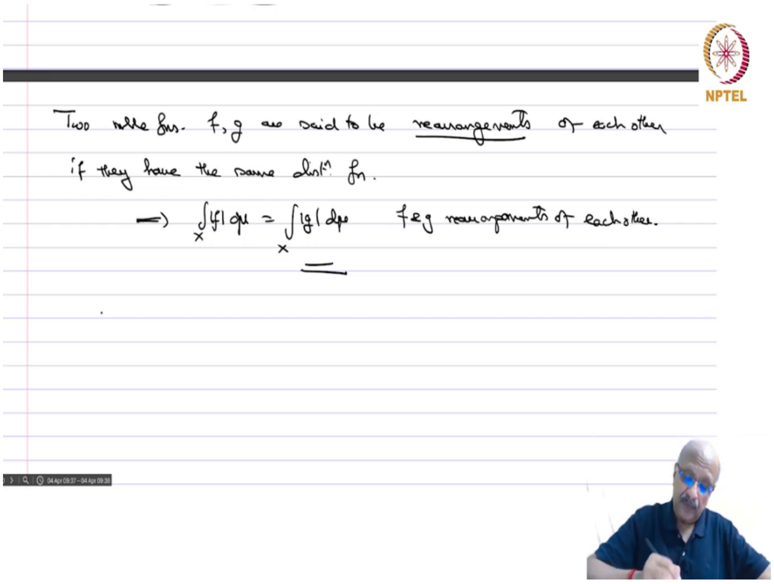

Now, two functions, two measurable functions f and g are said to be rearrangements of each other if they have the same distribution functions. And therefore, this implies that

$$\int_0^\infty F dm_1 = \int_X |f(x)|\, d\mu(x).$$

So, rearranging a function preserves the integral, so that is a nice thing and rearrangements are very useful, lots of applications in what are called comparison theorems for solutions of partial differential equations, so we, they use a lot of these tools.

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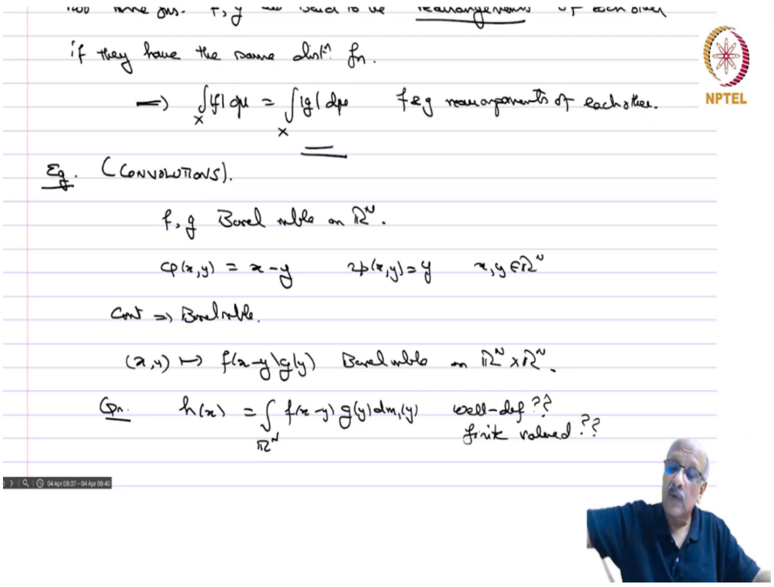

So, now we will have the next example very very important example this is the definition of convolutions.

Example: So, let us take f and g Borel measurable on $\mathbb{R}^N$. Now, you consider the function f of x, y going to $\varphi(x, y) = x - y$, and $\psi(x, y) = y$, so $x, y \in \mathbb{R}^N$. These are continuous implies they are also Borel measurable and we have seen when it involves Borel measurable functions then composition is also Borel measurable.

And therefore, you have (x, y)→ $f(x - y)g(y)$ is Borel measurable on $\mathbb{R}^N \times \mathbb{R}^N$. So, we want to know, so question

$$h(x) = \int_{\mathbb{R}^N} f(x - y)g(y)dm_1(y)$$

well defined or not? So, this is a question which you want to ask and we are going to try to answer the using Fubini's theorem.

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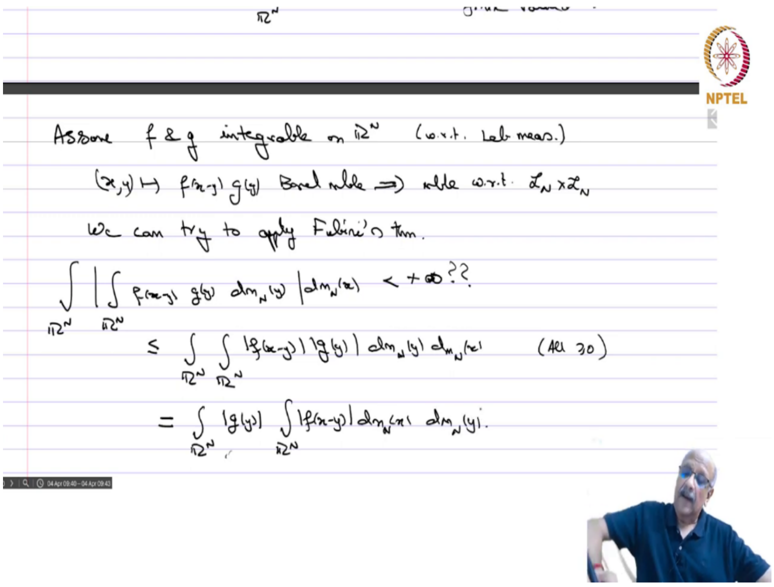

Now, assume, so assume f and g integrable on $\mathbb{R}^N$, of course, with respect to Lebesgue measures. Now, f and g, h, so $f(x - y)g(y)$, (x y) going to this is Borel measurable and therefore implies measurable with respect to $L^N \times L^N$, we have already seen that the Borel measurable on $\mathbb{R}^{2N}$ is contained in $L^N \times L^N$ which is contained in L^{2N}.

Therefore, it is also Borel measurable with respect to the product measure and therefore we can apply Fubini's theorem. So, we have a function which is measurable with respect to the product measure and so we can. Now, what is part B of Fubini's theorem which says how to find if a function is integrable or not.

So, we have to take

$$\int_{\mathbb{R}^N} |f(x - y)g(y)| dm_N(y)$$

this is a function of x and then we want to, sorry, take modulus of this function and then this is the x section and then we want to write integral $\mathbb{R}^N$ dm_N, is this finite? Then we can apply Fubini's theorem.

$$\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} |f(x - y)g(y)| dm_N(y) \right| dm_N(x) \le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x - y)||g(y)| dm_N(y) dm_N(x)$$

$$= \int_{\mathbb{R}^N} |g(y)| \int_{\mathbb{R}^N} |f(x - y)| dm_N(x) dm_N(y),$$

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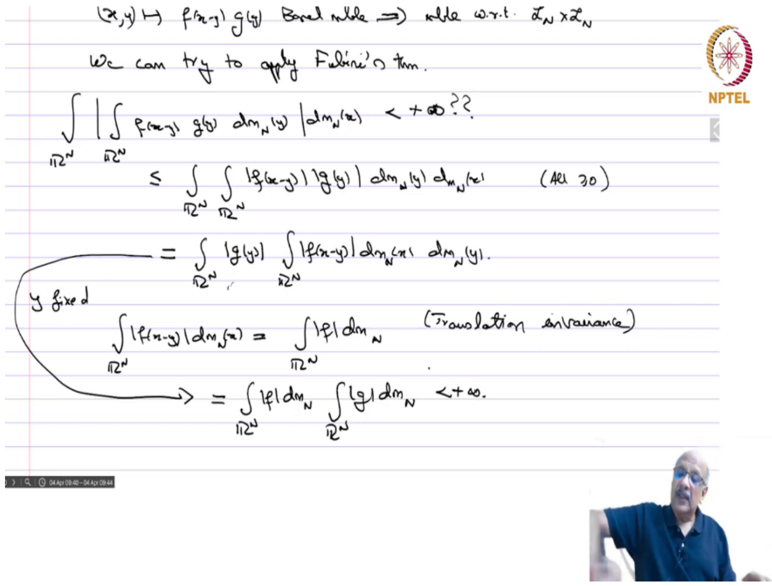

But $\int_{\mathbb{R}^N} |f(x - y)g(y)| dm_N(x) = \int_{\mathbb{R}^N} |f| dm_N$, because of translation invariance and therefore you have that this is now less than or equal to mod or equal to the other $\int_{\mathbb{R}^N} |f| dm_N$ that comes out and then what is remaining inside is nothing but $\int_{\mathbb{R}^N} |g| dm_N$ and we know that this is finite.

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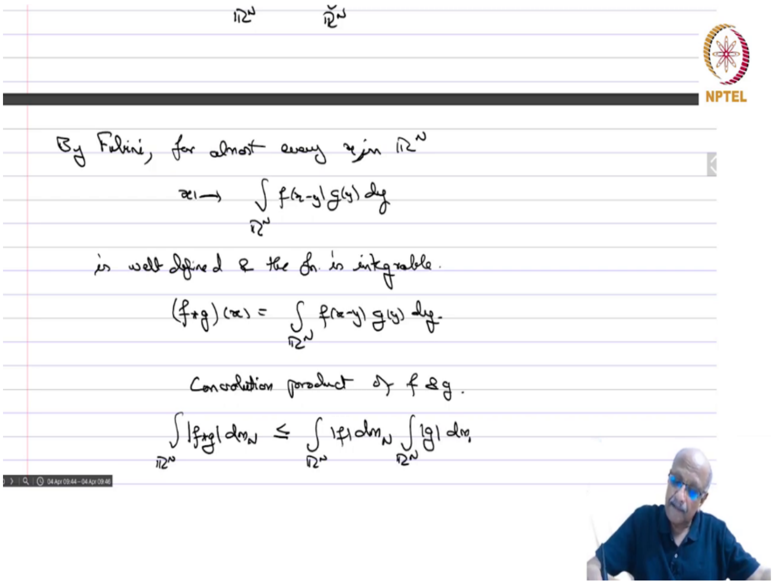

Therefore, by Fubini's theorem for almost every x in $\mathbb{R}^N$, $x \to \int_{\mathbb{R}^N} |f(x - y)g(y)| dm_N(y)$

is well defined and the function is integrable. So, we say, we define

$$(f * g)(x) = \int_{\mathbb{R}^N} |f(x - y)g(y)| d(y)$$

this is called the convolution product of f and g. And we also saw that

$$\int_{\mathbb{R}^N} |(f * g)(x)| dm_N(x) \leq \int_{\mathbb{R}^N} |f| dm_N \int_{\mathbb{R}^N} |g| dm_1.$$

So, this is coming from Fubini’s theorem.

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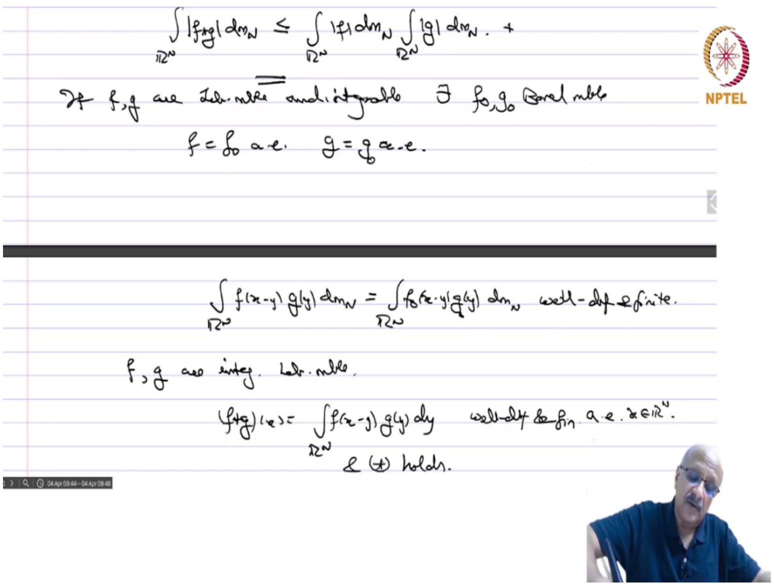

Now, if f and g are Lebesgue measurable and integrable there exists f_0, g_0 Borel measurable and $f = f_0$ $a.e.$, $g = g_0$ $a.e.$ and then because this we have done as an exercise, we have shown how to do it, first for characteristic functions, then simple functions and so on.

And therefore, we can so therefore, you have that

$$\int_{\mathbb{R}^N} f(x-y)g(y)\, dm_N = \int_{\mathbb{R}^N} f_0(x-y)g_0(y)dm_N$$

and this is well defined, this is well defined and finite. And therefore, you have that for if f and g are integrable Lebesgue measurable again you have

$$(f * g)(x) = \int_{\mathbb{R}^N} |f(x-y)g(y)| d(y)$$

is well defined and finite almost every x in $\mathbb{R}^N$, and you have integral and star holds. What is star? Star is this thing about integration. So, that is about convolution, convolution is a very important tool in analysis and we will see more about it a little later.