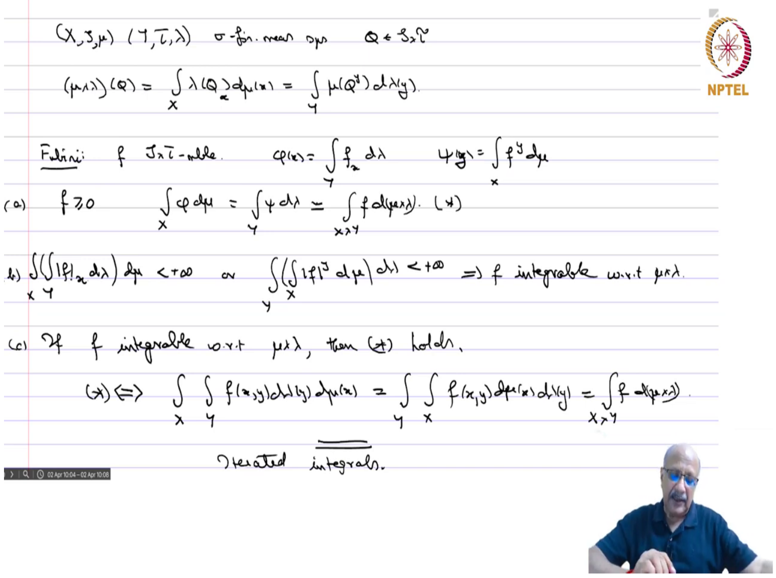
Measure and Integration
Professor S Kesavan
Department of Mathematics
The Institute of Mathematical Science
Lecture 54
Examples

(Refer Slide Time: 00:16)



So, we will now look at some **examples** of **Fubini's theorem**. So, let me just recall (X, S, μ), (Y, T, λ). So, these are σ finite measure spaces and then on $Q \in S \times T$ the product σ algebra, then $(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_Y \mu(Q^y) d\lambda(y)$.

So, this is the measure and then we have Fubini's theorem.

So, Fubini, so if f is non negative, then so f is $S \times T$ measurable, then you define

$$\varphi(x) = \int_Y f_x d\lambda, \; and \; \psi(y) = \int_X f^y d\mu.$$

And, then the Fubini's theorem says

(a): If $f \geq 0$ these functions are measurable and you have the

$$\int_X \varphi \, d\mu = \int_Y \psi d\lambda = \int_{X \times Y} f \, d(\mu \times \lambda).$$

(b): Then secondly, if

$$\int_X \Big(\int_Y |f|_x d\lambda\Big) d\mu < +\infty, \text{ or } \int_Y \Big(\int_X |f|_y d\mu\Big) d\lambda < +\infty$$

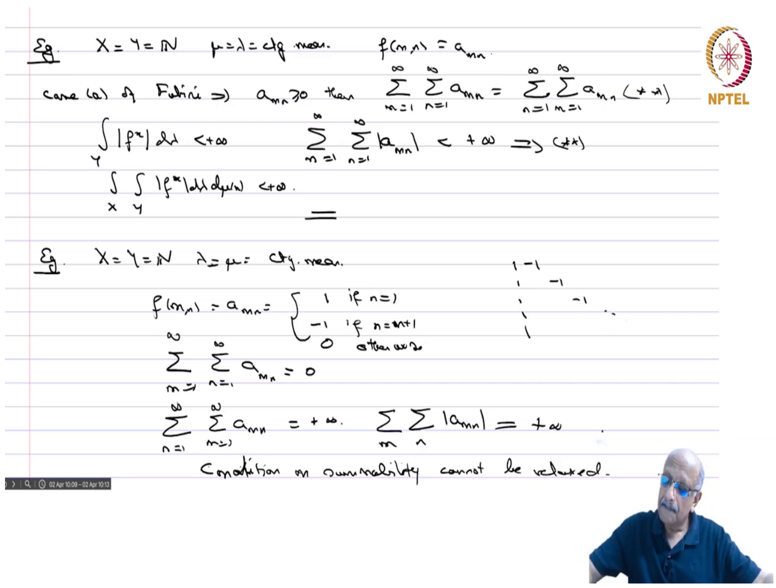
, then f is integrable with respect to $\mu \times \lambda$ and if so, this is (a), this was (b) and this is

(c): If f is integrable with respect to $\mu \times \lambda$ then star holds, star is this function and that star can be re-written in expanded form. So, this is integral

$$\int_X \int_Y f(x, y) d\mu(x) d\lambda(y) = \int_Y \int_X f(x, y) d\lambda(y) d\mu(x) = \int_{X \times Y} f \, d(\mu \times \lambda).$$

So, this is the Fubini's theorem, which we have so always to check if you can if it is non negative you apply blindly they may all be infinite but still it is true and if it is not, if it is of changing sign, then apply to mod f if they, these are called the iterated integrals. So, if one iterated integral of mod f is finite then f is integrable and then the integration can be carried out in this fashion. So, this is the reconnection.

(Refer Slide Time: 05:04)



So, now, let us look at examples.

Example: So, we have take $X = Y = \mathbb{N}$, $\mu = \lambda = counting\ measure$ and then f m, n it was a mn sum double series, then case a of Fubini implies a mn non negative then sigma m equals 1 to infinity sigma n equals 1 to infinity, a mn equals sigma n equals 1 to infinity, sigma m equals 1 to infinity a mn that is you can interchange the order we prove this using as a consequence of the monotone convergence theorem and this is also now case a of Fubini's theorem.

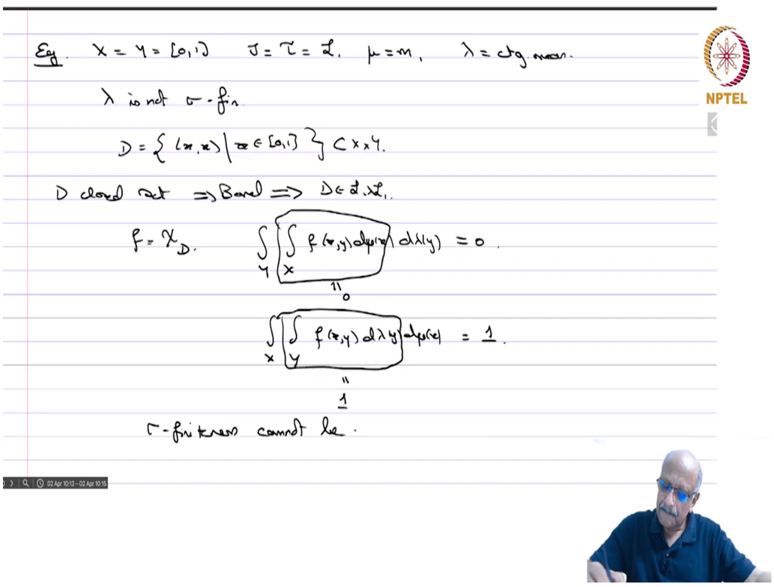
Now, what about case b so, we want integral mod fx so, that means, mod if x integral should be finite, so, y dλ should be finite. So, we say that integral and we wanted the integral X, integral Y, mod fx dλ dμ should be finite, this is the thing. Now, this condition if you look back and the example this is saying integral of m equals 1 to infinity sigma n equals 1 to infinity mod a mn is finite. So, we said that, this we called as b, bm and sigma bm is finite this was the condition which we put using and then this implies again implies star and this we prove using the dominated convergence theory.

So, that is the first example. So, we the cover what we prove using the monotone convergence theorem for the non negative case and the general case for the dominated convergence theorem. If you go back to that example, this is precisely the conditions which we used there using the monotone and dominated convergence theorems respectively.

Example: So, now, you have again $X = Y = \mathbb{N}$, $\lambda = \mu =$ equals counting measure and you have f m, n equals a mn equals 1, if n equals 1 that is the first column is always 1 minus 1 if n equals n plus 1 and 0 otherwise. So, we had 1 1 1 1 1, then minus 1 minus 1 minus 1, et cetera.

And therefore, now, if you take sigma m equals 1 to infinity, sigma n equals 1 to infinity then you have a mn. So, n equals 1 to infinity. So, if you are carrying then you get 0, and sigma n equals 1 to infinity sigma m equals 1 to infinity a mn, so, n equals 1 then you get infinity and this will be plus infinity. And this now, the previous condition sigma mod a mn this is equal to plus infinity because we put more or less everywhere then you get plus infinity and therefore, so condition on some ability cannot be the most cannot be relaxed. So, that tells you the condition.

(Refer Slide Time: 09:31)



So, this Fubini theorem which we said that if f is integrable then you can also here the, if and for that to be integrable we needed mod f to have the iterated integrals to be finite. And here mod f does not have iterated finite so if it is not integrable and Fubini theorem fails, so 3,

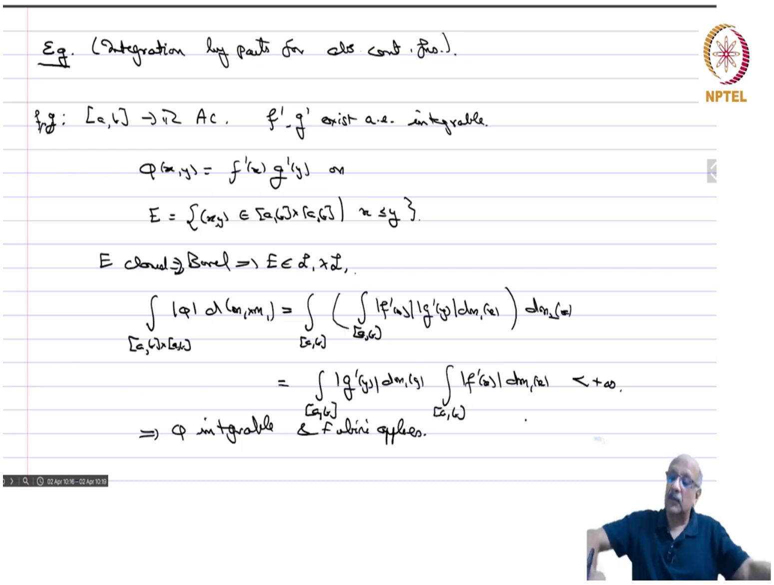
Example: $X = Y = [0, 1]$, $S = T = L_1$, $\mu = m_1$, but $\lambda =$ counting measure. Therefore, λ is not σ finite because it is uncountable set and so, counting measure.

So, now you are taking $D = \{(x, x) \mid x \in [0, 1]\} \subset X \times Y$.

So, D is a closed set implies Borel, implies D belongs to $L_1 \times L_1$ because we have seen this, Borel sets are contained in the product σ algebra, then f equals chi of D, then integral over Y integral over X, f of x, y, $d\mu(x)$ $d\lambda(y)$. Now, if you take the first integral it is supported at a single point which is of measure 0 for the Lebesgue measures. So, this is equal to 0 and therefore, this integral becomes 0 and if you take the integral over X, integral over Y, f of x, y, $d\lambda(y)$, $d\mu(x)$.

Now, if you take the first integral, this is a counting measure and it is supported at a single point. So, this integral is always equal to 1 because chi D and C equal to 1. So, it is 1 for all x and therefore, this integral is 1 therefore, so σ finiteness cannot be relaxed.

(Refer Slide Time: 12:02)



Example (Integration by parts): Then, another nice example. So, that is take so, this is called integration by parts, for absolutely continuous functions. So, if you want, if you know integration the parts for continuously differentiable functions then this integral $f^{'}$ is equal to the n values, my difference of the n values minus $f^{'}$, g will be difference of n values of f, g my plus minus integral d $f^{'}$. So, now, we want to show that even in the case of absolutely continuous functions which are differentiable almost everywhere. So, a, b and you have f and g from a, b to R absolutely continuous then $f^{'}$, $g^{'}$ exist almost everywhere and are integrable.

So, now, let us consider the function

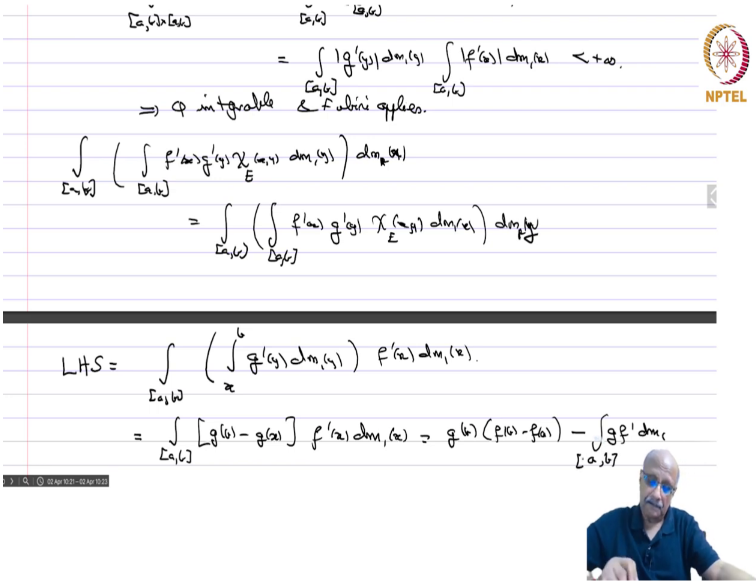
$$\varphi(x, y) = f^{'}(x)g^{'}(y) \ \ on \ \ E = \{(x, y) \in [a, b] \times [a, b] : \ x \leq y\}.$$

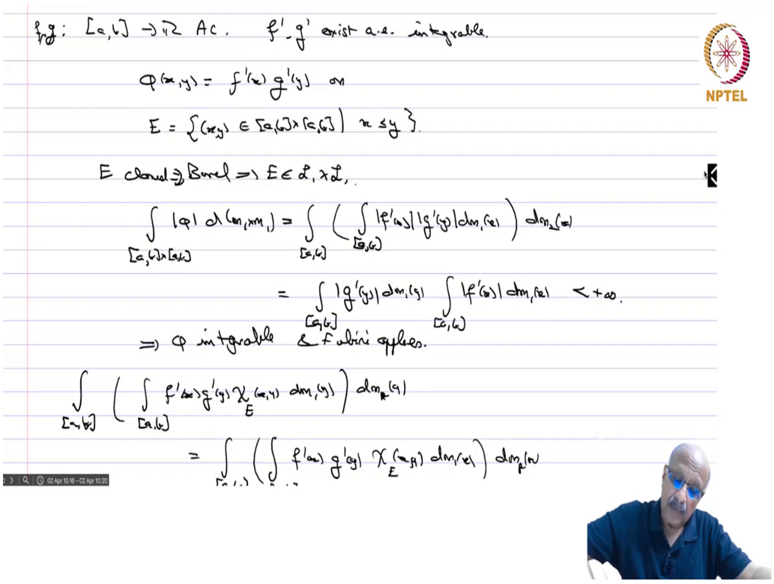
So, E is closed Borel and implies $E \in L_1 \times L_1$. So, we are assuming [a, b] with the Lebesgue measurable.

And, then integral $[a, b] \times [a, b]$ of mod phi d m1 cross m1 is equal to integral over the a b integral over a b mod $f^{'}(x)$, mod $g^{'}(y)$, dm x, d m 2x, but $g^{'}(y)$ comes out so the integral a b, mod $g^{'}(y)$ and then insane it is it comes up as a constant so then we can make dm 1 y into a b mod $f^{'}(x)$ dm 1x. And we know that this is finite. So, 1 iterated integral or both are finite and therefore, this means that phi is integrable and Fubini applies.

So, we always will do like this we given a function we want to apply Fubini's theorem, first we will take the modulus estimated see that the one of the iterated integrals is finite then you know the function is integrable and you can go ahead and apply Fubini's theorem.

(Refer Slide Time: 15:41)





So, what do you have that? By Fubini, so integral over [a b], integral over [a, b], $f'(x)$, $g'(y)$, chi x y, dm1 y, dm2 y equals integral over a b, integral over a b, $f'(x)$, $g'(y)$, chi over x y, dm1 x, dm2 x, m1 x. So, let us estimate the first one, so LHS, so we have to first integrate with respect to y and therefore, this becomes

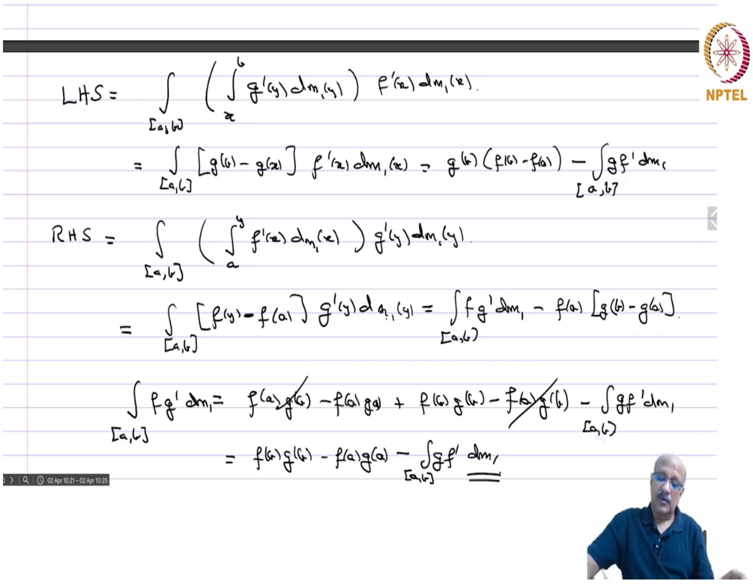
$$\text{LHS} = \int_{[a,b]} \left(\int_x^b g'(x)\, dm_1 \right) f'(y)\, dm_1(x)$$

$$= \int_{[a,b]} [f(b) - f(x)] f^{'}(x)\, dm_1(x) = g(b)[f(b) - f(a)] - \int_{[a,b]} g f^{'} dm_1(x).$$

integral a b, of integral a b, chi of x. So, this is becomes integral a to x, y will go from a to x, it will go from, because x is x is less than equal to y, sorry sorry. So, this should go from x is less than or equal to y.

So, y will go from x to b of $g^{'}(y)$ dm1 y into $f^{'}(x)$ dm 1x. So, now let us integrate the first one so, it is from x to b, $g^{'}(y)$ dm1 y, so that is equal to the integral a b since it is absolutely continuous function the integral of derivative is nothing but gb minus gx into f$f^{'}(x)$, dm1 x. Which, so, gb into $f^{'}(x)$, gb is a constant, come out integrative $f^{'}(x)$ will be fb minus fa. So gb into fb minus fa minus integral g, $f^{'}(x)$ dm1.

(Refer Slide Time: 19:17)



Now, let us in the same way write the RHS,

$$\text{RHS} = \int_{[a,b]} \left(\int_a^y f^{'}(x)\, dm_1 \right) g^{'}(y)\, dm_1(y)$$

$$= \int_{[a,b]} [f(y) - f(a)] g^{'}(y)\, dm_1(y) = \int_{[a,b]} f g^{'} dm_1 - f(a)[g(b) - f(a)].$$

$$\int_{[a,b]} f g^{'} dm_1 = f(a)g(b) - f(a)g(a) + f(a)g(b) - f(a)g(b) - \int_{[a,b]} f^{'} g\ dm_1.$$

$$= f(a)g(b) - f(a)g(a) - \int_{[a,b]} f^{'} g \ dm_1$$

Again this is integral over a b, yet because it is chi E, I am writing in this fashion, now this is nothing but fy minus fa into $g^{'}(y)$, dm1 y, and that is equal to integral a b, f $g^{'}$ dm 1 minus fa into gb and g, gb minus ga, and again and again using the fact that if the intent of an absolutely continuous function, then you have the fundamental theorem of calculus.

So, if you equate these two integral over a b, f$g^{'}$ dm1 equals fa, gb minus fa ga, and taking this to the other side plus fb gb minus fa gb and so, the first one, these two get canceled minus integral g $f^{'}$ dm1 and that is equal to fb gb minus fa ga minus integral g $f^{'}$ dm1 over a and this is exactly the integration by parts formula. Now, so, this so we continue with more examples next time.