Measure and Integration
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Lecture 53
Fubini's Theorem

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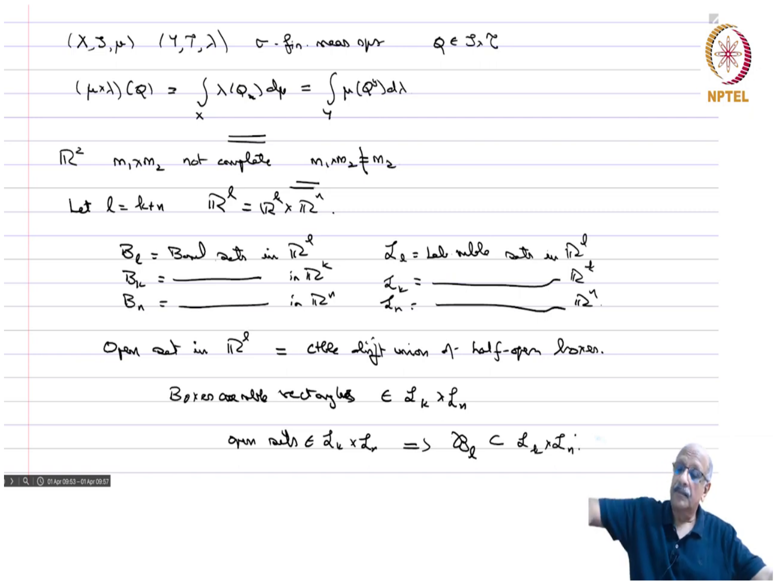

So, we were looking at the **product measure**. So, we have (X, S, μ), (Y, T, λ) σ finite measure spaces then we have $\mu \times \lambda$ is the measure defined on, so $Q \in S \times T$ the product σ algebra, then this was defined as

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)\, d\mu = \int_Y \mu(Q^y)\, d\lambda.$$

So, this is how we define the product measure and we saw that on our two for instance, $m_1 \times m_2$ is not complete and therefore, $m_1 \times m_2$ is not what we would have assumed is the measure. So, what is the relationship between the different σ algebras and the corresponding measures which we have on higher dimensional spaces?

So, we sketch the proof argument below. So, we will say let $l = k + n$ and then we consider $\mathbb{R}^l = \mathbb{R}^k \times \mathbb{R}^n$ then you have B^l equals Borel sets in $\mathbb{R}^l$, B_k equals Borel sets in $\mathbb{R}^k$ and B_n equals Borel sets in $\mathbb{R}^n$. Similarly, you have L_l equals Lebesgue measurable sets in $\mathbb{R}^l$ and then L_k equals Lebesgue measurable sets in $\mathbb{R}^k$ and L_n equals Lebesgue measurable sets in $\mathbb{R}^n$

So, now, given any open set in $\mathbb{R}^l$ equals countable disjoint union of half open boxes, we have seen this before any open set can be written as a countable disjoint union of half open boxes therefore, all since boxes are obviously, they are measurement rectangles and therefore, belong to $L_k \times L_n$ and therefore, open sets are also in, so open sets belong to $L_k \times L_n$ and this implies B_l Borel sets is contained in B $L_k \times L_n$.

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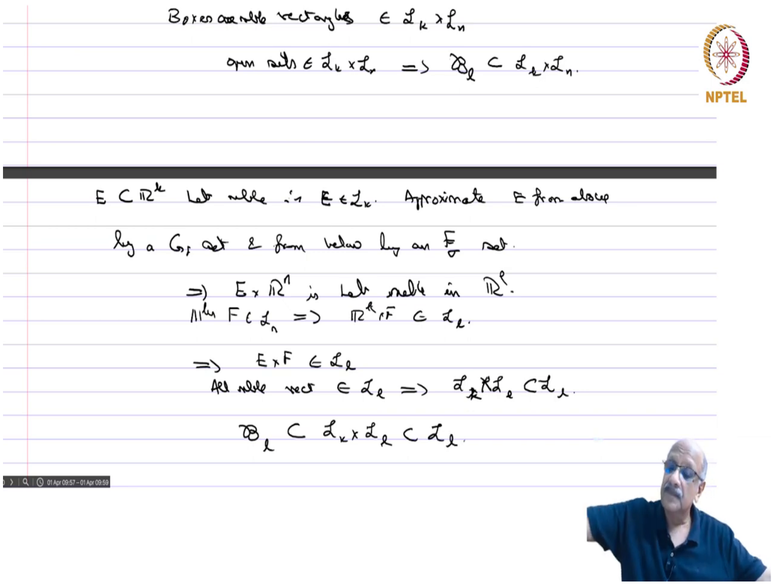

Now, if E is in $\mathbb{R}^k$ Lebesgue measurable that is $E \in L_k$. So, now, you can approximate E from above by d delta set and from below by an F σ set these are all the characterizations of this thing. Now, then this means that $E \times \mathbb{R}$ is Lebesgue measurable in $\mathbb{R}^l$. Why is that? Because you have $E \times \mathbb{R}$ will be what if you take g delta set then that will be a countable union of open set.

An open set cross $\mathbb{R}^n$ is again an open set therefore, it is Lebesgue measurable in $\mathbb{R}^l$ and therefore, countable union of open sets is also the big measurable in $\mathbb{R}^l$ and then F σ similarly for closed sets, intersection of closed sets et cetera is Lebesgue measurable $\mathbb{R}^n$ and these two closing on E and Lebesgue measurable is a completion and therefore, this implies that E cross $\mathbb{R}^n$ is, similarly, $F \in L_l$, L_n, implies $\mathbb{R}^k \times F$ belongs to L_n.

So, the intersection, so this implies the intersection of these two which is E cross F belongs to L_l. So, this means that all measurable rectangles belong to L_l and this implies that $L_k \times L_l$ is contained in L_l. Therefore, you have B_l is contained in the $L_k \times L_l$ which is contained in L_l.

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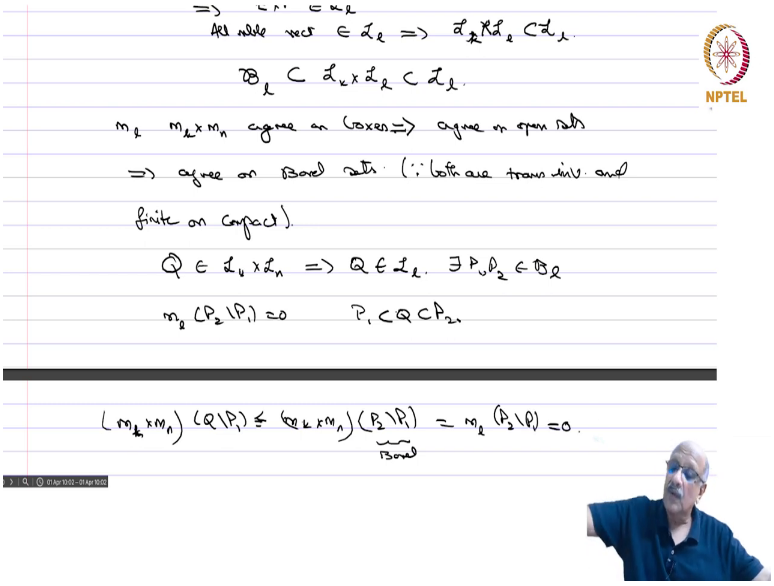

Now, m_l and $m_k \times m_n$ agree on boxes because it is whatever definition you give the because it is measurable rectangles boxes are measurable rectangles. So, the measure is just the product of the measures and then the same for the boxes in ml also. So, these agree on boxes and since every open set is countable disjoint union of boxes therefore, agree on open sets and therefore, this agree on Borel sets because this is translation both are, since both are translation invariant and finite on compact sets, both these are very easy to check.

And therefore, they agree on open sets, they are the translation within finite on compact sets then we are shown that the it is essentially the Lebesgue measure which is the thing, so ml is on Borel sets. So, they agree on Borel sets.

So, now, if $Q \in L_k \times L_n$ then this implies Q belongs to L_l. So, it is Lebesgue measurable. So, there exists P_1, P_2 in B_l Borel sets such that ml of $P_2 \setminus P_1$ equal to 0 and P_1 contained in Q contained in P_2 because, again the proposition about Lebesgue measurability. Now, $m_k \times m_n(Q \setminus P_1) \leq m_k \times m_n(P_2 \setminus P_1)$ and this is Borel set and therefore, that is equal to ml of $P_2 \setminus P_1$ and that is equal to 0.

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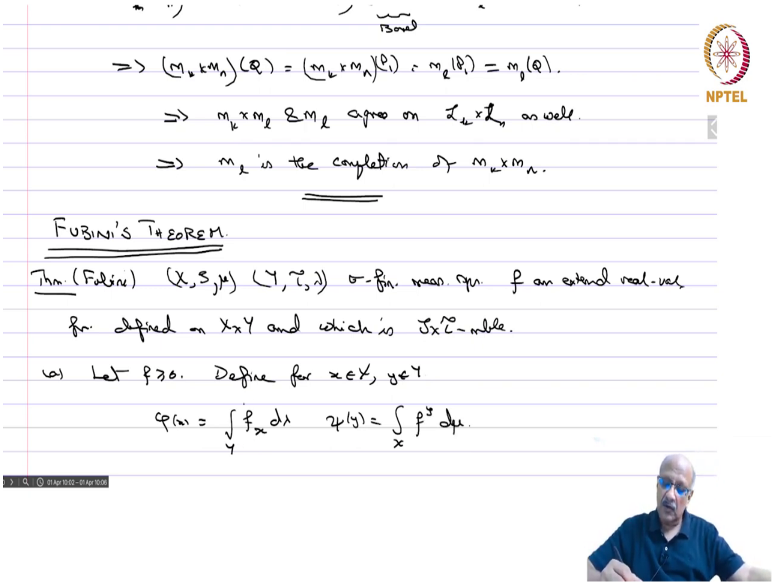

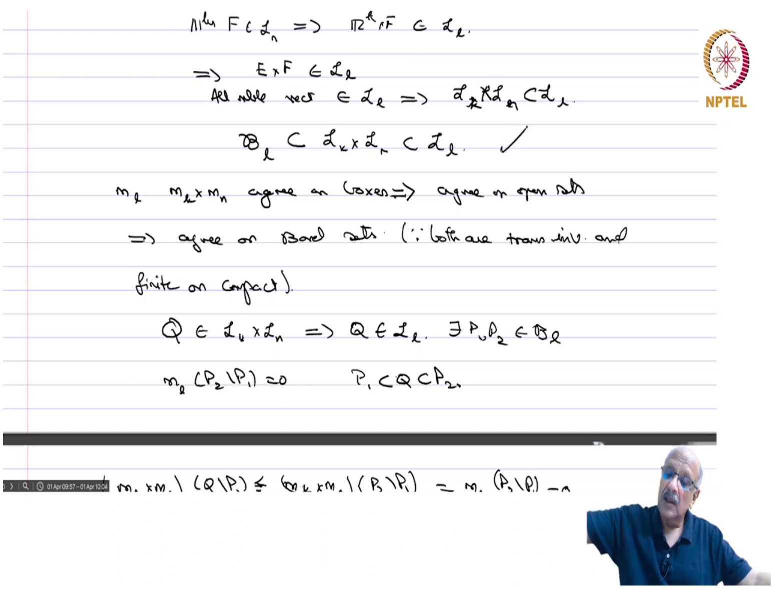

So, that implies that $m_k \times m_n(Q)$ equals $m_k \times m_n(P_1)$ which is $m_l(P_1)$ which is equal to $m_l(Q)$, and this implies that $m_k \times m_l$ and m_l agree on $L_k \times L_n$ as well. And therefore, this implies that, ml is the completion of $m_k \times m_n$. So, the Lebesgue measure in in a product space is the completion of the product of the Lebesgue measure which for many of the subspaces which you want to do, so, this is the relationship and we also have that B_k, B_l is contain $L_k \times L_n$ cross and L_l we have this relation L_n here sorry, and we have this relationship here.

And therefore, it works. So, now, we are going to prove one of the most important theorems again in the course. So, this is called **Fubini's Theorem**, it talks about integrability on the product space, so theorem Fubini

Theorem: Let (X, S, μ), (Y, T, λ) σ finite measure spaces. F an extended real valued function defined on $X \times Y$ and which is $S \times T$ measurable. It better be measurable in the product otherwise we are not talking about it.

So, a let f be non negative. Define for $x \in X,\ y \in Y$,

$$\varphi(x) = \int_Y f_x\, d\lambda \ \ and\ \psi(y) = \int_X f^y d\mu$$

remember these are the sections for this a function of y and this f^y is a function of x and therefore, you can integrate it on these spaces.

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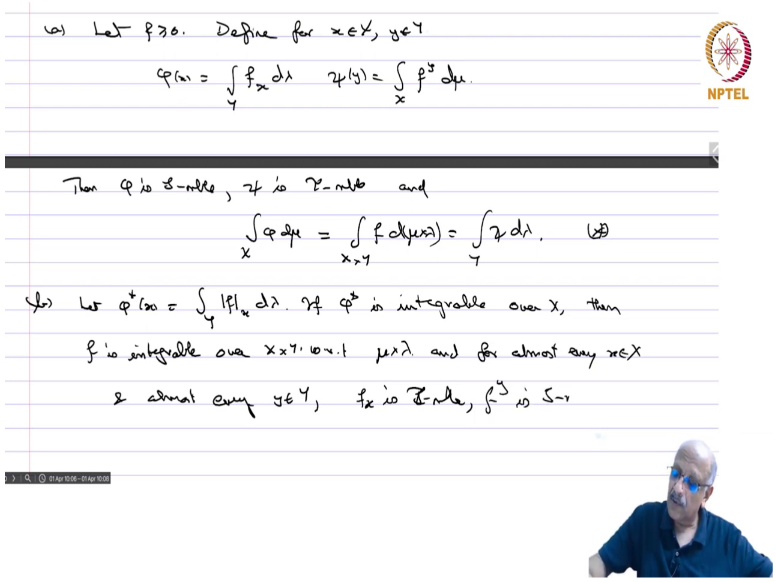

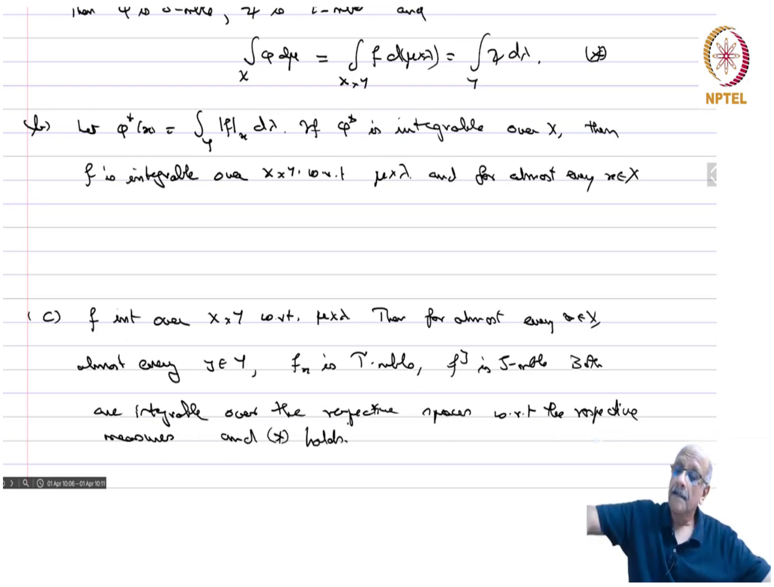

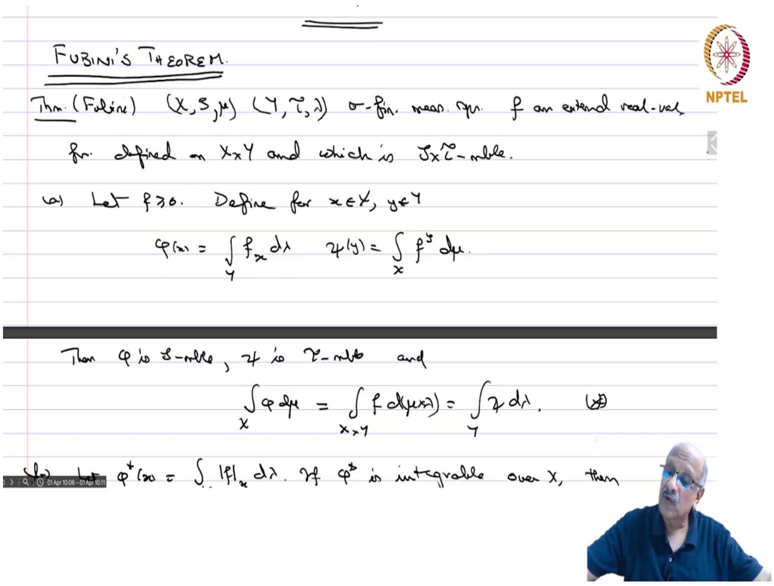

Then φ is S measurable, ψ is T measurable and integral over $\varphi d\mu$ over X is equal to integral f $d(\mu \times \lambda)$ over $X \times Y$ and that is equal to the integral over Y of ψ $d\lambda$. So, this is the important conclusion of Fubini's theorem. So, b let

$\varphi^{*}(x) = \int_{Y} |f|_{x} d\lambda$, if φ^{*} is S integrable,

integrable over X then f is integrable over $X \times Y$ with respect to the measure $\mu \times \lambda$ and for almost every $x \in X$, and almost every $y \in Y$ fx is S measurable, no sorry T measurable, fy is S measurable and star holds. c, f integrable over $X \times Y$ with respect to $\mu \times \lambda$ then that should be stated here then for almost every $x \in X$, $y \in Y$, fx is T measurable, fy is S measurable both are integrable over the respective spaces with respect to the respective measures and star holds.

So, let us look at this theorem a bit. So, first one says that if you have a non negative function, then you just do do not worry about anything star is true, all of them may be infinity, that is possible, but they are if one is infinity, everything is infinity one is finite, all the three are finite and they have to be equal that is, so, for non negative functions, no worries.

The second one tells you a condition when the function is integrable over the product space. So, if you take mod f and it is x section and y and that is integrable over X then the original function is integrable similarly, it is enough to check you can also do it for y you can take the mod of f over y if that is integral for x and then then also you can have the integrability.

And once you have integrability of the function of the product space, then the first part it is as though like it is in the case of non negative functions namely the sections are all measurable and integrable and you can write the general star formula which is the integral exactly.

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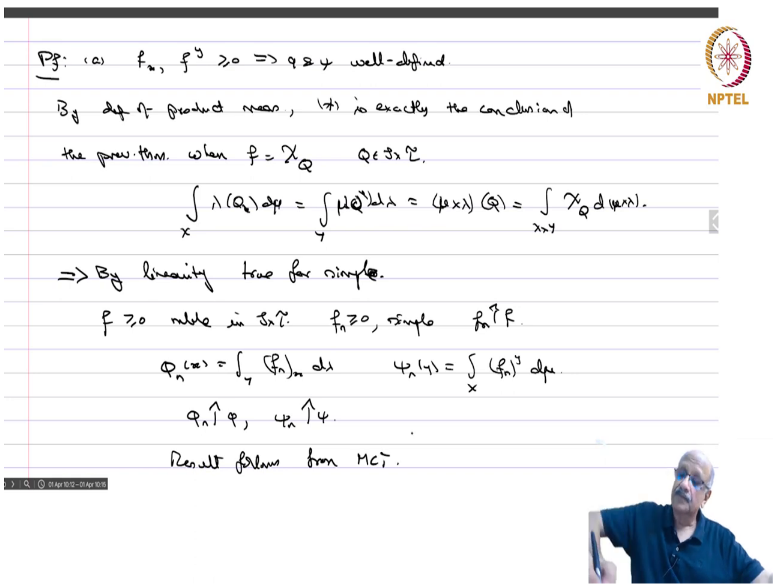

So, let us give the proof of this,

Proof: (a) f_x, f^y of non negative and therefore, φ, ψ are well defined. For non negative functions, you can always define the integral. Now, by definition of product measure star is exactly the conclusion of the previous theorem when $f = \chi_Q$, $Q \in S \times T$ because, this is exactly what we did because what was it conclusion integral of $d(Q_x)\ d\mu$ over X equals integral over y $\mu(Q^y)d\lambda$ and that is equal to $\mu \times \lambda(Q)$ which is nothing but integral over $X \times Y$ of chi Q.

So, this is exactly the theorem which we proved. So, we have proved it for characteristic functions then this implies by linearity for proof for simple functions, so, we are using the trick prove it for characteristic function, prove it for simple functions then use a limit theorem to prove it for any non negative function.

Now, f non negative function measurable in $S \times T$ then you have f_n, f_n increases to f. Then $\varphi_n(x)$ equals integral Y $f_n(x)\ d\lambda$, ψ_n of y equals the integral over X $f_n(y)\ d\mu$. Then, because these are simple functions and these φ_n increases to φ and ψ_n will increase to ψ. And therefore, result follows from the Monotone Convergence Theorem.

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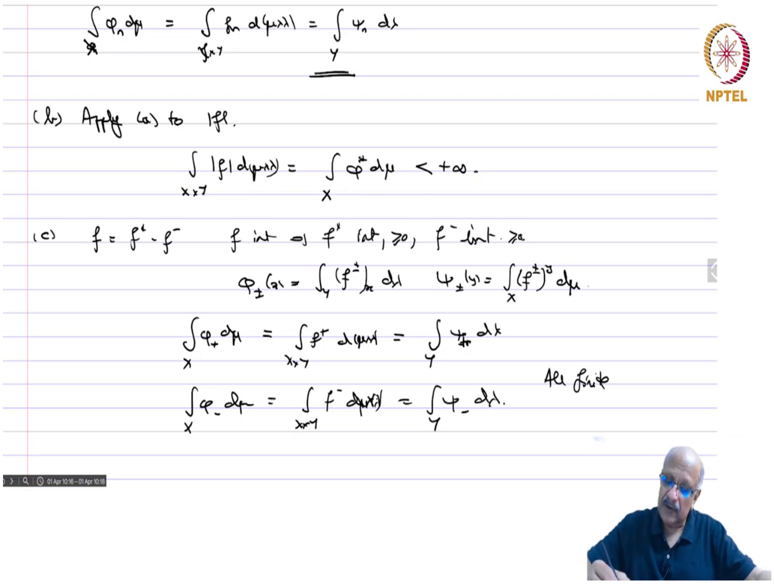

Because you have that integral f_n on $X \times Y$ f_n $d(\mu \times \lambda)$ equals integral ψ_n, $d\lambda$ over Y equals integral φ_n $d\mu$ over X and now, you just have to pass through the limit and then you will get this, so this proves this. b, so apply a to mod f, so we get the integral $X \times Y$ mod f $d(\mu \times \lambda)$ is equal to integral over X, φ^* $d\mu$ and that we know is finite and therefore, you have that f is integral.

c, so, now, you write $f = f^+ - f^-$ then f integrable implies f^+ integrable and equal to 0, f^- integrable and that is equal to 0, then you define φ plus minus x is equal to the integral Y f^+ minus x $d\lambda$ and ψ^{+-} y equals the integral over X, f^+ minus y $d\mu$.

Then what do you know? You know that integral f^+ $d(\mu \times \lambda)$ over $X \times Y$ is equal to integral ψ^+ $d\lambda$ over Y equals integral φ^+ $d\mu$ over X. And similarly, integral over x φ^- $d\mu$ is equal to the integral $X \times Y$, f^- $d(\mu \times \lambda)$ equals integral Y, ψ^- $d\lambda$. And all are finite, all finite because these things are integrable.

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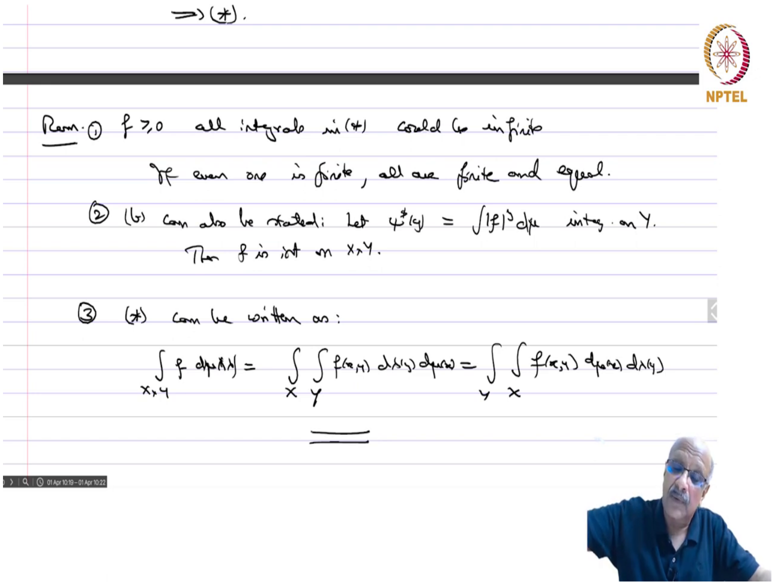

And therefore, they are finite and therefore, you can subtract and then you have $\varphi^{+}\varphi^{-}$ is exactly f^{+}, f^{-} so this, so this will give you, so this imply star just subtract back you will get some. So,

Remark: (1) If $f \geq 0$ all integrals in star could be finite, infinite if even one is finite all are finite and equal.

(2) Then, (b) can also be stated let $\psi^{*}(y) = \int |f|^{y} d\mu$ integrable on Y.

Then, f is integrable on $X \times Y$. So, you can check any one of the three.

(3) So, what is star? Star can be written as an expanded form integral

$$\int_{X\times Y} f d(\mu \times \lambda) = \int_{X}\int_{Y} f(x, y) d\lambda(y) d\mu(x) = \int_{Y}\int_{X} f(x, y) d\mu(x) d\lambda(y).$$

The first integral will give you the φ or the ψ definition and then next integral will give you the integral φ or the integral ψ definition. So, this is how you expand. So, essentially what we are doing, we used to do in double integrals in Riemann integral, Riemann integration namely the order of integration is unimportant that is what it says and that is true if f is integrable on the product space that is a Fubini's Theorem. So, we will see several examples of this in the subsequent lectures.