Measure and Integration
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Lecture 52
Product Measure

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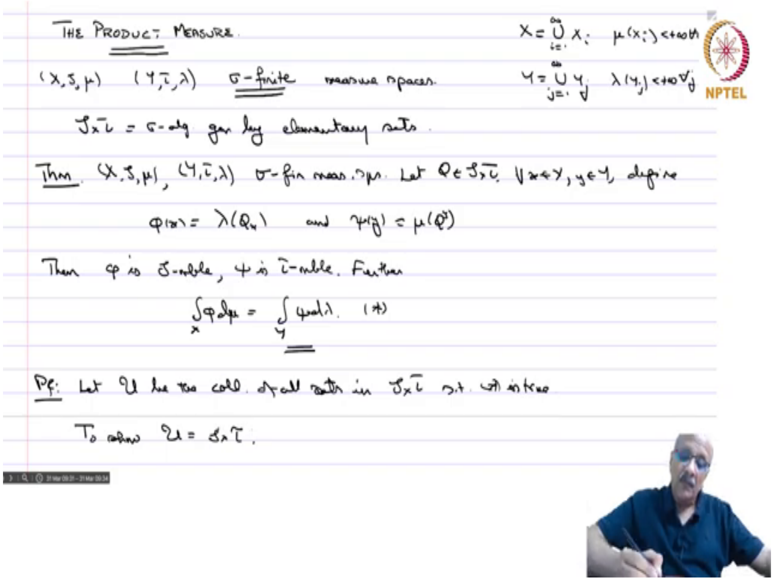

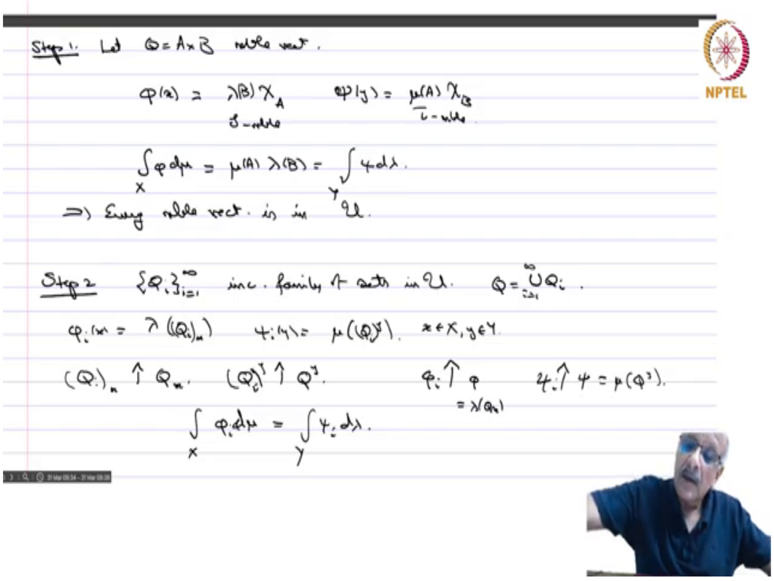

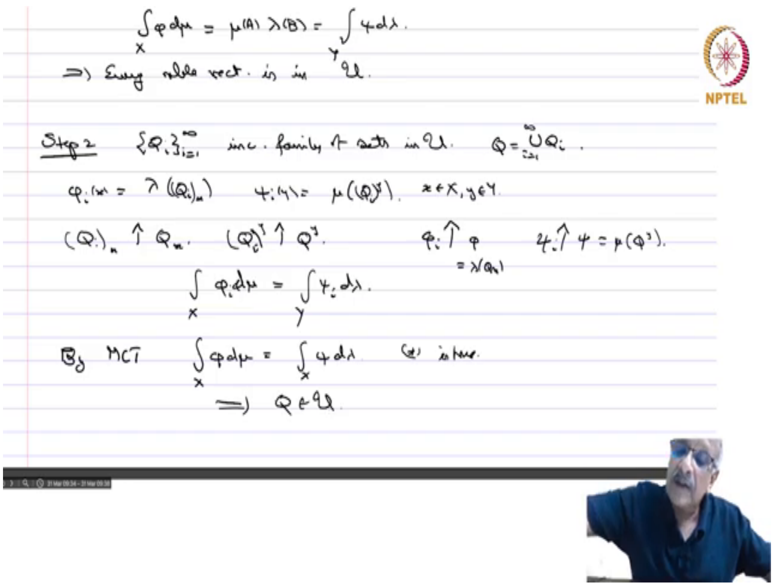

Today we will discuss the **product measures** so, we have (X, S, μ), (Y, T, λ) is σ finite measure spaces. So, what is σ finite? This means $X = \bigcup_{i=1}^{\infty} X_i$, $Y = \bigcup_{j=1}^{\infty} Y_j$ and $\mu(X_i) < \infty$, is finite for all i, and $\lambda(Y_i) < \infty$ is finite for all j.

This is what we mean by σ φ depends a countable union of sets of finite measure and you know what is $S \times T$ this is equal to σ algebra generated by elementary sets and we want to define now a measure on this σ algebra.

So, we start with the following theorem.

Theorem: Let (X, S, μ), (Y, T, λ), σ finite measure spaces. Let $Q \in S \times T$, that means it is measurable in the product σ algebra for $x \in X$, $y \in Y$ define

$$\varphi(x) = \lambda(Q_x) \textit{ and } \psi(y) = \mu(Q^y).$$

So, these are the sections.

Then φ is S-measurable, ψ is T-measurable further, there is the most important thing you have

$$\int_X \varphi \ d\mu = \int_Y \psi d\lambda.$$

Proof: Let U be the collection of all sets in $S \times T$, such that star is true, this is star and therefore, you take all these. So, our aim is to show that U to show $U = S \times T$. That will be that will be the theorem.

So, now, we will do it in various steps. So,

Step 1: let $Q = A \times B$ measurable rectangle, then that means each set is measurable in the corresponding space and you have the product. Then what is $\varphi(x)$, $\varphi(x)$ you remember the

$\varphi\ (x) = \lambda(B)\chi_{A}, \ \psi(y) = \mu(A)\chi_{B}.$

So, both of them are so this is S measurable and this T measurable. And what is

$$\int_X \varphi \ \ d\mu = \int_Y \psi d\lambda.$$

So, we have therefore, this implies that every measurable rectangle is in U.

Step 2, $\{Q_i\}_{i=1}^{\infty}$, increasing family of sets in U and you set Q equal to Union Q_i, i equals 1 to infinity, question is $Q \in U$ or not. So, $\varphi_i(x)$ is nothing but $\lambda((Q_i)_x)$ and Psi i y equals $\mu((Q_i)^y)$, $x \in X, y \in Y$.

Then $(Q_i)_x$ increases to Q of x there is no doubt about that and similarly, $(Q_i)^y$ increases to Q of y and for each of these since Q_i are all in U, so, you have integral $\varphi_i(x)$, φ_i i $d\mu$ equals integral φ_i dλ, but then φ increases to φ and psi increases to psi. φ is what $\lambda(Q_x)$ and this is equal to $\mu(\ Q^y)$ o.

So, by monotone convergence theorem you have a sequence of non negative function increasing. So, integral x of φ $d\mu$ is equal to integral of psi $d\lambda$ and therefore, R is true and this implies that $Q \in U$. So, if you have an increasing sequence in U then this union is also in U.

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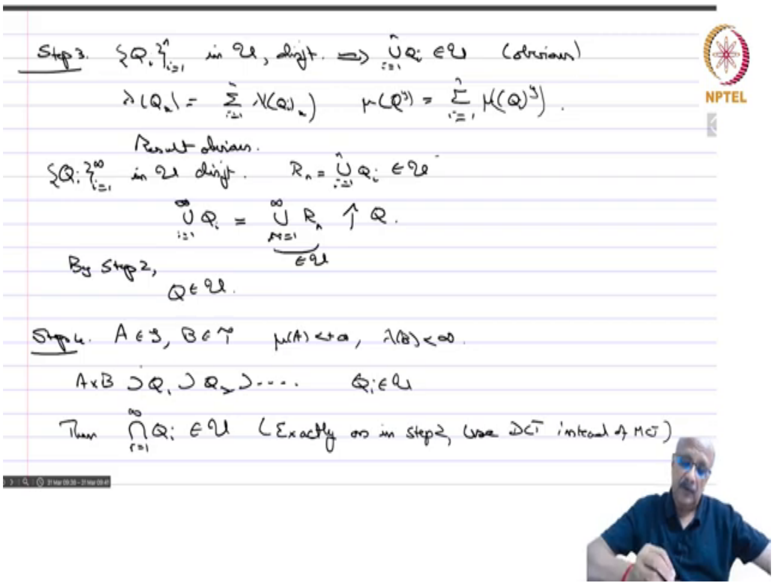

Step 3 $\{Q_i\}_{i=1}^{n}$ in U disjoint, then this implies this is obvious, so, that union $\{Q_i\}_{i=1}^{n}$ belongs to U. So, this is obvious because some of the integrals is the integral of the sum and therefore, you have nothing to leave it through. So, these are all disjoint. So, in fact $\mu((Q_i)_x)$, is U n σ $\mu((Q_i)_x)$, $\mu(Q_x)$ is so, $\mu(Q_x)$, sorry λ is σ_i i equals 1 to n $\lambda((Q_i)_x)$ and then $\mu(Q^y)$ equals σ i equals 1 to n because they are disjoint, $\mu((Q_i)^y)$ and therefore, now result is obvious so, now, result is obvious.

So, then therefore, if you have $\{Q_i\}_{i=1}^{\infty}$ in U disjoint then R n equals union i equals 1 to n, $\{Q_i\}_{i=1}^{\infty}$ belongs to U by what we just saw and therefore, Q unionQ_i, i equals 1 to n, Q_i infinity Q_i equals union i n equals 1 to infinity of R n and this increases to Q and by step 2 you have elements in U, which I increase into Q, we know that $Q \in U$. So, if you have a countable disjoint union of sets in U, where union is also in U.

So, step 4, A in S, B in T, $\mu(A)$ finite λ(B) finite and $A \times B$ contains Q 1 contains Q2 etcetra, Q_i in U then intersection i equals 1 to infinity, Q_i belongs to U. So, exactly as in step 2 use dominated convergence theorem instead of monotone convergence theorem, because you have that is where you put this finiteness condition on $A \times B$, you just have to the repeat the proof and you will see that by dominated convergence theorem you have this. Now, since this is step what 4.

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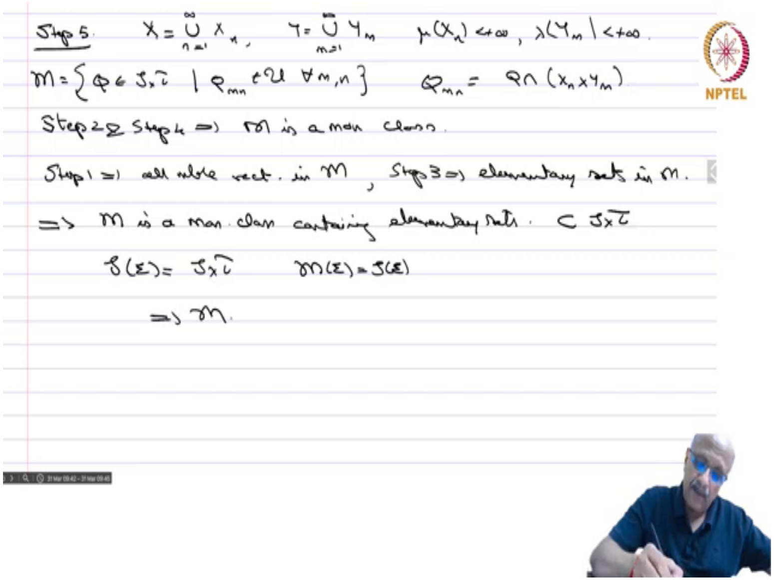

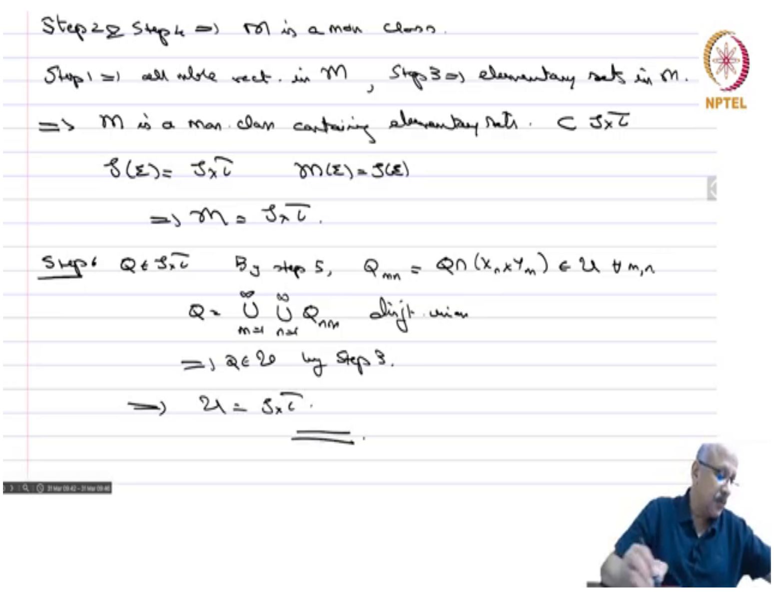

Step 5, you have now $X = \bigcup_{i=1}^{\infty} X_i$, $Y = \bigcup_{j=1}^{\infty} Y_j$ and $\mu(X_i) < \infty$, is finite for all i, and $\lambda(Y_i) < \infty$ is finite for all j, because X and Y are σ finite measure spaces and now, you take Q, so, now, you define M, to be set of all Q and $S \times T$ such that Q_{mn} belongs to U for all m, n. What is Q_{mn}, Q_{mn} is equal to Q intersect $X_n \times Y_m$.

So, now, step 2 and step 4 implies M is a monotone class, step 2 says any increasing in Q is in U, any increasing set family the union is in Q, U. And then the second 1 decreasing means it is all contained in a finite set and that is in Q_{mn}'s are all contained in $X_n \times Y_m$ which are finite measure and therefore, by step 4 the intersection will also be in U. Therefore, by step 1, 2 and 4, the M is a monotone class.

Now, step 1 implies all measurable rectangles in M and step 3 implies elementary sets in M therefore, you have that M is a monotone class containing elementary sets but what is a S of elementary sets this is $S \times T$ and M is of course is contained in $S \times T$ because you are taking all elements in $S \times T$ for which something is true and therefore, an M of E equals S of E. This we also know by earlier proposition which we prove and therefore, this implies that M equals $S \times T$.

Step 6: If $Q \in S \times T$. So, by step 5, Q_{mn}, which is Q intersection $X_n \times Y_m \in M$, belongs to U for all m, n and Q is nothing but the disjoint union M equals 1 to infinity. Union n equals 1 to infinity, Q nm and this is a disjoint union therefore, the implies $Q \in U$ by step 3. So, this implies that U is same as $S \times T$. So, this proves the entire Theorem.

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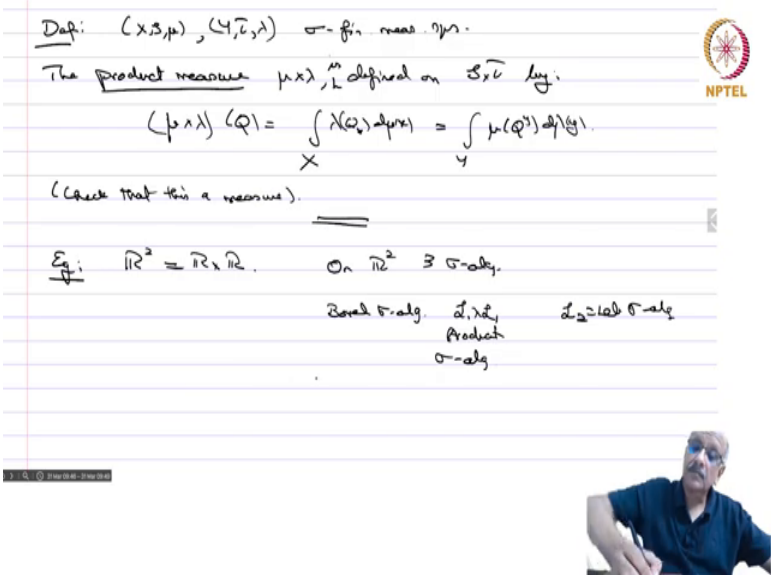

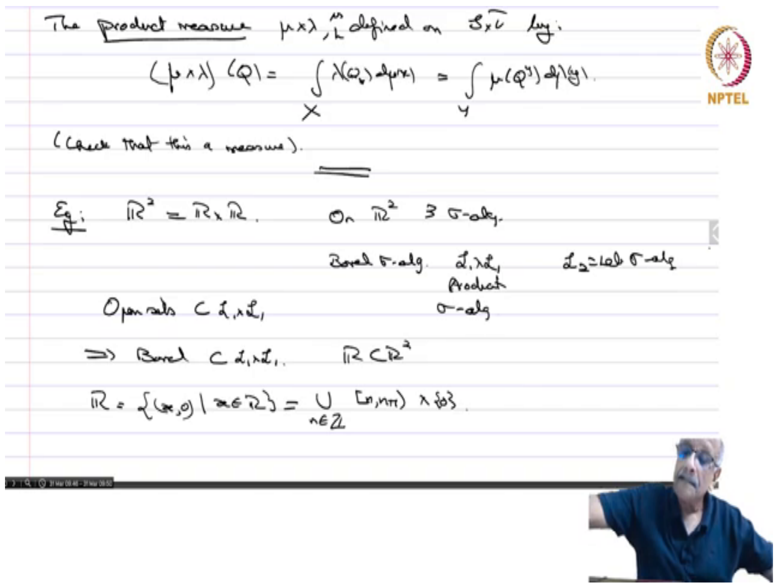

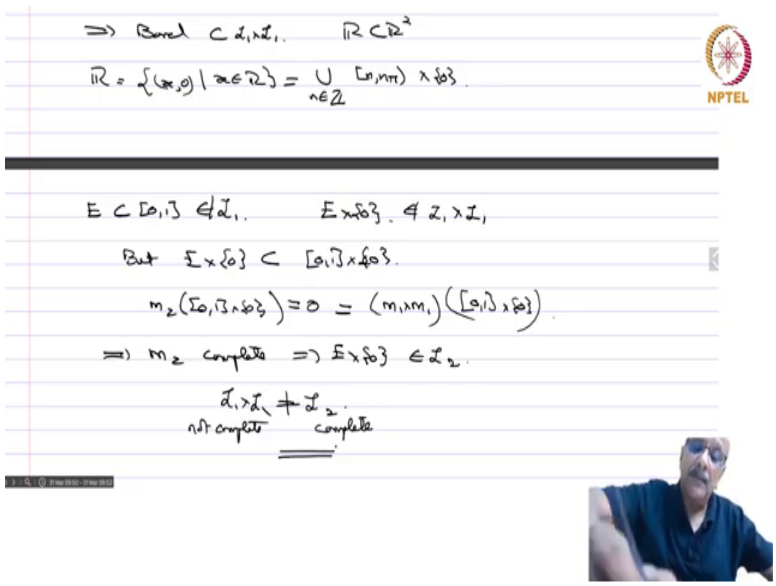

Definition: (X, S, μ), (Y, T, λ), σ finite measure spaces the product measure

$\mu \times \lambda$ defined on $S \times T$,

$$\mu \times \lambda(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_Y \mu(Q^y) d\lambda(y).$$

So, this it is easy to check it is a measure because there all you have to do is non negative and you only have to check countablity which is immediate because you are defining it in terms of the integral and disjoint sets the integrals non negative functions there is no problems. So, this is obviously measure. So, check that this is a measure, it is a very simple checking to do and then the. So, this is how we define the product measure on the product σ algebra.

So, we have to, we that is why we need to do this work. So, now example

Example: Let us take $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$. We have 3 σ algebras we have the Borel σ algebra, we have the Lebesgue σ algebra and we also have $L_1 \times L_1$ which is the product σ algebra, Now, the question is, is are these how are these related?

Now so, clearly since all open sets, you can write them in terms of product of open sets here each thing and open sets are in the corresponding space they are all measurable rectangles and therefore, open sets contained in $L_1 \times L_1$ and this implies Borel, this contained in $L_1 \times L_1$. Now, let us take $\mathbb{R}$, $\mathbb{R}$ equals set of all x, 0. $x \in \mathbb{R}$ and this is as a subset of $\mathbb{R}$ is contained in $\mathbb{R}^2$ this equal to the Union over n belonging to Z of n, n plus 1, cross 0.

Now, you take $E \subset [0, 1]$ non measurable then if you take $E \times \{0\} \notin L_1 \times L_1$, because the section of the 0 section will be E and that is not in L_1, that is not a measurable rectangle and therefore, this is not in L_1, but $E \times \{0\} \subset [0, 1] \times \{0\}$ and you know that $m_2([0, 1] \times \{0\})$ we have done this computation already this is 0 and by the whatever we have done now for the product measure this also $m_1 \times m_1([0, 1] \times \{0\})$.

So, this implies since m_2 is complete the Lebesgue measure is complete and this implies that E cross 0 belongs to L_2, therefore, $L_1 \times L_1$ is not equal to L_2, this is not complete, not complete and this is complete, it is not complete we have just now seen you have a set whose product measures is 0, but it has a subset which is not measurable and therefore, this so, this is so, we have to be careful the product measure is not the lebesgue measure in the highest base then what is it how are they all related? That we will see next step.