Measure and Integration
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Science
Lecture 51
Product spaces: Measurable functions

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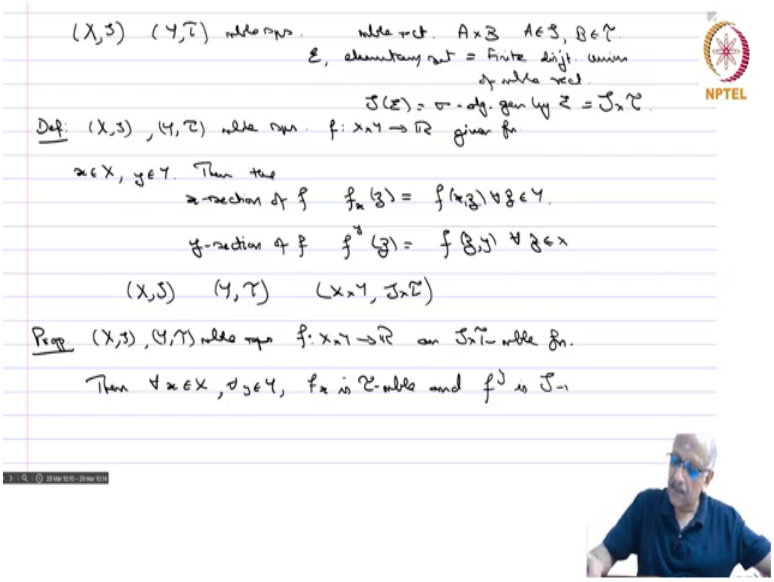

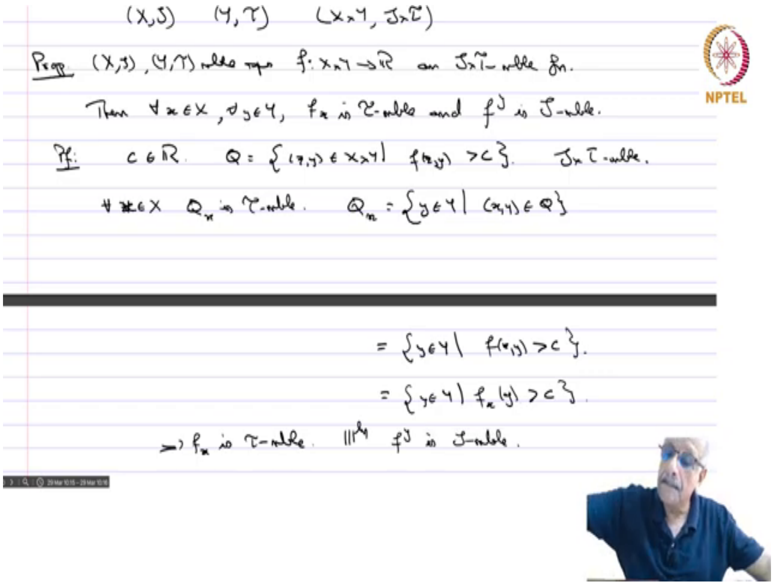

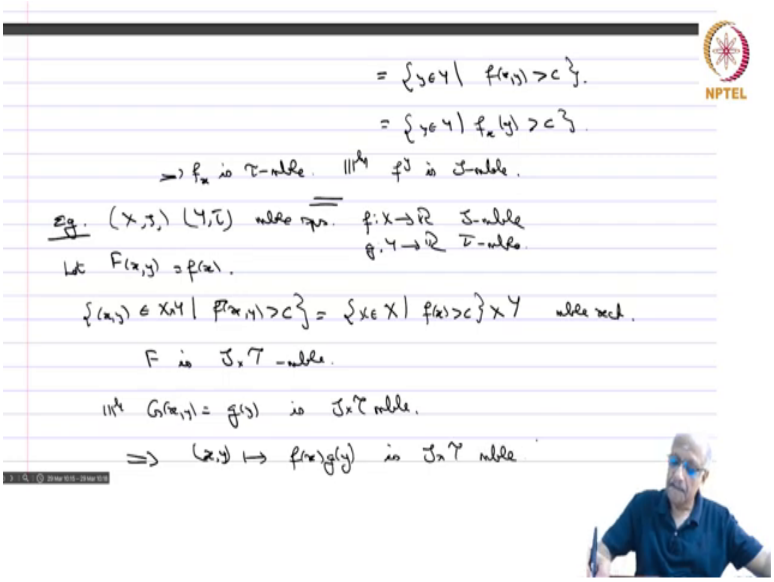

So, we have now looking at product spaces. So, we have (X, S) and (Y, T) measurable spaces. So, measurable rectangle is of the form $A \times B$, $A \in S$, $B \in T$ and then elementary set E equals finite disjoint union of measurable rectangles and then so, this is E elementary sets, then $S(E)$ the σ algebra generated by E, σ algebra generated by E and that is denoted by $S \times T$. So, this is the σ algebra we put on X cross S.

So, now we have definition

Definition: Let (X, S), (Y, T) measurable spaces and $f: X \times Y \rightarrow \mathbb{R}$ given a function, $x \in X$, $y \in Y$ then the x-section of f, denoted f_x

$$f_x(\xi) = f(x, \xi), \ \xi \in Y.$$

Similarly, the y-section of f,

$$f^y(\xi) = f(\xi, y), \ \xi \in X.$$

So, this is these are the 2 sections of the function just as we define x and y-section of sets we have sections of functions. Now, we have 3 measurable spaces. So, we have (X, S) we have (Y, T), then we have $X \times Y, S \times T$. So, when I say measurable, I will have to say in where it is measurable. So, I will say S measurable, T measurable or $S \times T$ measurable depending on the context one can understand then, where these functions are and how they measurable.

So, now, we have a proposition just as we had a proposition for the test of measurability of sets this proposition gives you a test of measurability of a function. So,

Proposition: Let (X, S), (Y, T) measurable spaces $f: X \times Y \to \mathbb{R}$ and $S \times T$, measurable function. Then for every $x \in X$ for every $y \in Y$, f_x and is T measurable and f^y is S measurable.

Proof, Y is this measurability. If either of these sections are not even for 1 x or 1 y the corresponding section is not measurable, then the original function cannot be measurable. So, this is how we use it to test for measurability.

So, let c belong to R, then you take Q equals to set of all $(x, y) \in X \times Y$ such that f of x y is bigger than c. So, this is $S \times T$ measurable, then for every $x \in X$, Q_x is T measurable but what is Q_x? So, this is set of all $y \in Y$ set $(x, y) \in Q$.

Which is equal to set of all $y \in Y$. So, set f of x, y is bigger than C that is equal to set of all $y \in Y$. So, said f_x of y is bigger than C. So, this is T measurable and therefore, therefore, this implies that f_x is T measurable.

So, for every c in R this set is measurable and therefore, the by the definition f x. So, similarly f^y is also measurable. So, now example (X, S), (Y, T), measurable spaces f X to R, S measurable, g from y to R, T measurable. Now, let capital F of x y equal to f of x, we are defining it as independent of y. So, so, the for all x y belongs to $X \times Y$, such that f of x y is bigger than C. Set of all $x \in X$ such that f of x is bigger than c cross Y and this is a measurable rectangle and therefore.

Capital F is $S \times T$ measurable. Similarly, G of x y equals g y is $S \times T$ measurable. The product of measurable functions is measurable, so, x y going to effects the g y variable separable case is $S \times T$ measurable.

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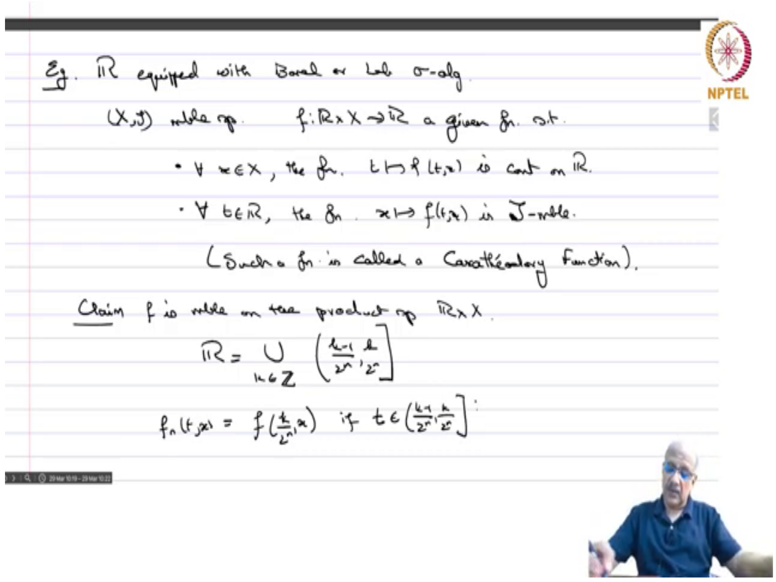

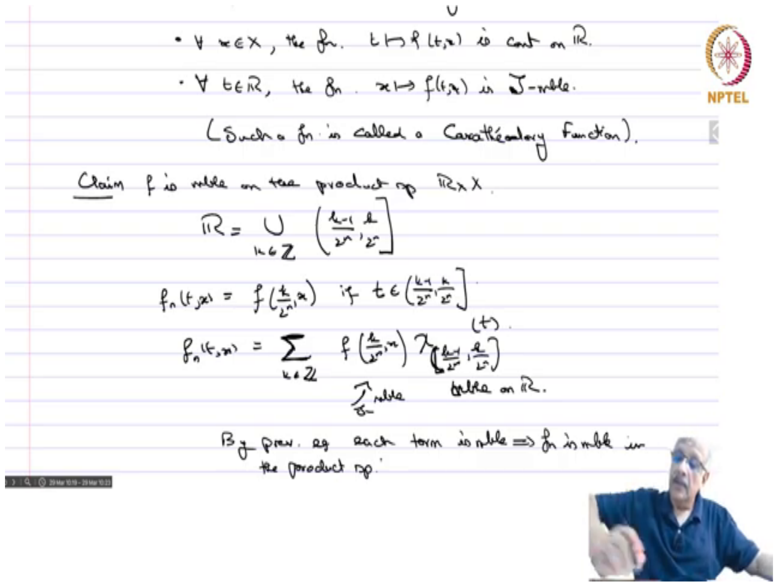

So, you take a measurable function in 1 variable measurable function another variable multiply them you will still get a measurable function in the next. Now, R equipped with Borel or Lebesgue σ algebra X, S measurement space. Now, $f: R \to X$, $R \times X$ to R, given function such that for every $x \in X$ the function t going to $f(t\ , x\ )$, $f(t\ , x)$ is continuous for every t in R, the function x going to f of t x is measurable, S measurable.

We have seen an example of thing earlier. So, such a function is called a caratheodory function. A function in 2 variables continuous in the real variable and measurable in the other variable is called a caratheodory function. So, claim f is continuous on the product, sorry F is measurable on the product space. Whether you put Borel cross x or Lebesgue cross x, it does not matter, it whatever it is, it is measurable.

So, we want to prove this. So, R equals union K belong to the integers of k minus 1 by 2 power n, k by 2 power n, you can take it either way like so, let us take it open here and take it closed here. So, I am dividing the real line into sub intervals each of length 1 by 2 power n very small when n is large, and then R is the union of all these semi open intervals.

Now, you define f n of t x equals f of k by 2 power n, x if t belongs to K minus 1 by 2 power n, K by 2 power n. T will belong to exactly one of these intervals and therefore, you have that it is given by this. So, we can write f n of T x, is σ k in Z, f of K by 2 power n, x into kai K minus 1 by 2 power n, k by 2 power m.

So, this is measurable, S measurable and this is of course continue T measurable whether it is Lebesgue or Borel it does not matter. Now, by example, by previous example each term is measured and that implies that f n is measurable in the product space, because you take any finite combination that is measurable and then you take in limited partial sums are all measurable and then you let n tend to infinity the limit is measurable.

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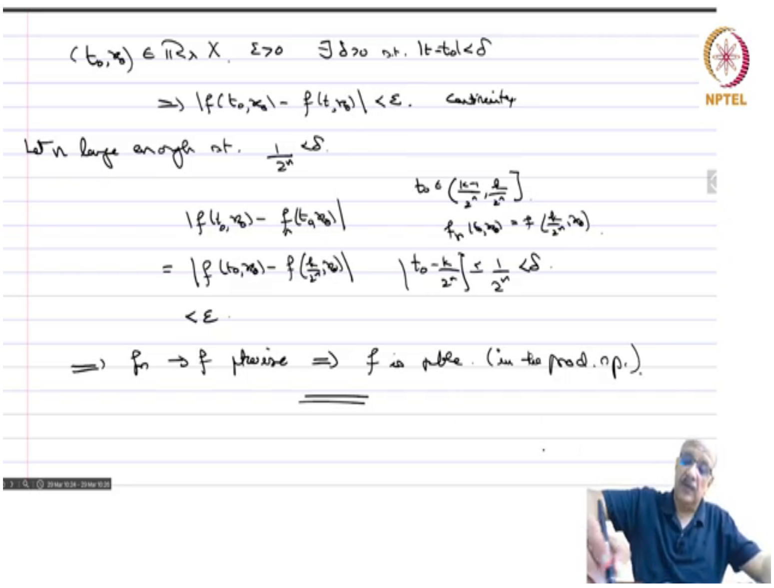

So, now, you take $(t_0, x_0) \in R \times X$, $\epsilon > 0$, given then there exists a $\delta > 0$ is that $|t - t_0| < \delta$, implies mod f of $|f(t_0, x_0) - f(t_0, x)| < \epsilon$, because the function is continuity. Comes from the continuity.

Now, n large enough such that $(\frac{1}{2})^n < \delta$. Then what is $|f(t_0, x_0) - f_n(t_0, x_0)|$. Now, you have to see. So, t_0 will belong to some k minus 1 by 2 the power of n, k by 2 power n and then f n of t naught, x_0 is f of k by 2 power n, x naught.

So, this is equal to mod f of t naught, x naught minus f of k by 2 power n, x naught, but T naught minus K by 2 power n is less than or equal to 1 by 2 power n and that is less than delta and therefore, you have that this is less than epsilon. So, this implies that f n converges to f pointwise implies f is measurable in the product space so, it is a important example.

So, if you have continuity in one variable and real line has Borel or Lebesgue σ algebra, measurability in the other variable, then the function is measurable in the product space. So, we will continue. So next time we will try to construct a measure on the product σ algebra, which comes from the 2 measures μ *and* λ on the regional spaces.