## Measure and Integration Professor S Kesavan Department of Mathematics Institute of Mathematical Science Lecture 51 Product spaces: Measurable functions

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(X,3) (Y,T) where representations and a rest Ax3 Ax3, Bx7. E, elementary out = Finite shift with Jef: (X,3), (Y, 2) when you f: XxY - IR given for acx, yer. Then the a-rection of f fx (8) = f(x8) 48 er. K-redion of f (3) = f By HBEX (X,S) (Y,T) (X,Y, J,E) Proge (X,3), (Y,7) where my f: Xn7 - Sit an Jai- where for. Then YREX, duer, fris 2-able and fo is J-1 Q, | () 29 Mar 1010-29 Mar 10 M (X,S) (Y,T) (X,Y, J,T) Pege (X,3), (Y,7), allo man f: Xx7 - Sir an Jx7- whe for. Then HEEK, DSEY, Fris 2-when and fi is J-when. NPTEL ME COR Q= ELANGEXXY from 7C3 JAT-ulle. HIEX Q in Carlle. Q = { 364 ( Carle Q} = {yey + f(+,3)>c}. = {yey) fely > c3. -> for in T-when IIIthen = 3 in I-whele.

= { 564 | f(1) > c }. = Syey) f. (3) > c3 => f is T-mlke Illy I's is I-able Eq. (X,J) (Y,T) when yo. f: X-3R J-while Lot F(z,y) = P(z) 5(x,3) & X,4 | Prain1>c3 = 5xex1 f(x)>c3x4 when reck . F is J.T - le INte Grent = fis) is Jx ? mble. => (x.) IN fraggy in Jn? where

So, we have now looking at product spaces. So, we have (X, S) and (Y, T) measurable spaces. So, measurable rectangle is of the form  $A \times B$ ,  $A \in S$ ,  $B \in T$  and then elementary set *E* equals finite disjoint union of measurable rectangles and then so, this is *E* elementary sets, then S(E) the  $\sigma$  algebra generated by E,  $\sigma$  algebra generated by E and that is denoted by  $S \times T$ . So, this is the  $\sigma$  algebra we put on X cross S.

So, now we have definition

**Definition:** Let (X, S), (Y, T) measurable spaces and  $f: X \times Y \to \mathbb{R}$  given a function,  $x \in X$ ,  $y \in Y$  then the x-section of f, denoted  $f_x$ 

$$f_{x}(\xi) = f(x,\xi), \ \xi \in Y_{\xi}$$

Similarly, the y-section of f,

$$f^{y}(\xi) = f(\xi, y), \ \xi \in X.$$

So, this is these are the 2 sections of the function just as we define x and y-section of sets we have sections of functions. Now, we have 3 measurable spaces. So, we have (X, S) we have (Y, T), then we have  $X \times Y, S \times T$ . So, when I say measurable, I will have to say in where it is measurable. So, I will say S measurable, T measurable or  $S \times T$  measurable depending on the context one can understand then, where these functions are and how they measurable.

So, now, we have a proposition just as we had a proposition for the test of measurability of sets this proposition gives you a test of measurability of a function. So,

**Proposition:** Let (X, S), (Y, T) measurable spaces  $f: X \times Y \to \mathbb{R}$  and  $S \times T$ , measurable function. Then for every  $x \in X$  for every  $y \in Y$ ,  $f_x$  and is T measurable and  $f^y$  is S measurable.

**Proof,** Y is this measurability. If either of these sections are not even for  $1 \ge 1 \le 1$  y the corresponding section is not measurable, then the original function cannot be measurable. So, this is how we use it to test for measurability.

So, let c belong to R, then you take Q equals to set of all  $(x, y) \in X \times Y$  such that f of x y is bigger than c. So, this is  $S \times T$  measurable, then for every  $x \in X$ ,  $Q_x$  is T measurable but what is  $Q_x$ ? So, this is set of all  $y \in Y$  set  $(x, y) \in Q$ .

Which is equal to set of all  $y \in Y$ . So, set f of x, y is bigger than C that is equal to set of all  $y \in Y$ . So, said  $f_x$  of y is bigger than C. So, this is T measurable and therefore, therefore, this implies that  $f_x$  is T measurable.

So, for every c in R this set is measurable and therefore, the by the definition f x. So, similarly  $f^y$  is also measurable. So, now example (X, S), (Y, T), measurable spaces f X to R, S measurable, g from y to R, T measurable. Now, let capital F of x y equal to f of x, we are defining it as independent of y. So, so, the for all x y belongs to  $X \times Y$ , such that f of x y is bigger than C. Set of all  $x \in X$  such that f of x is bigger than c cross Y and this is a measurable rectangle and therefore.

Capital F is  $S \times T$  measurable. Similarly, G of x y equals g y is  $S \times T$  measurable. The product of measurable functions is measurable, so, x y going to effects the g y variable separable case is  $S \times T$  measurable.

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Eq. IR equipped with Boral or Lab or-alg (X, J) when go f: RxX > 22 a given gr. st · V reex, the fr. ELSP lt, a) is can't on IR. · Y ber, the 8n . 21- flip) in J-while (Such a for in called a Casathéodory Function). Claim fie ruble on the product op TicxX. TIC = U (Loc e) TIC = U (2000)  $f_n(t_{2^n}) = f(\frac{t}{2^n}) \quad it t \in \left(\frac{k+1}{2^n}, \frac{k}{2^n}\right)$ · V wex, the fr. LISP ( +, v) is can't on R. · Y ten, the 8n . 21 flin in J-male. NPTEL (Such a for in called a Carathéodory Function). Chaim fie mble on the product op TRXX. TR = U (Lor )2  $f_{n}(t_{j}x) = f\left(\frac{k}{2^{n}}x\right) \quad iq \quad t \in \left(\frac{k+1}{2^{n}},\frac{k}{2^{n}}\right)$  $F_{n}(k,m) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2},m\right) \frac{1}{2} \left(\frac{k}{2},m\right) \frac$ By preve ag each torm is able => & is able in the product DD.

So, you take a measurable function in 1 variable measurable function another variable multiply them you will still get a measurable function in the next. Now, R equipped with Borel or Lebesgue  $\sigma$  algebra X, S measurement space. Now,  $f: R \to X, R \times X$  to R, given function such that for every  $x \in X$  the function t going to f(t, x), f(t, x) is continuous for every t in R, the function x going to f of t x is measurable, S measurable.

We have seen an example of thing earlier. So, such a function is called a caratheodory function. A function in 2 variables continuous in the real variable and measurable in the other variable is called a caratheodory function. So, claim f is continuous on the product, sorry F is measurable on the product space. Whether you put Borel cross x or Lebesgue cross x, it does not matter, it whatever it is, it is measurable.

So, we want to prove this. So, R equals union K belong to the integers of k minus 1 by 2 power n, k by 2 power n, you can take it either way like so, let us take it open here and take it closed here. So, I am dividing the real line into sub intervals each of length 1 by 2 power n very small when n is large, and then R is the union of all these semi open intervals.

Now, you define f n of t x equals f of k by 2 power n, x if t belongs to K minus 1 by 2 power n, K by 2 power n. T will belong to exactly one of these intervals and therefore, you have that it is given by this. So, we can write f n of T x, is  $\sigma$  k in Z, f of K by 2 power n, x into kai K minus 1 by 2 power n, k by 2 power m.

So, this is measurable, S measurable and this is of course continue T measurable whether it is Lebesgue or Borel it does not matter. Now, by example, by previous example each term is measured and that implies that f n is measurable in the product space, because you take any finite combination that is measurable and then you take in limited partial sums are all measurable and then you let n tend to infinity the limit is measurable.

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( to, x) & IRX X. ETO 3620 Dr. 1t-toled => \f(to,x\_1- f(t,x\_0) < E. construity Let v large an orgh st.  $\frac{1}{2^{n}} < \delta$ .  $| f(l_0, r_0) - f(l_0, r_0)|$   $l_0 \in \left(\frac{|k-n|}{2^{n}}, \frac{\rho}{k^n}\right]$   $= | f(l_0, r_0) - f(\frac{l_0}{2^{n}}, \frac{\rho}{k^n})|$   $| t_0 - \frac{k}{2^{n}} \int \frac{1}{2^{n}} < \delta$ . =) for the province =) f is public (in the prod. op.)

So, now, you take  $(t_0, x_0) \in R \times X$ ,  $\epsilon > 0$ , given then there exists a  $\delta > 0$  is that  $|t - t_0| < \delta$ , implies mod f of  $|f(t_0, x_0) - f(t_0, x)| < \epsilon$ , because the function is continuity. Comes from the continuity.

Now, n large enough such that  $(\frac{1}{2})^n < \delta$ . Then what is  $|f(t_0, x_0) - f_n(t_0, x_0)|$ . Now, you have to see. So,  $t_0$  will belong to some k minus 1 by 2 the power of n, k by 2 power n and then f n of t naught,  $x_0$  is f of k by 2 power n, x naught.

So, this is equal to mod f of t naught, x naught minus f of k by 2 power n, x naught, but T naught minus K by 2 power n is less than or equal to 1 by 2 power n and that is less than delta and therefore, you have that this is less than epsilon. So, this implies that f n converges to f pointwise implies f is measurable in the product space so, it is a important example.

So, if you have continuity in one variable and real line has Borel or Lebesgue  $\sigma$  algebra, measurability in the other variable, then the function is measurable in the product space. So, we will continue. So next time we will try to construct a measure on the product  $\sigma$  algebra, which comes from the 2 measures  $\mu$  *and*  $\lambda$  on the regional spaces.