Measure and Integration
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Lecture 50
Product Spaces

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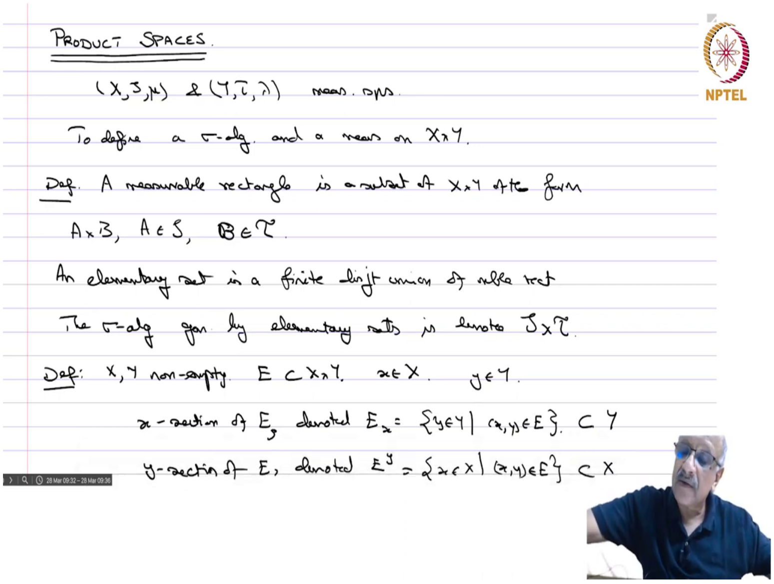

So, now we will start a new topic, this is **Product spaces**. So, let us take (X, S, μ) and (Y, T, λ) measure spaces. So, we want to define a σ algebra and measure on $X \times Y$, which is compatible with the structures on X and Y and also we want to relate the process of integration on $X \times Y$ with the process of integration on X and on Y.

So, our first aim of course, is to study the σ algebra which is there and so, we will do that to start with, then we will have to look at measurable functions and then finally, integration. So, let so, definition a measurable rectangle is a subset of $X \times Y$ of the form $A \times B$, $A \in S$, $B \in T$, an elementary set is a finite disjoint union of measurable rectangles. The σ algebra generated by elementary sets is denoted $S \times T$, so $S \times T$ is a σ algebra on $X \times Y$ and it is generated by the elementary sets, the elementary sets is the, finite disjoint unions of measurable rectangles, which are just products of sets taken from the 2 individual σ algebras.

Definition: X, Y non-empty of course, always non-empty so, we do not have to say this again, E contain in $X \times Y$ and $x \in X$. Then, we say the x-section of E and y in Y denoted

$$E_x = \{y \in Y \mid (x, y) \in E\} \subset Y.$$

So, this is a subset of Y remember. Similarly, the y-section of E denoted

$$E^{y} = \{x \in X \mid (x, y) \in E\} \subset Y.$$

and this of course, is contained in X.

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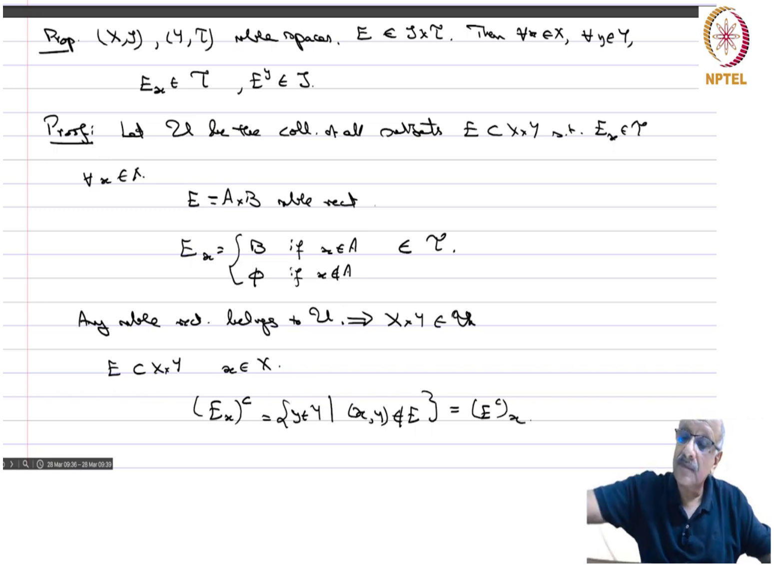

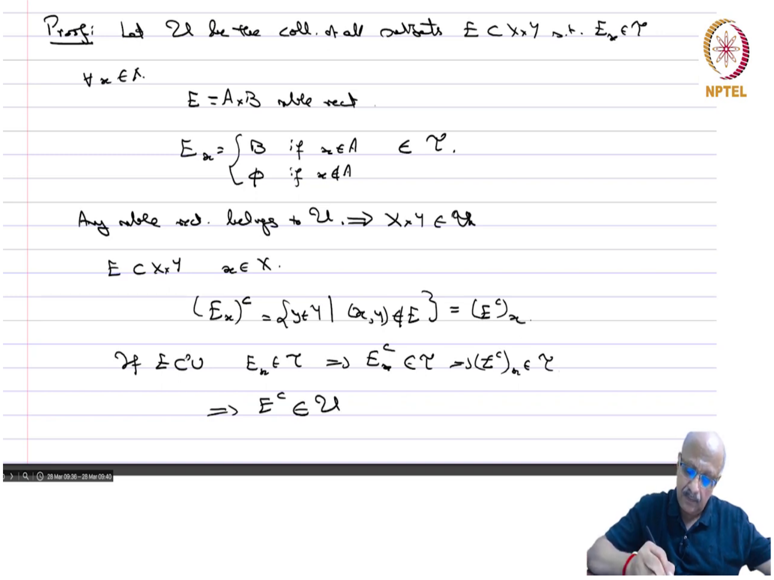

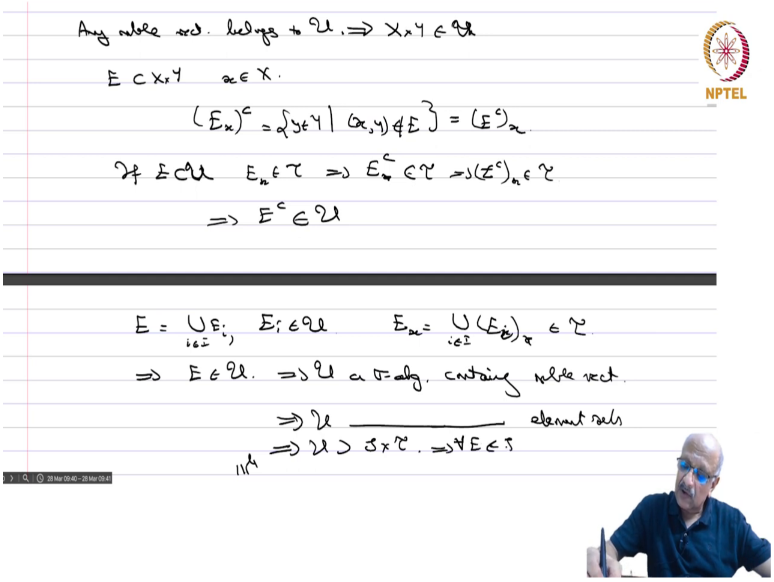

Proposition: So, (X, S), (Y, T), measurable spaces $E \subset S \times T$. The σ algebra and then $for\ every\ x \in X,\ y \in Y$, we have $E_x \in T,\ E^y \in S$.

So, this is a test of measurability if if this does not happen suppose you have a set E whose x-section does not belong to T for some x or for y-section does not belong to S for some y, then the set is not measurable because if it is an $S \times T$, both of these must happen.

So, this is just a test of measurability so proof. So, let U equals be the collection of all subsets E in $X \times Y$ such that $E_x \in T$, following the $x \in X$. So, now, we want to see what kind of set this U is. So, if you have $A \times B$, $E = A \times B$ measurable rectangle then, what is E_x? $E_x = B$, if x belongs to A and will be the empty set if x is not in A because $E = A \times B$.

If x is not in A then for no y, x y will be in the this. Similarly, if x is in A then all of y anyway in B will be in E and therefore, you have E x is this and therefore, this of course, belongs to T, therefore any measurable rectangle belongs to U. Now, in particular $X \times Y$ it says belongs to U. Now, if E is contained in $X \times Y$ and x is in X, then what is E_x complement the set of all y in Y, such that $(x, y) \notin E$, y is not in E x that means $(x, y) \notin E$ and this is nothing but E_x^c.

So, if E belongs to U, then E_x will belong to E implies, $E_x^c \in T$, implies $E_x^c \in T$, and therefore, this implies E complement belongs to U.

Similarly, $E = \cup E_i$, with $E_i \in U$, then E_x is nothing but union i in I, E x, E ix, and all this will belong to T, because each of them is in U and therefore, you have that E belongs to U. Therefore, U is closed under countable unions and complementation it contains $X \times Y$ therefore, U is a σ algebra containing measurable rectangles.

If it is a σ algebra containing measurable rectangles is going to contain all the elementary sets, implies U is a σ algebra containing elementary sets and so, U contains $S \times T$. So, every element in $S \times T$ is in U, that means $E_x \in T$. Similarly, so, this implies for every E in $S \times T$ for every $x \in X$, you have $E_x \in T$.

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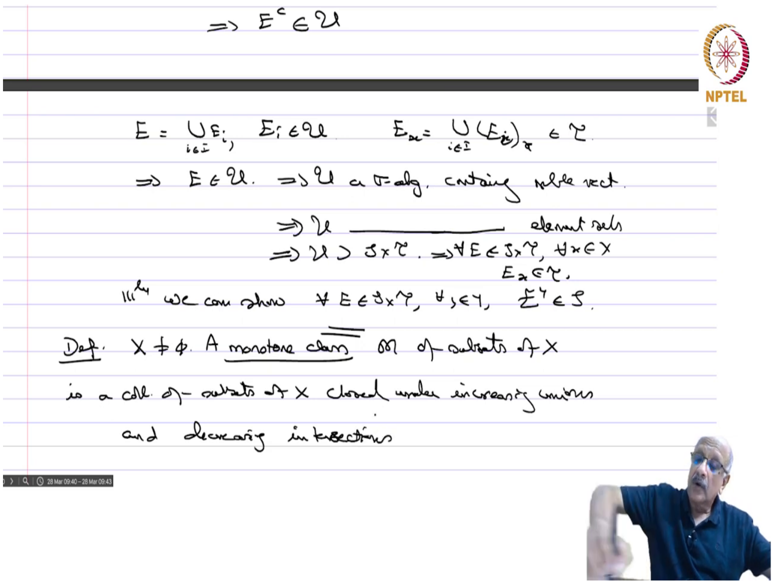

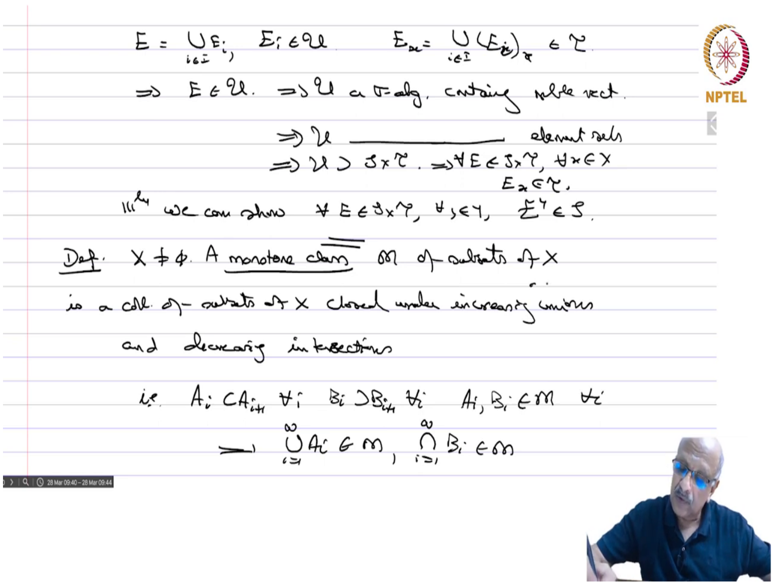

So, similarly we can show for every $E \in S \times T$, for every y in Y, you have $E^{y} \in S$. So, that completes the proof of that thing. So, now, we have seen rings, we have seen algebras, we have seen σ algebra and so on. Now, we are going to define 1 more collection of subsets of a given set x.

Definition: Let $X \neq \Phi$ is a non-empty monotone class, $M \subset X$, is a collection of subsets of X closed under increasing unions and decreasing intersections.

What does that mean? That is $A_i \subset A_{i+1}$, for all i, $B_i \subset B_{i+1}$ for all i, $A_i, B_i \in M$ for all i. Implies $\cup A_i \in M$ *and* $\cap A_i \in M$; $B_i \in M$. Increasing countable closed and increasing countable unions and decreasing countable intersections. Such a set is called monotone class.

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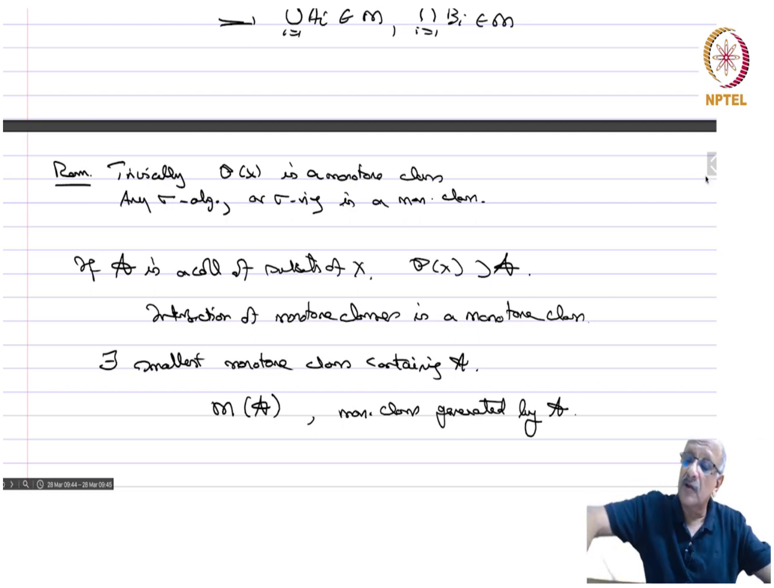

Remark: trivially B_x is a monotone class, any σ algebra or σ ring is a monotone class. So, if it is A collection of subsets of x then P(x) is a monotone class which contains A. Now, intersection of monotone classes, it is a monotone class that is easy to see. Therefore, there exists the smallest monotone class containing A, and this is called $M(A)$ and this is also called monotone class generated by A. So, given any collection you also have this.

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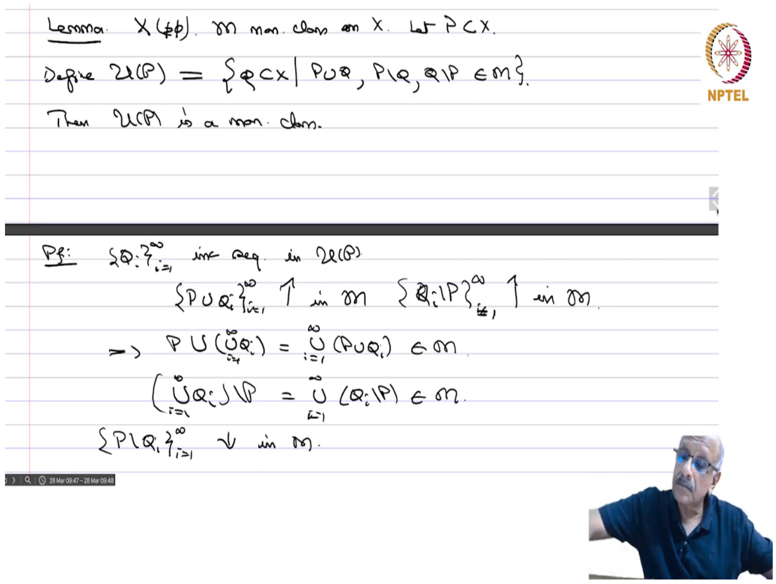

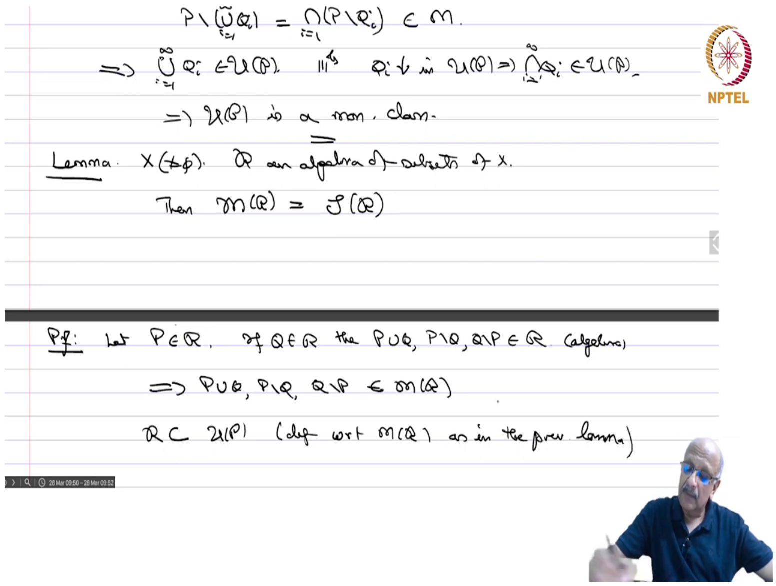

Lemma: X non-empty set and M monotone class in on X, let P be contained in X. Define

$U(P) = \{Q \in X\colon\ P \cup Q,\ P\backslash Q,\ Q\backslash P \in M\}$.

Then $U(P)$ is a monotone class.

Proof: Let us take $\{Q_i\}_{i=1}^{\infty}$, increasing sequence in $U(P)$, then $P = \bigcup_{i=1}^{\infty} Q_i$ is increasing in M and $Q_i\backslash P_i$ equals 1 to infinity is also increasing and in M.

And since M is a monotone class this implies P union, union Q_i, i equals 1 to infinity, which is union P union Q_i, equals 1 to infinity belongs to M and similarly, union Q_i, i equals 1 to infinity minus P is equal to Union i equals 1 to infinity Q_i minus P and this is increasing and therefore, this again in M.

Now P minus Q_i, is decreasing in M and P minus union Q_i, i equals 1 to infinity is nothing but intersection P minus Q_i, i equals 1 to infinity this is a decreasing thing and therefore, this also belongs to M. So, this implies that union Q_i belongs to U. Similarly, if Q_i is decreasing in $U(P)$ then intersection i equals 1 to infinity of Q_i is also in $U(P)$, therefore, $U(P)$ is a monotone class. Same type of proof elementary set theoretic arguments.

Lemma: Let $X \neq \Phi$ and R algebra of subsets of X, then $M(R)$, this is nothing but the smallest monotone class containing R, $M(R) = S(R)$, this smaller σ ring containing of R, so, monotone class and the σ ring generated by an algebra are one and the same, they are not 2 different objects.

Proof: Let $P \in R$, now if Q belongs to R, then $P \cup Q$, $P\backslash Q$, $Q\backslash P$ all belong to R because R is an algebra and this implies that $Q\backslash P$, $P\backslash Q$, all belong to $M(R)$, because if they belong to R they belong to $M(R)$, because $M(R)$ is bigger.

And therefore, this means that U have P. This means, R is contained in $U(P)$. Because for every $Q \in R$ these 3 have and R is contained in $U(P)$. And if U have P is defined as in the previous lemma defined with respect to $M(R)$ as in the previous lemma. $M(R)$ is a monotone class so, we can define like this.

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Pf: Let $P \in \mathcal{R}$. If $Q \in \mathcal{R}$ then $P \cup Q, P \setminus Q, Q \setminus P \in \mathcal{R}$ (algebra)

$\Rightarrow P \cup Q, P \setminus Q, Q \setminus P \in \mathcal{M}(\mathcal{R})$

$\mathcal{R} \subset \mathcal{U}(P)$ (def wrt $\mathcal{M}(\mathcal{R})$ as in the prev lemma)

$\mathcal{U}(P)$ is a mon. class $\supset \mathcal{R}$

$\Rightarrow \mathcal{U}(P) \supset \mathcal{M}(\mathcal{R})$.

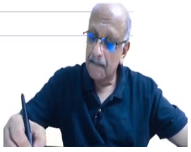
Let $Q \in \mathcal{M}(\mathcal{R})$. $P \in \mathcal{R}$, $Q \in \mathcal{U}(P)$

By symm. $\Rightarrow P \in \mathcal{U}(Q)$. $\Rightarrow \mathcal{R} \subset \mathcal{U}(Q)$.

$\Rightarrow \mathcal{M}(\mathcal{R}) \subset \mathcal{U}(Q)$.

$\forall P, Q \in \mathcal{M}(\mathcal{R}) \Rightarrow P \cup Q, P \setminus Q, Q \setminus P \in \mathcal{M}(\mathcal{R})$

i.e. $\mathcal{M}(\mathcal{R})$ is an algeb



$\Rightarrow \mathcal{M}(\mathcal{R}) \subset \mathcal{U}(Q)$.

$\forall P, Q \in \mathcal{M}(\mathcal{R}) \Rightarrow P \cup Q, P \setminus Q, Q \setminus P \in \mathcal{M}(\mathcal{R})$

i.e. $\mathcal{M}(\mathcal{R})$ is an algebra.

Now $\{E_i\}_{i=1}^{\infty}$ ctble coll. in $\mathcal{M}(\mathcal{R})$.

$F_n = \bigcup_{i=1}^{n} E_i \in \mathcal{M}(\mathcal{R})$ (alg)

$F_n \uparrow \Rightarrow \bigcup_{i=1}^{\infty} E_i = \bigcup_{n=1}^{\infty} F_n \in \mathcal{M}(\mathcal{R})$

$\Rightarrow \mathcal{M}(\mathcal{R})$ σ-alg. $\supset \mathcal{R}$

$\mathcal{M}(\mathcal{R}) \supset \mathcal{S}(\mathcal{R})$

$\mathcal{S}(\mathcal{R})$ is a mon. class $\supset \mathcal{R} \Rightarrow \mathcal{S}(\mathcal{R}) \supset \mathcal{M}(\mathcal{R})$

$F_n = \bigcup_{i=1}^{n} E_i \in \mathcal{M}(\mathcal{R})$ (alg)

$F_n \uparrow \Rightarrow \bigcup_{i=1}^{\infty} E_i = \bigcup_{n=1}^{\infty} F_n \in \mathcal{M}(\mathcal{R})$

$\Rightarrow \mathcal{M}(\mathcal{R})$ σ-alg. $\supset \mathcal{R}$

$\mathcal{M}(\mathcal{R}) \supset \mathcal{S}(\mathcal{R})$

$\mathcal{S}(\mathcal{R})$ is a mon. class $\supset \mathcal{R} \Rightarrow \mathcal{S}(\mathcal{R}) \supset \mathcal{M}(\mathcal{R})$

$\Rightarrow \underline{\underline{\mathcal{S}(\mathcal{R}) = \mathcal{M}(\mathcal{R})}}$

So, R is in $U(P)$, R is and, since $U(P)$ is a monotone class containing R, implies $U(P)$, contains M, R because $M(R)$ is the smallest monotone class. So, now let Q belong to $M(R)$. Then if P is in R then we just saw Q belongs to $U(P)$ because $M(R)$ is contained in $U(P)$ and therefore, Q belongs to $U(P)$ by the symmetry by symmetry you have P belongs to $U(P)$. So, this implies that R is contained in $U(P)$ and this a monotone class this implies $M(R)$ is contained in $U(Q)$.

So, if you have for all P and Q in $M(R)$, $M(R)$ is contained in $U(P)$ this implies that $P \cup Q$, $P \backslash Q$, $Q \backslash P$ belongs to $M(R)$, that is $M(R)$ this is an algebra. Now, E i, i equals 1 to infinity countable collection in $M(R)$, then F i equals to union, F n is equals to union i equals 1 to n, E i belongs to a $M(R)$, because this is an algebra and F n is increasing this implies union i equals 1 to infinity E i, equals union i equals 1 to infinity, n equals 1 to infinity F n belongs to MR.

So, this implies that $M(R)$ is a σ algebra and it contains R and therefore, you have $M(R)$ contains S of R on the other hand $S(R)$ is a σ, is a monotone class and this implies $S(R)$ contains $M(R)$. So, $M(R)$ is contained $S(R)$, $S(R)$ is contained in $M(R)$ and therefore, this implies that $S(R) = M(R)$. So, if you have an algebra then whether you make the monotone class or is σ algebra it does not matter.

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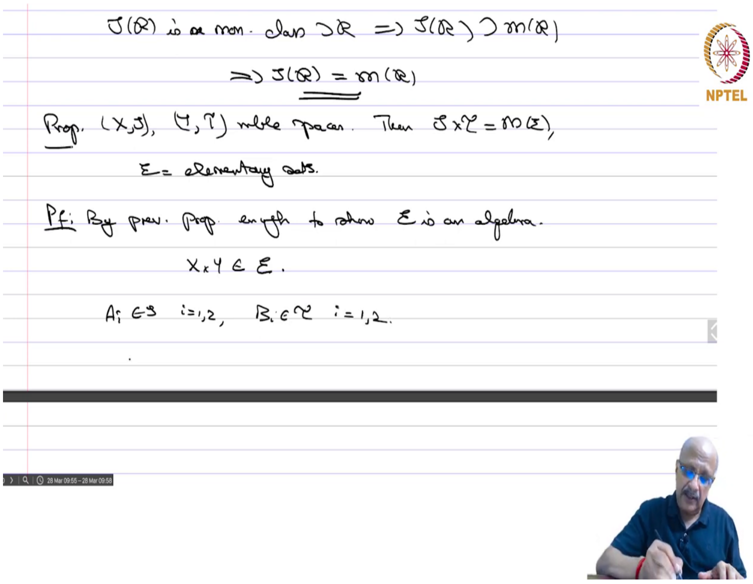

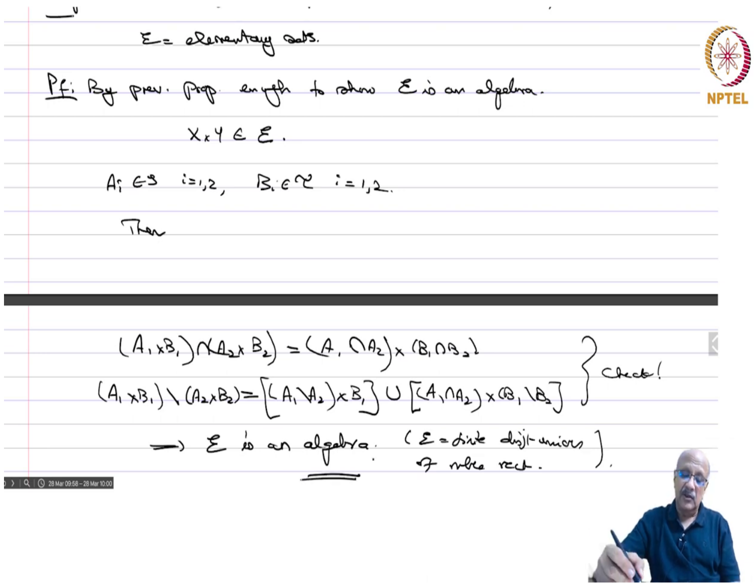

So, next proposition

Proposition: (X, S), (Y, T) measurable spaces then $S \times T = M(E)$, where E is equal to eliminatory sets. So, if you take all the elementary sets, then the σ algebra generated by elementary sets and σ algebra generated and the monotone class generated by the elementary sets both are 1 and the same.

Proof: By previous proposition enough to show E is an algebra. So, of course $X \times Y$ is a measurable rectangle.

So $A_i \in S,\ i = 1, 2$, and $B_i \in T,\ i = 1, 2$ then you must this is some set theory, which you should check $A_1 \times B_1 \cap A_2 \times B_2$ it is nothing but $A_1 \times A_2 \cap B_1 \times B_2$ and $A_1 \times B_1 \backslash A_2 \times B_2 = A_1 \backslash A_2 \times B_1 \backslash B_2 \cup A_1 \cap A_2 \times B_1 \backslash B_2$ this is check and therefore, the intersection of any measurable rectangle is a measurable rectangle and the difference of measurable rectangles is the disjoint union of measurable rectangles, and therefore, it is an elementary set.

Therefore, you have that so, this implies so, if you take any elementary sets then the different union and difference will just be again and elementary set because of these 2 relationships which we have used and therefore, E is an algebra, what is E? E equals finite disjoint unions of measurable rectangles and therefore, that is again an algebra and then by the previous theorem we know that the σ algebra generated this and monotone class generated by this are one in the same so we will continue this next time.