Measure and Integration
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences

Lecture No- 5
1.5 – Measurable Sets

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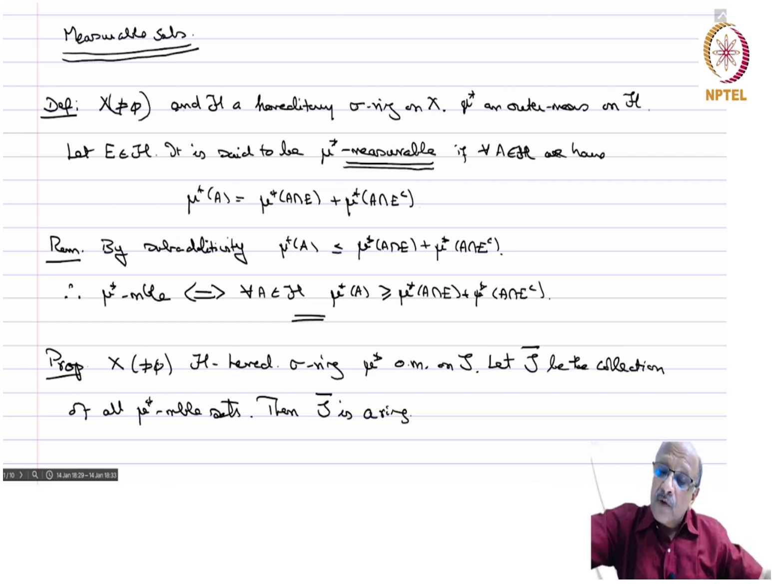

We will now talk about measurable sets. Let, so define it.

Definition: So, let X be a non-empty set, and H a hereditary σ-ring on X, μ * an outer measure on H. Let $E \in H$ is said to be μ *-measurable, ff for every $A \in H$, we have

$$\mu*(A) = \mu*(A \cap E) + \mu*(A \cap E^c).$$

Remark: By sub-additivity, we have that $\mu*(A) \leq \mu*(A \cap E) + \mu*(A \cap E^c)$. Therefore, μ *-measurable $\Leftrightarrow \forall A \in H$, $\mu*(A) \geq \mu*(A \cap E) + \mu*(A \cap E^c)$.

So, now we start with the nice proposition.

Proposition: So, X non empty, H hereditary σ-ring and μ * outer measures on S. Let $\bar{S}$ be the collection of all μ *-measurable sets. Then, $\bar{S}$ is a ring.

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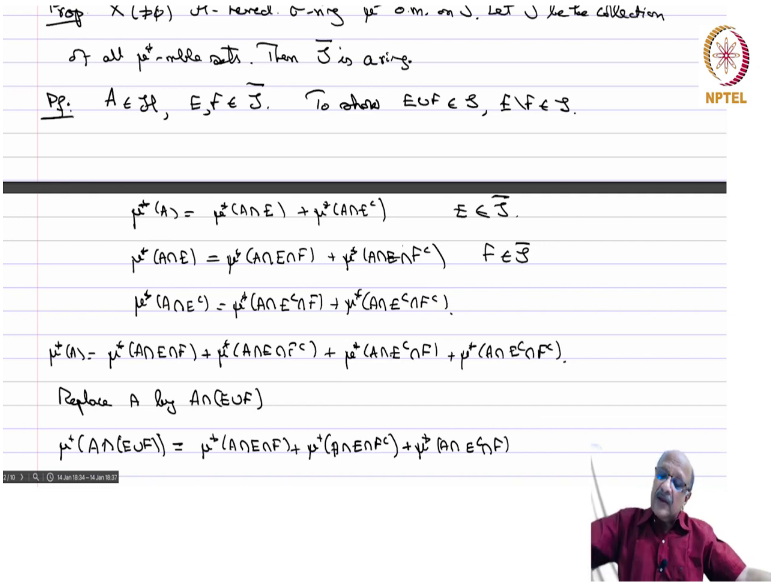

proof. Let us take $A \in H$, $E, F \in \overline{S}$. We want to show that $E \cup F \in S$, $E \setminus F \in S$.

Now, $\mu*(A) = \mu*(A \cap E) + \mu*(A \cap E^c)$, $E \in \overline{S}$.

Also, $\mu*(A \cap E) = \mu*(A \cap E \cap F) + \mu*(A \cap E \cap F^c)$, $F \in \overline{S}$ and

$$\mu*(A \cap E^c) = \mu*(A \cap E^c \cap F) + \mu*(A \cap E^c \cap F^c).$$

So, if you now substitute for this in all that, so you get

$$\mu*(A) = \mu*(A \cap E \cap F) + \mu*(A \cap E \cap F^c) + \mu*(A \cap E^c \cap F) + \mu*(A \cap E^c \cap F^c).$$

Now, replace A by $A \cap (E \cup F)$. Then if you substitute A intersection E union F in these three:

$$\mu(A \cap (E \cup F)) = \mu*(A \cap E \cap F) + \mu*(A \cap E \cap F^c) + \mu*(A \cap E^c \cap F).$$

$$\Rightarrow \mu*(A) = \mu*(A \cap (E \cup F)) + \mu*(A \cap E^c \cap F^c)$$

$$= \mu*(A \cap (E \cup F)) + \mu*(A \cap (E \cup F)^c)$$

$$\Rightarrow E \cup F \in \overline{S}.$$

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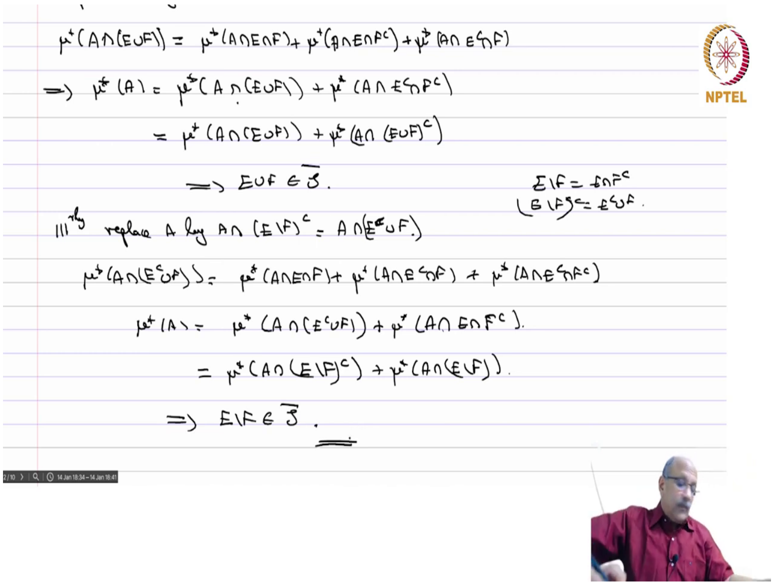

So, similarly, replace $A \cap (E\backslash F)^c = A \cap (E^c \cap F)$. So, now we replace it again in that relationship. So, then you get that

$$\mu * (A \cap (E^c \cap F)) = \mu * (A \cap E \cap F) + \mu * (A \cap E^c \cap F) + \mu * (A \cap E^c \cap F^c)$$

So, you get $\mu * (A) = (A \cap (E^c \cup F)) + \mu * (A \cap E \cap F^c)$

$$= \mu * (A \cap (E\backslash F)^c) + \mu * (A \cap (E\backslash F))$$

and from this we deduce that $E\backslash F \in \overline{S}$.

So, you have that $\overline{S}$ is a ring.

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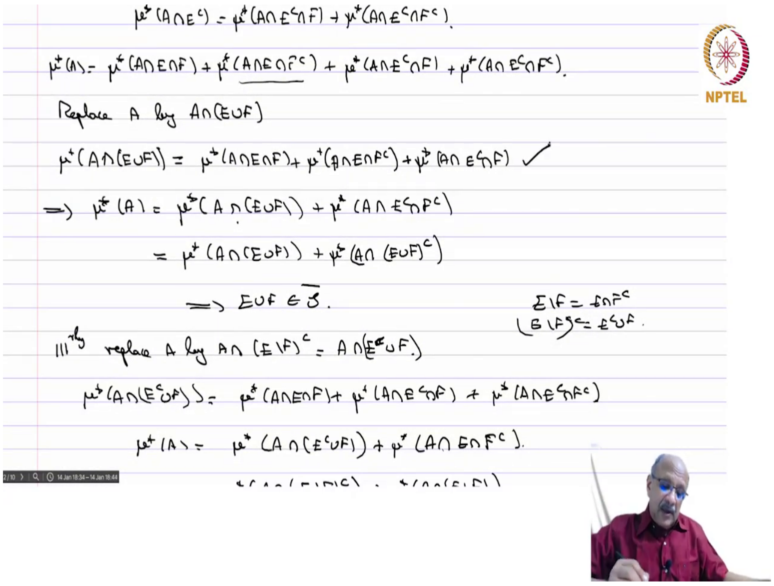

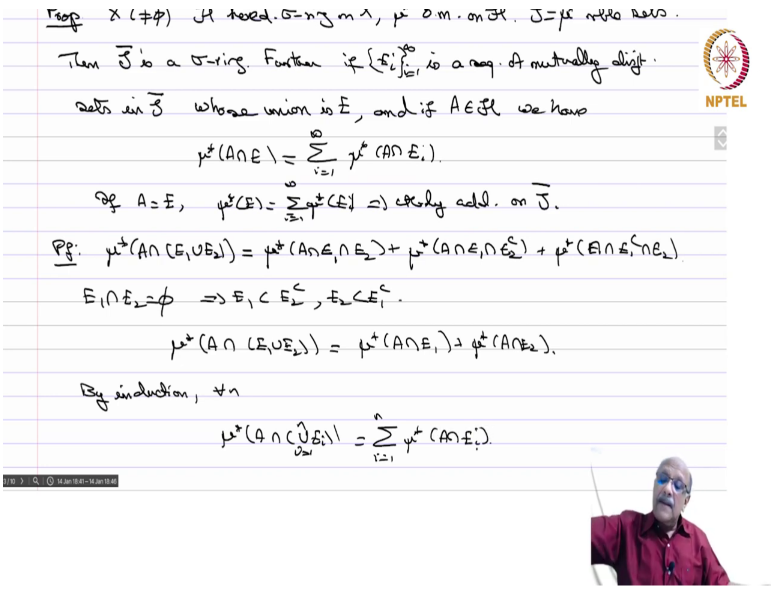

So, now, in fact we can prove much more than this. So, the next proposition.

Proposition: X non-empty set, H hereditary σ-ring, μ * outer measure on H. Then $\overline{S}$ bar is a σ-ring. Further, $\{E_i\}_{i=1}^{\infty}$ is a sequence of mutually disjoint sets in $\overline{S}$, whose union is E and if $A \in H$, we have $\mu^*(A \cap E) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$.

In particular, if $A = E$, $\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E_i) \Rightarrow$ countably additive on $\overline{S}$.

proof. So, we already saw

$\mu * (A \cap (E_1 \cup E_2)) = \mu * (A \cap E_1 \cap E_2) + \mu * (A \cap E_1 \cap E_2^{\ c}) + \mu * (A \cap E_1^{\ c} \cap E_2)$.

So, then, if $E_1 \cap E_2 = \phi$, then what do you have? That implies $E_1 \subset E_2^{\ c}$, $E_2 \subset E_1^{\ c}$.

So, you apply all these three things.

So, $\mu * (A \cap (E_1 \cup E_2)) = \mu * (A \cap E_1) + \mu * (A \cap E_2)$.

So, by induction, for any n we have

$$\mu * (A \cap (\cup_{i=1}^{\ n} E_i)) = \sum_{i=1}^{n} \mu * (A \cap E_i).$$

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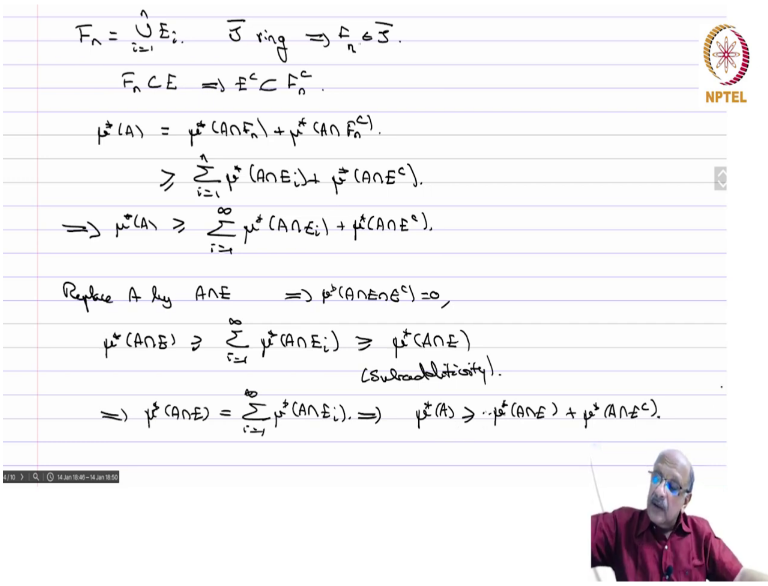

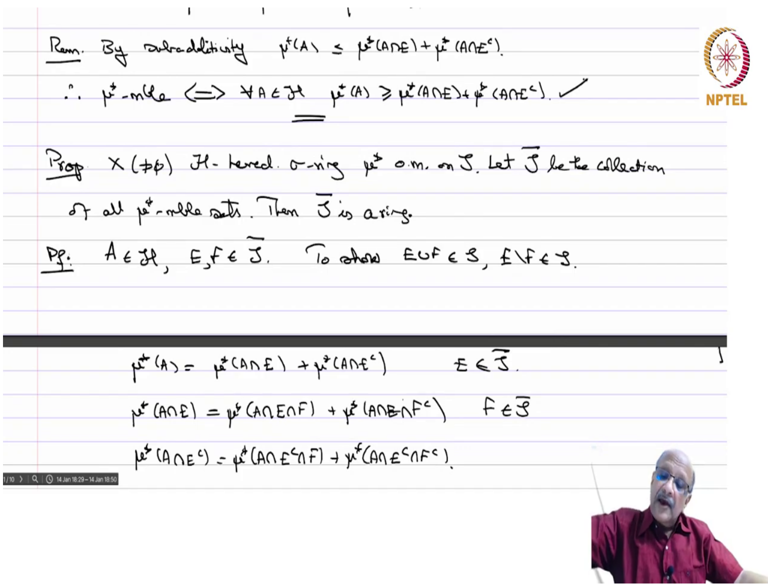

So, now, let us set $F_n = \cup_{i=1}^{n} E_i$. Now, we know that $\overline{S}$ is a ring $\Rightarrow F_n \in \overline{S}$. Further,

$F_n \subset E \Rightarrow E^c \subset F_n^c$. So, we now have

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$$

$$\geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E^c).$$

Therefore, $\mu^*(A) \geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E^c)$. Now, replace A by $A \cap E$. So, then $\mu^*(A \cap E \cap E^c = 0)$ and

$$\mu^*(A \cap E) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) \geq \mu^*(A \cap E).$$

$$\Rightarrow \mu^*(A \cap E) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$$

$$\Rightarrow \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

And I already made this remark long ago, that it is enough to prove the greater than or equal to inequality here.

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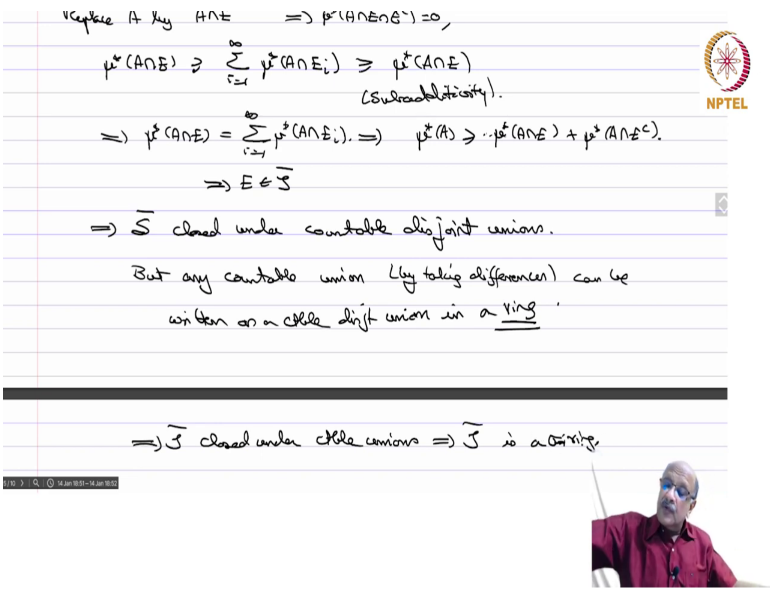

And therefore, this implies that $E \in \overline{S} \Rightarrow \overline{S}$ is closed under countable disjoint unions. But any countable union by taking differences we have already done can be written as a countable disjoint union in a ring. This implies that $\overline{S}$ is closed under countable unions, and this implies $\overline{S}$ is a σ-ring.

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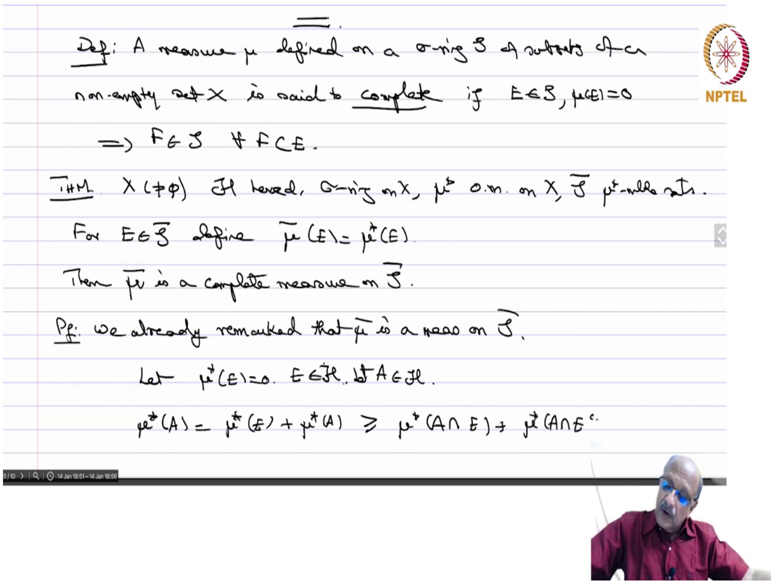

Definition: A measure μ, defined on a σ-ring S of subsets of a non-empty set X is said to be complete if $E \in S, \mu(E) = 0 \Rightarrow F \in S, \forall F \subset E$.

So, we have the following theorem which you have almost proved.

Theorem: X non-empty set, H hereditary σ-ring on X, μ *-outer measure on X, $\overline{S}$ μ *-measurable sets. For $E \in \overline{S}$, define $\overline{\mu}(E) = \mu^*(E)$. Then $\overline{\mu}$ is a complete measure on $\overline{S}$.

proof: we already remarked that $\overline{\mu}$ is a measure on $\overline{S}$. So, now let $\mu^*(E) = 0$, $E \in H$. Let $A \in H$. Then, you have $\mu^*(A) = \mu^*(E) + \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

So, this is true.

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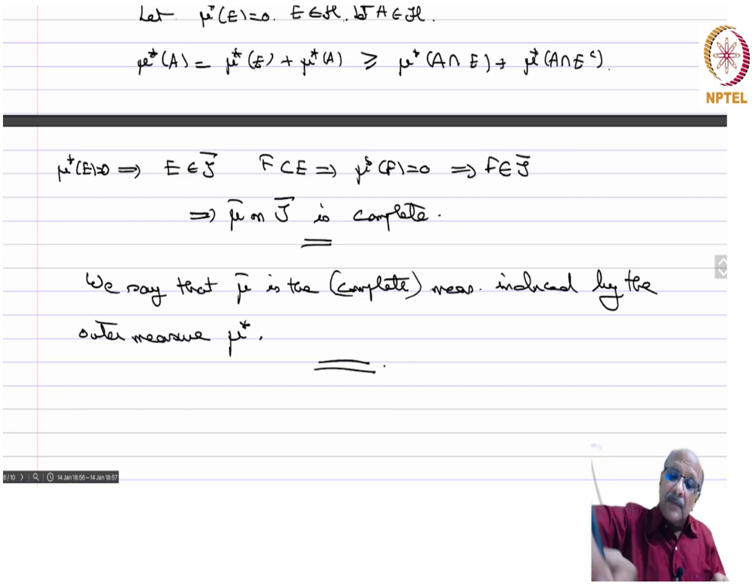

So, this implies that, $\mu * (E) = 0 \Rightarrow E \in \overline{S}$. Then $F \subset E \Rightarrow \mu * (F) = 0 \Rightarrow F \in \overline{S}$.

So, this implies that $\overline{\mu}$ on $\overline{S}$ is complete.

This completes the proof of the theorem.

So, we say that $\overline{\mu}$ is the complete measure induced by the outer measure $\mu *$.

So, our next step is clear, we know that we have a measure on a ring, then you can have an outer measure in a very natural way, and then you know how to construct a mu bar. Now, the only question is are the elements of the original ring also in the hereditary sigma-ring, are they also in the, are they all measurable or not? So, that is a question which we have to answer. And therefore, once we answer that, then we will know how to extend the measure to something bigger. So, we will do that next time.