Measure and Integration
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Lecture 49
Exercises

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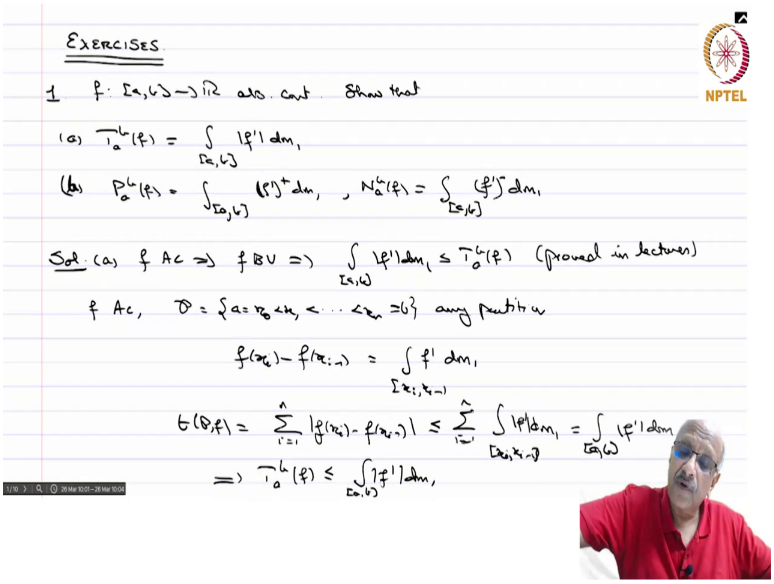

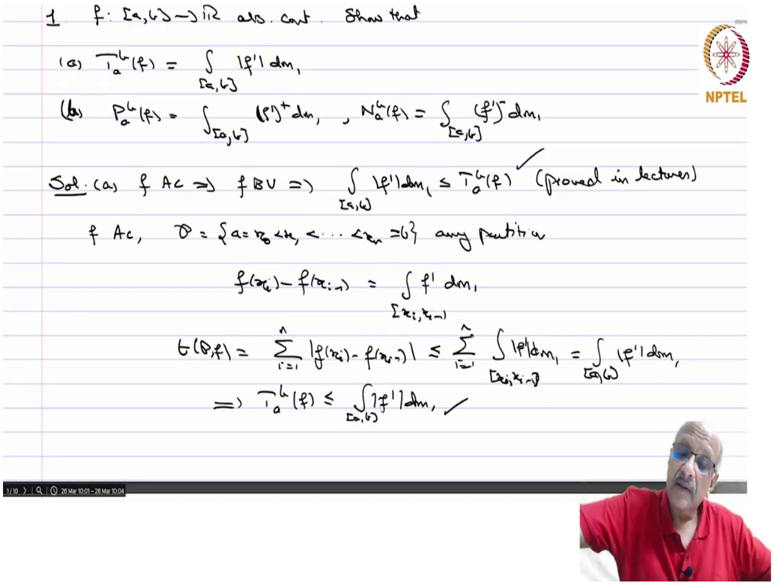

So, now, let us do some exercises. So, first 1.

1: $f: [a, b] \to \mathbb{R}$ absolutely continuous, show that

$$(a)\ T^b_a(f) = \int_{[a,b]} |f'|\, dm_1$$

So, we proved this for continuously differentiable functions. So, it is also true for absolutely continuous functions.

(b) $P^{b}_{a}(f) = \int_{[a,b]} (f^{'})^{+} \ dm_{1}$

(c) $N^{b}_{a}(f) = \int_{[a,b]} (f^{'})^{-} dm_{1}$

Solution (a) f absolutely continuous implies $\int_{[a,b]} |f^{'}| \, dm_{1} \leq T^{b}_{a}(f)$. So, proved in lectures you already saw this. Now f is absolutely continuous.

So, $\wp = \{a = x_{0} < x_{1} < x_{2} <..... < x_{n} = b\}$ any partition then you have

$f(x_{i}) - f(x_{i-1}) = \int_{[x_{i-1}, x_{i}]} f^{'} \, dm_{1}$

that is just by absolute continuity and therefore you have $T^{b}_{a}(f)$ which you have to take the modulus which is equal to sigma i equals 1 to N mod of x i minus f x i minus 1 that less than equal to sigma i equals 1 to N integral x i x i minus 1 mod $f^{'}$ d m1 which is equal to integral a b mod $f^{'}$ dm1.

And therefore this implies a $T^{b}_{a}(f)$ it is a supremum is also less than mod $f^{'}$ d m1 over a **(b).** So, that completes the proof you have both inequalities you have 1 here you have 1 here and therefore that does the trick.

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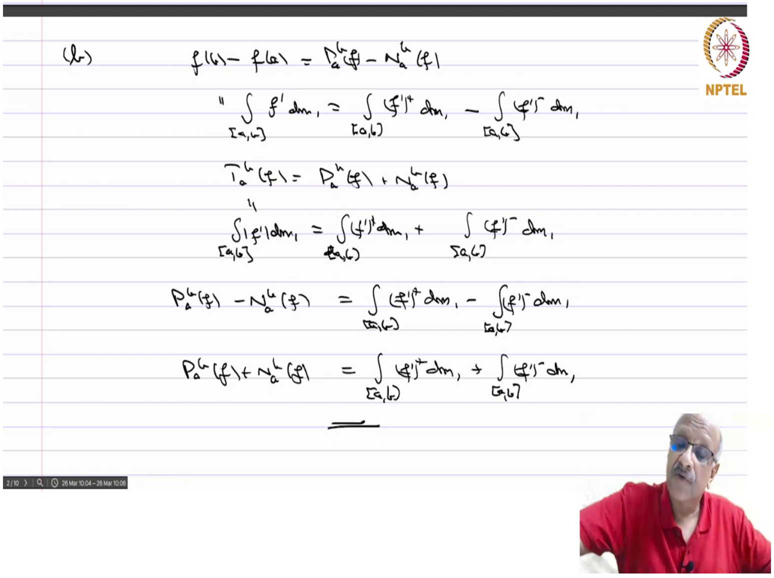

(c). So, you have that f b minus f a equals p a b of f minus $N^{b}_{a}(f)$ but that is equal to integral mod $f^{'}$ d m1 over a b equals integral sorry not mod equals $f^{'}$ d m1 and therefore that is equal to integral a b $f^{'}$ plus d m1 minus integral over a b f' minus d m1 and you also have $T^{b}_{a}(f)$ equals p a b of f plus $N^{b}_{a}(f)$ and that is equal to integral mod f' d m1 by a and that is equal to integral f' plus over a b d m1 plus integral over a b f' minus d m1.

And consequently you have p a b f minus $N^{b}_{a}(f)$ equals integral over a b f' plus d m1 minus integral f' minus d m1 over a b and p a b of f plus $N^{b}_{a}(f)$ equals integral a b f' plus d m1 plus integral a b f' minus d m1 and that completes the proof because you simply solve for p a b and N^{b}_{a} from these two.

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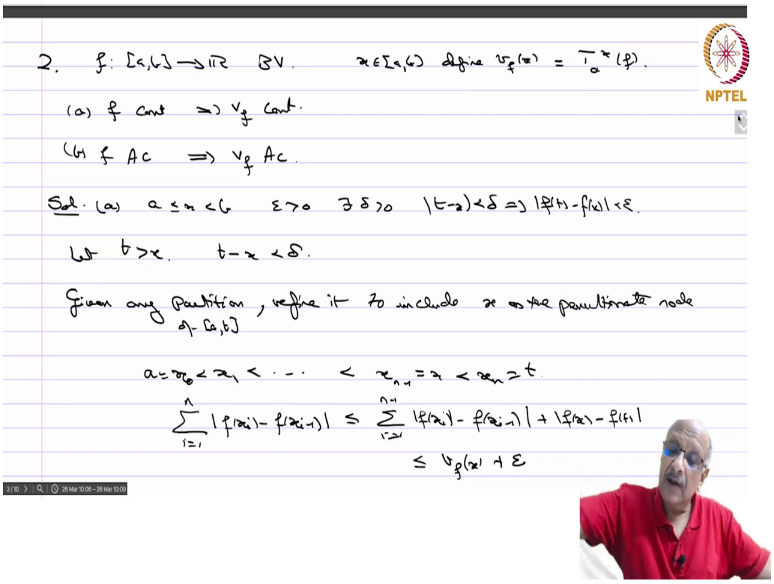

(2): Second question $f: [0, 1] \to \mathbb{R}$ bounded variation for $x \in [a, b]$ define $V_f(x) =$ equal to $T^b{}_x(f)$.

(a), f continuous implies V_f continuous.

(b), f absolutely continuous implies V_f absolutely.

Solution: We will prove the continuity by showing the left and the right continuity. So, let a less than equal to x less than b epsilon greater than 0 then there exists a delta positive such that T minus x less than delta implies mod f T minus f x less than epsilon this is the usual continuity.

So, now, let T belong to x T be bigger than x and T minus x less than delta. So, given any partition refine it to include partition of 80 refine it include x as the penultimate node. So, what you want to do you have a equals x naught less than x 1 less than etcetera less than x N minus 1 which will be equal to x less than x N which will be equal to t. Then sigma i equals 1 to N mod f x i minus f of x i minus 1 is less than equal to sigma i equals 1 to N minus 1 mod f of x i minus f of x i minus 1 plus mod f x minus f T and that is less than equal to this will be $V_f(x)$ because the last node is for here this up to N minus 1 that is equal to x.

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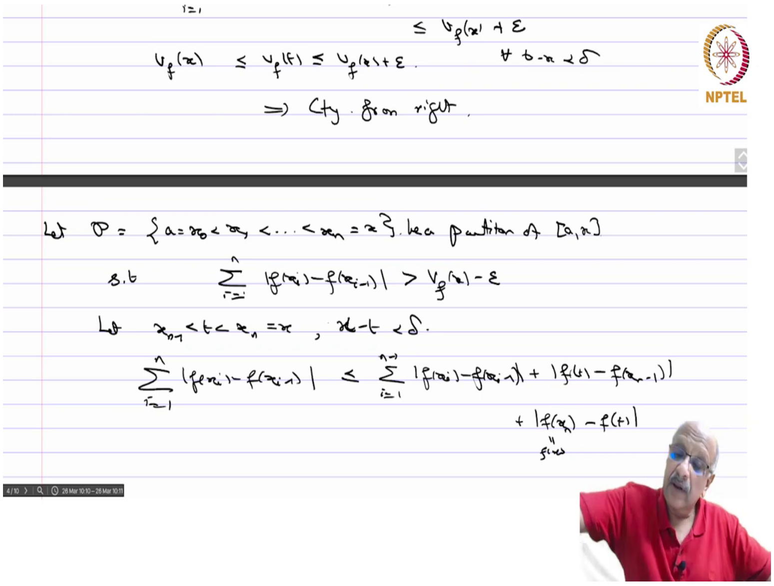

So, that is less than equal to v of x plus f x minus f T is less than epsilon and this can be done for any partition and therefore you have this implies that $V_f(T)$ is less than equal to $V_f(x)$ plus epsilon but on the other hand you know that v f the total variation is of course a monotonic function the more longer the interval even more partitions are there for the supremum and therefore $V_f(x)$ is anyway less than this and this is true for all T minus x less than delta and this implies continuity from right.

Now, for the continuity from left. So, you take any partition p equals a equals x not less than x x1 less than etcetera lesson x N equal to x partition of a x and now such that. So, let b a partition of a x such that sigma i equals 1 to N mod f of x i minus f of x i minus 1 is bigger than $V_f(x)$ minus epsilon because the partition is a supremum the supremum is $V_f(x)$. So, i can always find the partition which is bigger than v f x minus epsilon.

Now let x N minus 1 less than T less than x N equal to x and T my x minus T less than delta. So, sigma i equals 1 to N mod f of x i minus f of x i minus 1 is less into sigma i equals 1 to N minus 1 mod f of x i minus f of x i minus 1 plus mod f of x f of T minus f of x N minus 1 plus mod f of x N minus f T and this is equal to f of x remember I am just use the triangle inequality.

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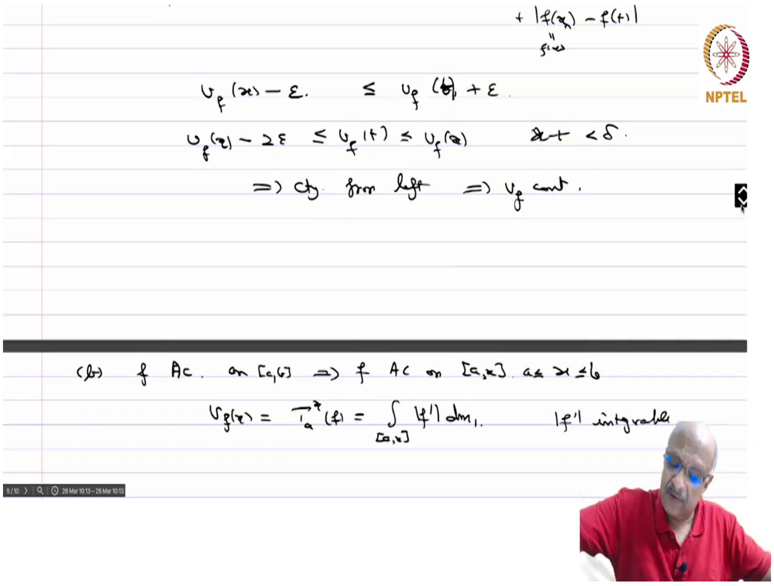

So, this is less than equal to $V_f(x)$, $V_f(T)$ because I have a partition here and then I added T as a point b f of T and the last term is less than epsilon and the left hand side is bigger than $V_f(x)$ minus epsilon. So, $V_f(x)$ minus 2 epsilon is less than equal to $V_f(T)$ and that is less than equal to $V_f(x)$ because x is bigger than t.

So, x is bigger than T and therefore once again this is for all x minus T less than delta and this implies continuity from left. So, this implies that v f is continuous. b, f is absolutely continuous on a b and that is implies f is absolutely continuous on a x for any x less than equal to a less than equal to x s. So, you have that $V_f(x)$ is $T^b{}_x(f)$ which is equal to integral a x mod f' d m1 now mod f' is of course integrable.

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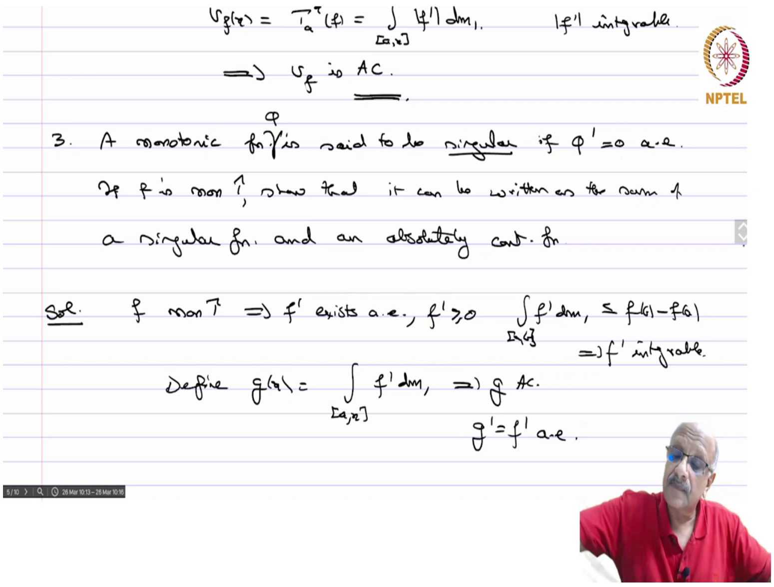

And so, $V_f(x)$ is nothing but the indefinite integral of an integrable function and therefore this implies that v f is A C.

(3): A monotonic function is said to be singular if f' the monotonic function f, $f^{'} = 0$ a.e.. So, example is the cantor function which is a monotonic increasing function which is the derivative 0 almost everywhere if f is monotonic increasing show that it can be written.

So, let me first not to give you confusion between the two f's here. So, this is for the if f is 1 turning sure it can be written as the sum of a monotonic increasing function. Sorry, a singular function and an absolutely continuous. Solution so, f monotonic increasing implies f' exists almost everywhere f' greater than equal to 0 and integral f' d m1 over a b is less than equal to f b minus f a implies f' integrable.

Define g of x equals integral a to x f' d m1 and this implies that g is absolutely continuous and also you know that g dash equal to f' almost every.

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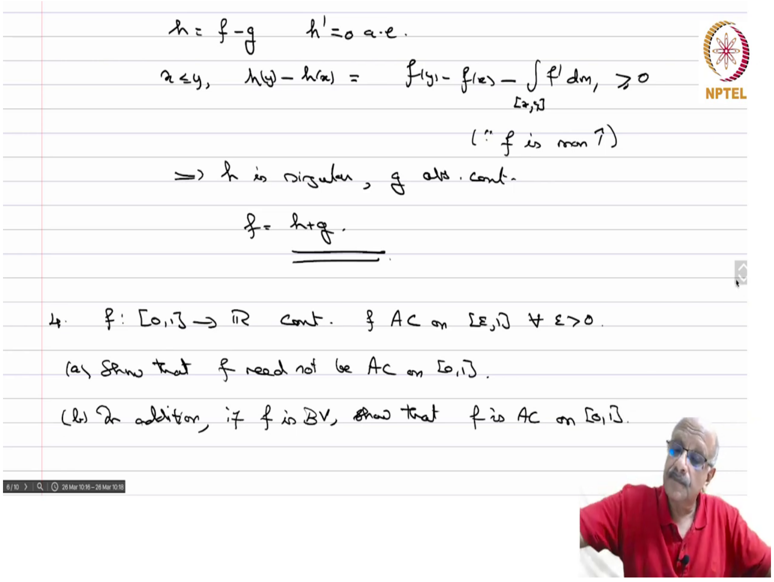

Now, you set h equals f minus j then h dash equal to 0 almost everywhere and you have if x is less than equal to y, h y minus h x is the same as $f(x)$ minus $f(x)$, h is $f(x)$ minus integral x to y f' d m1 but that is great or equal to 0 since f is monotonic increase the integral of a monotonic increasing function is less of the derivative is less than or equal to the values of the end points. So, this is this thing. So, this implies that h is singular because it is monotonic increasing it is monotonic and has zero derivative g is absolutely continuous and f equals h plus g.

(4): $f: [0, 1] \to \mathbb{R}$ continuous and f absolutely continuous on epsilon 1 for every epsilon positive.

(a), show that f need not be absolutely continuous on [0 1].

(b), in addition if f is Bv show that f is absolutely continuous on 0.

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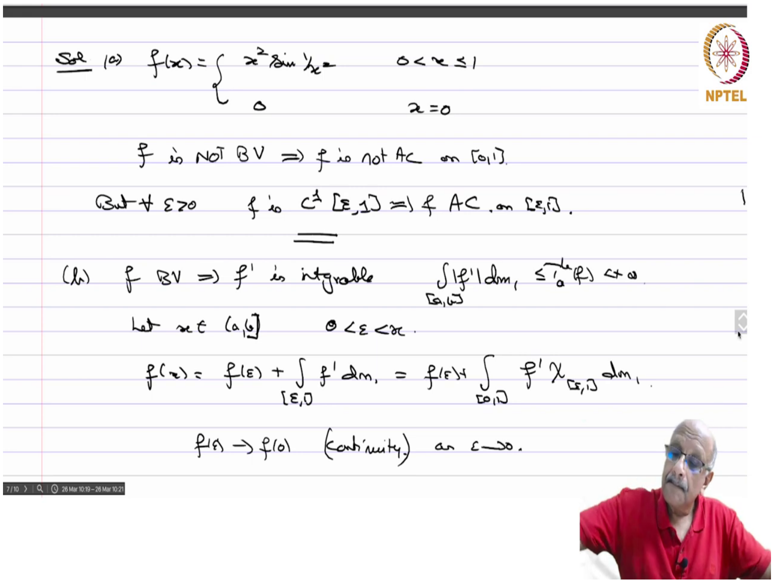

Solution. **(a)** So, the first 1 we will give a counter example. So, let us take

$$f(x) = x^2 sin\left(\frac{1}{x}\right), \quad 0 < x \leq 1;$$

$$= 0 \qquad\qquad x = 0.$$

then we know f is not BV we have already seen and this implies f is not absolutely continuous on [0 1] but for every epsilon positive f is c 1 on epsilon 1 implies f is absolutely purpose. So, this is [a b].

(b): So, now, we have to show now we are given that f is BV implies $f^{'}$ is integral in fact you know the integral mod f' d m1 over [a, b] we saw this in the beginning also is this equal to $T^b_a(f)$ which is less than plus infinity. So, now let $x \in [a, b]$ and let 0 less than epsilon less than x. So,

$$f(x) = f(\epsilon) + \int_{[\epsilon,x]} f^{'} dm_1 = f(\epsilon) + \int_{[0,b]} f^{'} \chi_{[\epsilon,x]} dm_1.$$

now as ϵ , $f(\epsilon) \to f(0)$ this is continuity as $\epsilon \to 0$.

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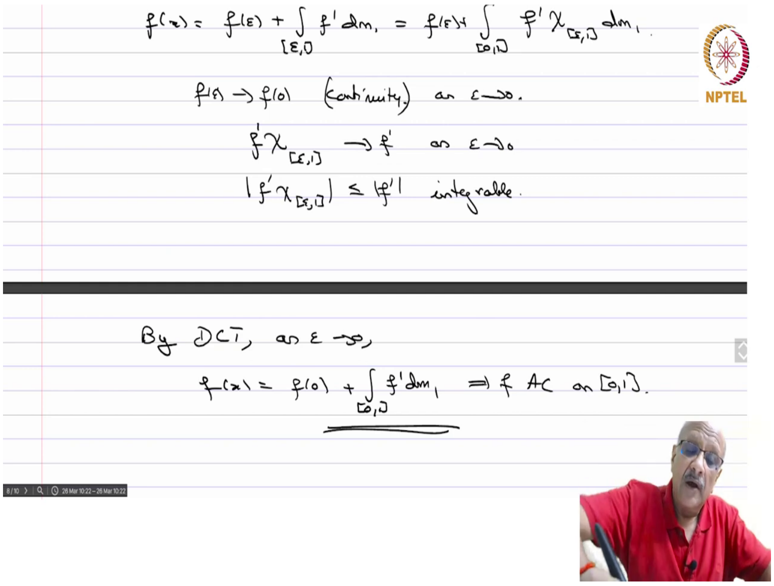

$f(\epsilon) \rightarrow f(0)\ as\ \epsilon \rightarrow 0$ and $f^{'}\chi_{[\epsilon,1]} \rightarrow f^{'}(0)\ as\ \epsilon \rightarrow 0,\ |f^{'}\chi_{[\epsilon,1]}| \leq |f^{'}|$

and this is integral therefore by Dominated convergence theorem as $\epsilon \rightarrow 0$ we have

$$f(x) = f(0) + \int_{[0,x]} f^{'} dm_1 \Rightarrow f\ is\ absolutely\ continuous\ on\ [0,1].$$

So, with this we will conclude this chapter next time we will start a new topic.