Measure and Integration
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Lecture 48
Absolute continuity

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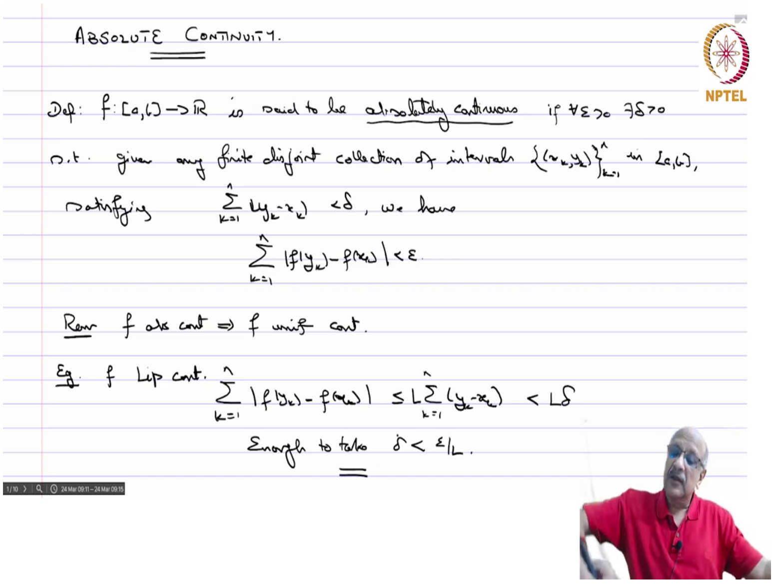

Absolute Continuity:

So, we were looking at the question as to when the function can be written as an indefinite integral of an integrable function and we saw that a necessary condition for this was that it should be uniformly continuous and of bounded variation. Today, we are going to introduce a new concept which will give us a necessary and sufficient condition for a function to be written as an indefinite integral of an integrable function and of course as we have already seen that this integrable function will be equal almost everywhere to the derivative of the original function.

So, we are now going to study the concept of Absolute Continuity. So,

Definition, $f:[a,b]\to\mathbb{R}$ is said to be absolutely continuous. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that given any finite collection, finite disjoint collection of intervals $\{(x_k, y_k)\}_{k=1}^{n}$ in [a, b] satisfying $\sum_{k=1}^{n} y_k - x_k| < \delta$ we have

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon.$$

.

So, let us understand this definition. So, given any epsilon you have a delta. Now, uniform continuity will say this mod y minus x is less than delta then mod f y minus f x is less than epsilon. Now, we are going a little further, namely given any disjoint collection of intervals in a b such that the total length of all these intervals is less than delta then this corresponding variation is also less than epsilon.

So, it is obvious. So,

Remark f absolutely continuous implies f uniformly continuous because you can just take one interval instead of a collection which is less than delta and then you will have the definition of uniform continuity. So, example

Example: f is Lipschitz continuous. So, then you have

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq L \sum_{k=1}^{n} |y_k - x_k| < L\delta.$$

So, enough to choose, enough to take delta less than epsilon by L and then you have. So, every Lipchitz continuous function is absolutely continuous.

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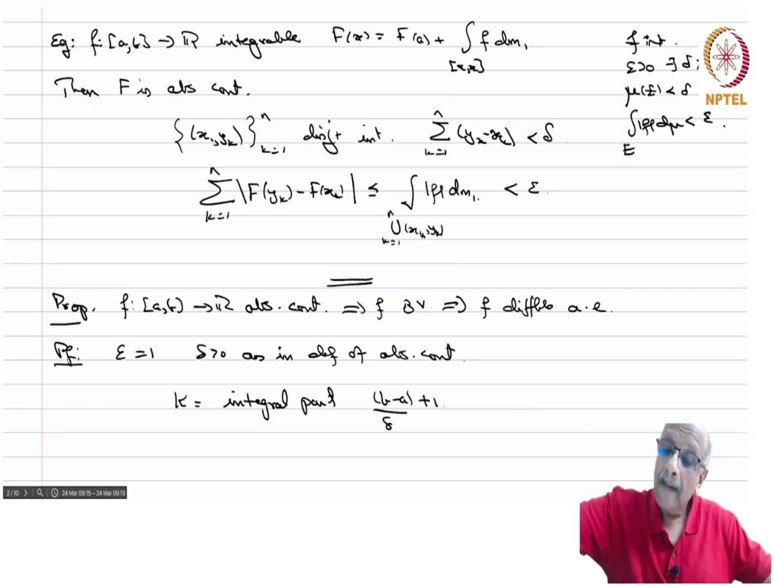

Example, $f: [a, b] \rightarrow \mathbb{R}$ and $F(x) = F(a) + \int_{[a,x]} f\, dm$

then F is absolutely continuous. So, you see it is much more than uniform quantity, we have proved that this is uniformly continuous and bounded variation. Now, we are saying it is absolutely convenient which is a step further than uniform continuity and therefore this is again a necessary condition for f to be written as an indefinite integral of an integral function and we will later see in this session that this is also a sufficient condition.

So, let us prove this. So, if you have $\{(x_k, y_k)\}_{k=1}^{n}$ n disjoint intervals and $\sum_{k=1}^{n} y_k - x_k| < \delta$ we have

$$\sum_{k=1}^{n} |F(y_k) - F(x_k)| \leq \int_{\bigcup_{k=1}^{n}(x_k, y_k)} |f| dm_1 < \varepsilon.$$

. So now, we know that the indefinite integral is absolutely convinced, we have made this statement before by what we mean given any epsilon there is a delta.

So, so given any epsilon. So, f integrable given in epsilon there exists a delta positive such that mu e less than delta implies integral mod f over e, d mu is less than epsilon. So, this was called the absolute continuity of the integral and this is exactly the reason for calling it that is precisely what we are seeing now. Namely the indefinite integral is absolutely continuous in the sense.

So, if this measure is less than delta this has to be less than epsilon and therefore we have the. So, this is the reason why we call it. So, our aim now is to show that this is in fact a sufficient condition. Now, first proposition is that

Proposition: $f: [a, b] \to \mathbb{R}$ **is** absolutely continuous implies $f: [a, b] \to \mathbb{R}$ is BV and of course that implies $f: [a, b] \to \mathbb{R}$ is differentiable almost everywhere.

Proof, You take $\epsilon = 1$ and δ as in the definition of absolute continuity.

So, corresponding to $\epsilon = 1$ there will be a δ which tells you something happens and that is precisely we are taking that delta. Now, you take capital K equals the integral part of $(b - a)/\delta + 1$.

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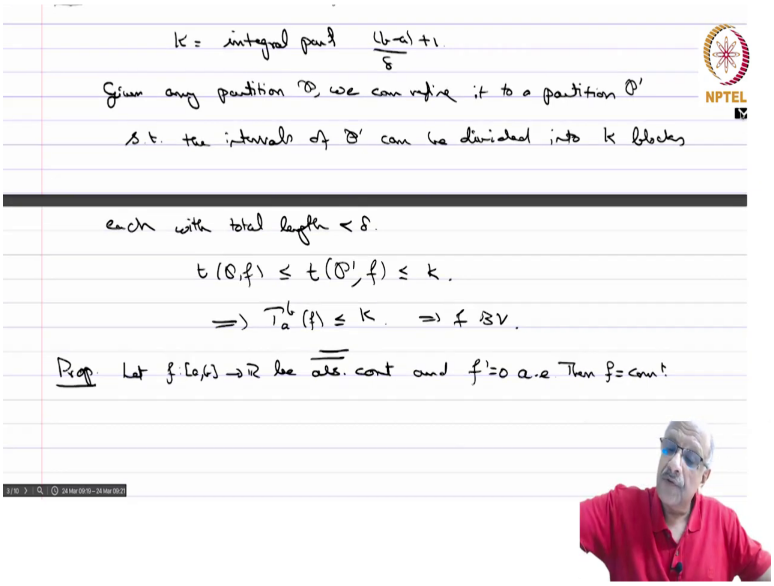

So, given any partition $\mathscr{P}$, we can refine it to a partition $\mathcal{P}'$. So, what do you mean by refine it? We are going to add some more extra points such that the intervals of $\mathcal{P}'$ can be divided into K blocks each with total length less than delta because K is b minus a by delta plus 1. So, if you have K blocks then each of them will have to be of length less than delta. So, you can add extra points as many as you wish and then you have.

So now, $t(\mathcal{P}, f)$ given any refinement you can, it is very easy, it is always less than $t(\mathcal{P}', f)$. Now, this you in each if you take the in each block the variation since the measure is total length is less than delta the variation will be less than 1 which is epsilon. So, there are K such blocks and therefore this is less than equal to K and this implies that t a b of f is less than equal to capital K implies f BV.

So, this is a very nice property. So, the next important property proposition. So, you have a function f which is differentiable everywhere and if $f' = 0$ and the set is connected then you know that f has to be equal to a constant. Now, what happens if its f is differentiable almost everywhere and the derivative is 0. So, that is the next proposition:

Proposition: $f: [a, b] \to \mathbb{R}$ **is** absolutely continuous and $f' = 0$ almost everywhere then $f = c$ constant.

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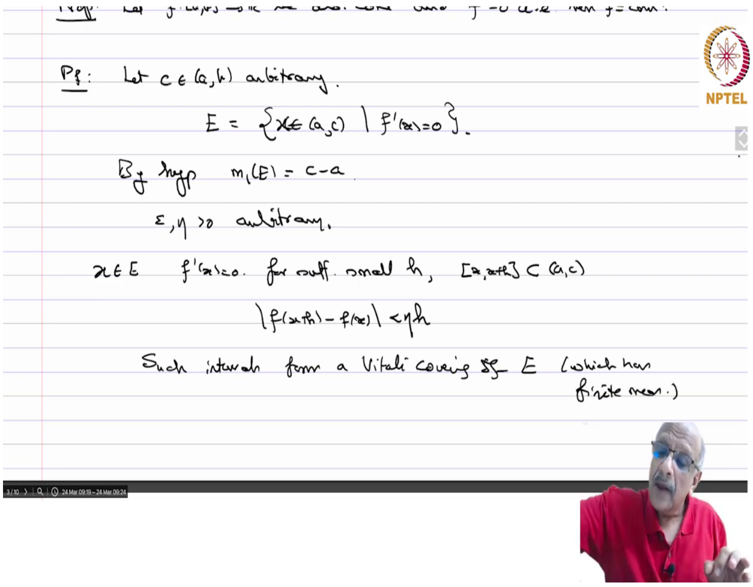


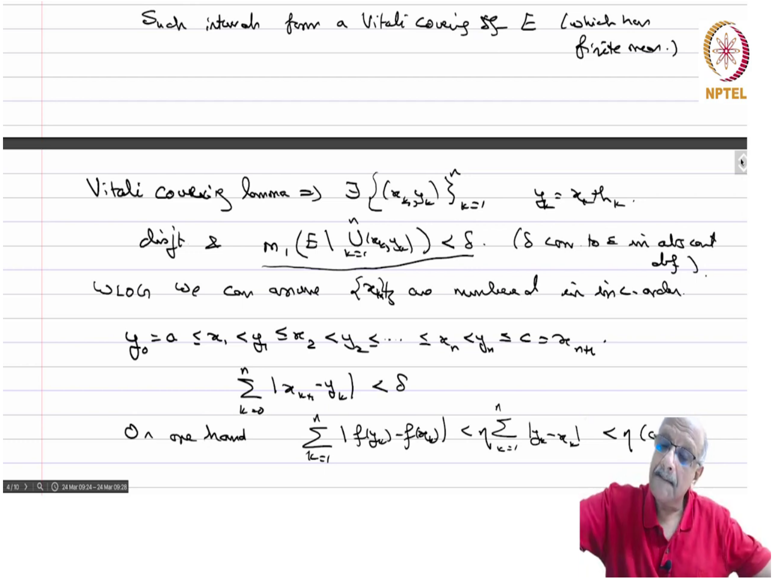
So, if you look at the cantor function cantor function is a function which is differentiable almost everywhere and the derivative is 0 because almost except for the cantor set the complement of the cantor set consists of flat parts of various intervals in each interval f is a constant, a different constant and $f^{'} = 0$. So, the counter function is certainly naught a constant but it is f dash is 0 almost everywhere and therefore cantor function is not absolutely continuous. So, that is this. So now,

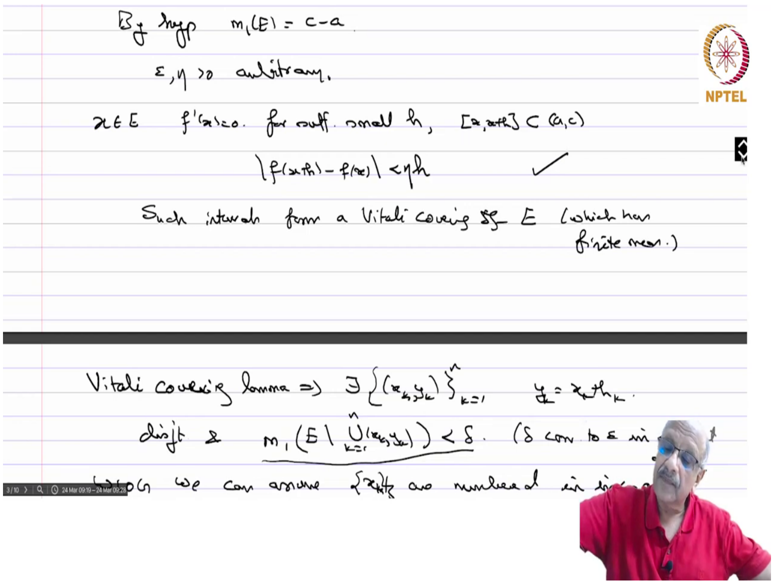
Proof: Let $c \in [a, b]$ arbitrary.

Now, you take $E = \{x \in (a, c) \mid f^{'}(x) = 0\}$. So, by hypothesis $m_1(E) = c - a$, because it is function $f^{'} = 0$ almost everywhere. So, this measure must be the same as the full measure. So, now $\varepsilon, \eta > 0$ is arbitrary. Suppose x belongs to E then what does it mean $f^{'}(x) = 0$ then for sufficiently small h we have x, x plus h will belong to a c and $f\ (x + h) - f(x) < \eta h$

because the derivative is x, f of x plus h minus $f(x)$ by h is the difference quotient which goes to 0 because $f^{'} = 0$.

So, it can be made arbitrarily small for h arbitrarily small. So, therefore this and this you have such intervals and h is arbitrarily small and this is true such intervals form a vitally covering and therefore which has finite measure.





So, therefore by the Vitali covering lemma implies that exists $\{(x_k, y_k)\}_{k=1}^{n}$, $y_k = x_k + h$ disjoint and such that the measure of e minus union (x_k, y_k); k equals 1 to n, this can be made less than delta. So, δ corresponds to ε in absolute continuity definition. So, that is the thing. Now, without loss of generality we can assume x_k, are numbered in increasing order then what do you have.

So, you have y naught equal to a. less than equal to x_1, strictly less than y_1, less than equal to x_2, strictly less than y_2 etcetera less than equal to x_n, strictly less than y_n, less than equal to c, equal to $x_n + 1$. So, we have numbered it this way and strictly less because they are all open intervals and then they have this joint and therefore x_2 will always be greater or equal to y_1.

So, this is what we then and sigma k equal to 0 to n of x_k plus 1 minus y_k will be less than δ that is coming from this condition here. So, these x_k plus 1 and y_k y_k to x_k plus 1 this is the portion which is not covered by those intervals and therefore you have the sum of all those is less than delta. So, on one hand we have sigma mod f of y_k minus f of x_k; k equals 1 to n is less than eta times sigma k equals 1 to n mod y_k minus x_k which is less than eta times c minus a. This is just where we have done the vitally covering from this condition here.

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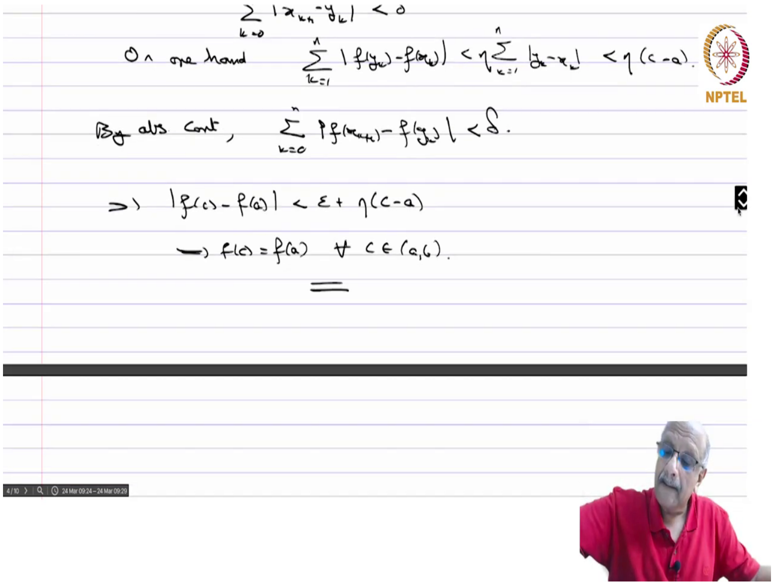

Now, by absolute continuity we have sigma k equals 0 to n mod $f(x_k)$ plus 1 minus $f(y_k)$ is less than delta. So now, if you add all these two together by the triangle inequality. So, this implies that mod fc minus fa is less than epsilon plus eta times c minus a and this implies that fc equal to fa and for all c in a b and by continuity also for b also and therefore we have that function is a constant.

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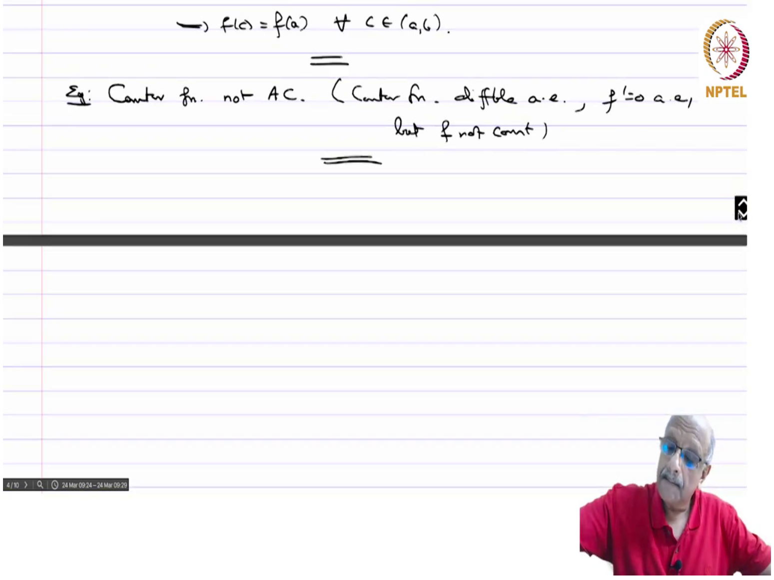

Example: Cantor function not absolutely continuous because cantor function differentiable almost everywhere $f^{'} = 0$ almost everywhere but f not constant.

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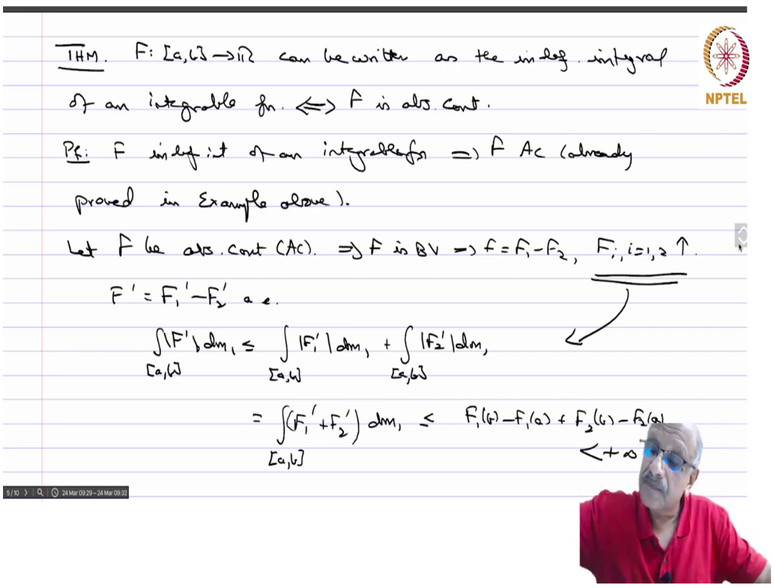

So now, we have the final theorem, which settles this question. So,

Theorem: $F: [a, b] \to \mathbb{R}$ can be written as the indefinite integral of an integrable function if and only if F is absolutely continuous.

Proof. So, F indefinite integral of an integrable function implies F absolutely continuous already seen proved in an example about how we already did this.

So now, we only have to show the sufficiency. So, let F be absolutely continuous. So, it is, so this implies F is BV implies $F = F_1 - F_2$, F_i, $i = 1, 2$ monotonic increasing then $F' = F_1' - F_2'$ almost everywhere and so,

$$\int_{[a,b]} |F'| dm_1 \leq \int_{[a,b]} |F_1'| dm_1 + \int_{[a,b]} |F_2'| dm_1 = \int_{[a,b]} (F_1' + F_2') dm_1$$

but F_1 and F_2 are monotonic increasing and therefore this implies the derivatives have to be non-negative.

$\leq F_1(b) - F_1(a) + F_2(b) - F_2(a)$ and that is of course finite.

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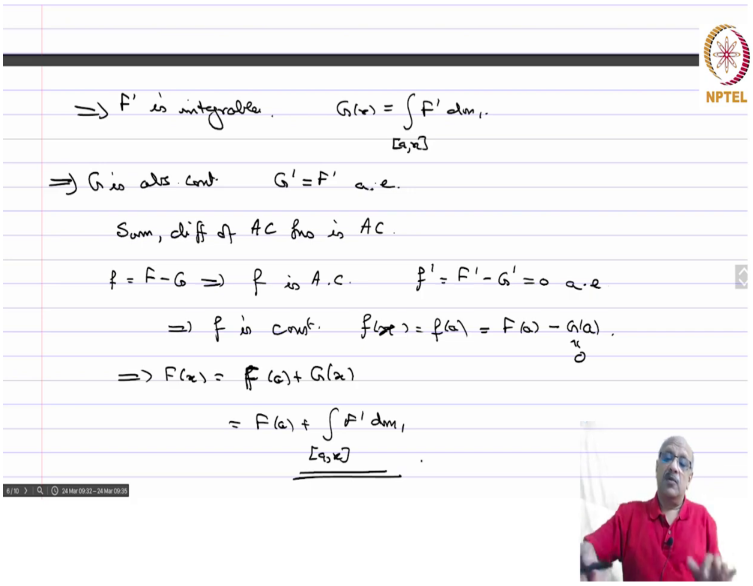

So, this implies that F' dash is integrable. Now, you said

$$G(x) = \int_{[a,x]} F' dm_1$$

Then G is absolutely continuous already seen in the example and also this first part of this theorem. So, G is absolutely conscious and $F' = G'$ almost everywhere we have seen last time. If you have an integrable function then the indefinite integral is differentiable and the derivative is equal to f' almost everywhere.

So now, if you have. So, some difference of AC functions is absolutely continuous. you can easily check all you have to take is the delta to be the smaller of the two deltas and that will work for you or you just use triangle inequality. So, if F is absolutely continuous. So, you write $f = F - G$ then capital F is absolutely continuous G is absolutely continuous, place F is absolutely continuous and you have $f' = F' - G' = 0$ which is equal to 0 almost everywhere.

So, this implies f is a constant. So, f equals let us say, so $f(x) = f(a) = F(a) - G(a)$, $G(a) = 0$ and therefore it is equal to capital F and therefore this implies that

$$F(x) = F(a) + G(x) = F(a) + \int_{[a,x]} F' \, dm_1$$

and that tells you that it is the indefinite integral of an integrable function and what is that integrable function it is equal to the derivative almost everywhere. So, a function can be written as an indefinite integral of an integral function if and only if. So, this is the final answer, it is absolutely continuous and the integrand in that case will be the derivative almost everywhere.