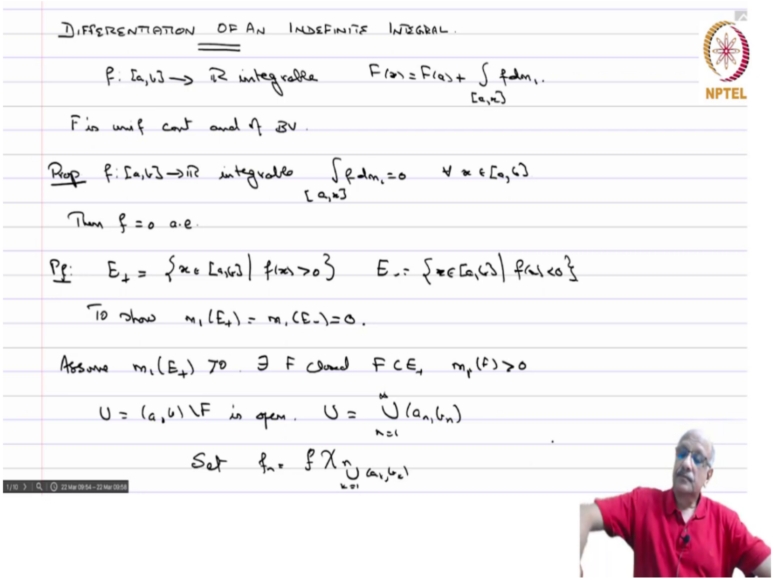
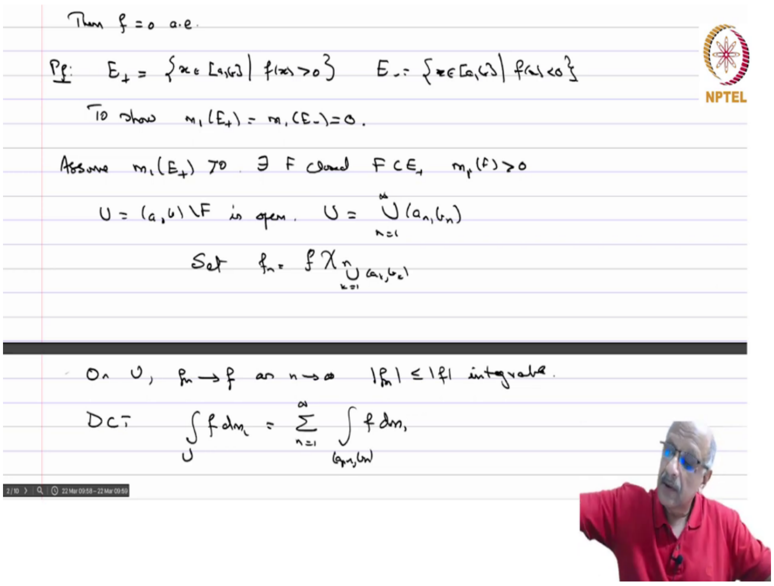
Measure and Integration
Professor. S. Kesavan
Department of Mathematics
Indian Institute of Technology Madras
Lecture -47
Differentiation of an indefinite integral

(Refer Slide Time: 00:16)





Differentiation of an Indefinite Integral:

So, we will now continue with the investigation of the fundamental theorem of calculus. So, we want to know the Differentiation of an Indefinite Integral so, what is an indefinite integral? It is so $f: [a, b] \to \mathbb{R}$ integrable when we define $F(x) = F(a) + \int_{[a,x]} f dm_1$. So, this is called the indefinite integral and we want to see if it is derivative is going to be equal to f almost everywhere at least like in the fundamental theorem of calculus.

So, we already know that F is uniformly continuous and of bounded variation. So, this much we know. So, now, let us before we proceed we need the following technical

Propositions. Let $f: [a, b] \to \mathbb{R}$ integrable and integral $\int_{[a,x]} f dm_1 = 0$, $\forall x \in [a, b]$. Then $f = 0$ $a.e.$

Proof: We define $E_+ = \{x \in [a, b] \mid f(x) > 0\}$, $E_- = \{x \in [a, b] \mid f(x) < 0\}$. So, to show $m_1(E_+) = m_1(E_-) = 0$, then we have because then $f = 0$ $a.e.$ So, assume so, we will do it for the m plus the proof for m minus is similar. So, assume $m_1(E_+) > 0$ all these sets are measurable, so, there is no problem.

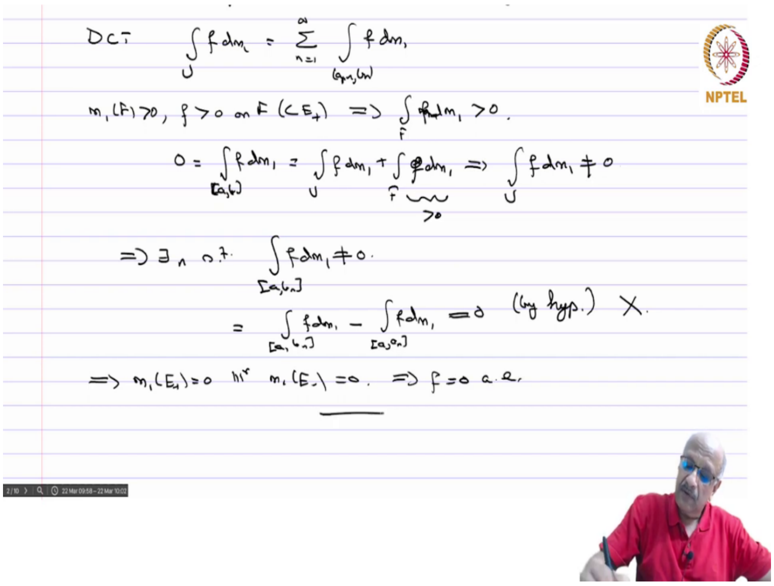
So, then $\exists F$ $closed$ $F \subset E_+$ $such\ that$ $m_1(F) > 0$ in fact you can show that it is measured as close to E pluses to measure the E plus as possible, because everything is finitely measured here. And therefore, in particular we are using a very weak condition here namely m1 of f is strictly positive. Now, $U = (a, b) \backslash F$ is open of course. So, then $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$.

So, that a set $f_n = f\chi_{\bigcup_{k=1}^{n} (a_k, b_k)}$. Then on U we have a $f_n \to f$ because as n tends to infinity and $|f_n| \leq |f|$ integrable. Therefore, by the Dominant convergence theorem integral on

$$\int_U f\, dm_1 = \sum_{n=1}^{\infty} \int_{(a_n, b_n)} f\, dm_1.$$

which is the integral of chi union n equals 1 to n an bn. So, that is nothing but integral an bn of f dm 1 and summation over n equals 1 to infinity.

(Refer Slide Time: 05:40)

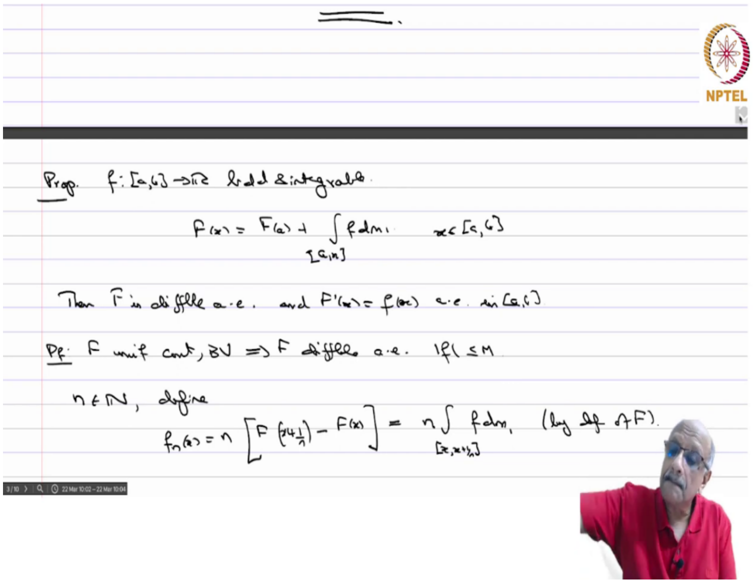


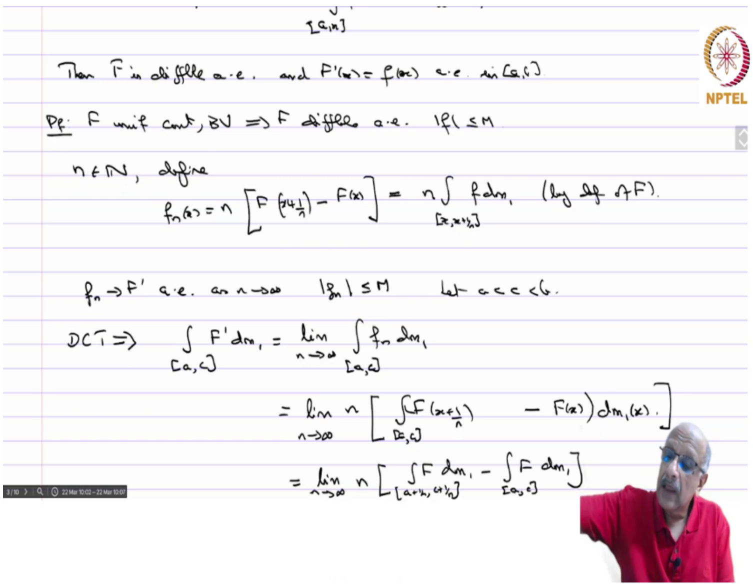
So, now $m_1(F) > 0,\ f > 0\ on\ F(\subset E_+) \Rightarrow \int_F f\, dm_1 > 0$ otherwise, because you know if the integral of a non-negative function is 0 then the function has to be 0 almost everywhere therefore, you have that 0 equals integral ab that is given.

Now, it does not matter if we put the endpoints as closed or open because those are of measure 0 and they do not matter. So, what is it really does not matter and that is equal to integral over the U f dm 1 let us say integral ab same as integral over open ab and then plus integral over F f dm 1 and this one is strictly positive and therefore, you have integral over U f dm 1 cannot be equal to 0.

But, what is integral over there for f dm 1 it is the sum of the various things therefore, that exists n such that integral over an bn f dm 1 is not equal to 0, but what is the c equal to the c equal to integral a to bn f dm 1 minus integral a to an f dm 1 and that is given to be 0 by hypothesis as I said we have that is it. and let us therefore a contradiction and consequently this implies that m1 of E plus equal to 0. Similarly, m1 of E minus is also equal to 0 implies f equal to 0 almost everywhere.

(Refer Slide Time: 08:24)





Proposition: $f: [a, b] \to \mathbb{R}$ bounded and integral

$$F(x) = F(a) + \int_{[a,x]} f \, dm_1, \quad x \in [a, b].$$

As I repeatedly say, it does not matter what they put open or close at either end in such intervals because the integral over points are all 0. Then F is differentiable almost everywhere that we do already know and $F'(x) = f(x)\ a.e.\ in\ [a, b]$.

Proof: F is uniformly continuous and bounded variation implies F differentiable almost everywhere we know this already. Now, let us assume $|f| \leq M$. So, if $n \in \mathbb{N}$ then define

$$f_n(x) = n[F(x + 1/n) - F(x)] = n \int_{[x,x+1/n]} f \, dm_1$$

So, by definition of F then $f_n \to F' \ a.e.$ as n tends to infinity that is by just definition and you also have mod $|f_n| \leq M$ because mod f is less than equal to M. So, if we integrate over this this will be less than equal to n times 1 by n into n and therefore, this is also the same.

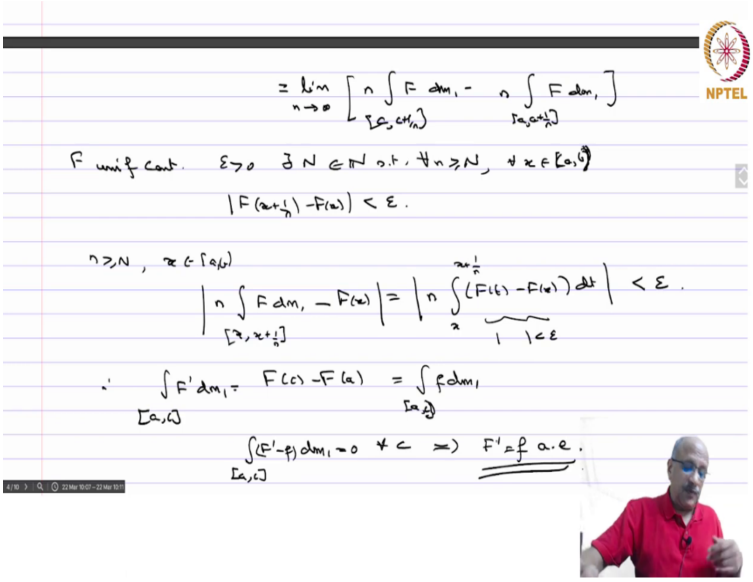
Hence again by the Dominated convergence theorem that integral a to let $a < c < b$. So,

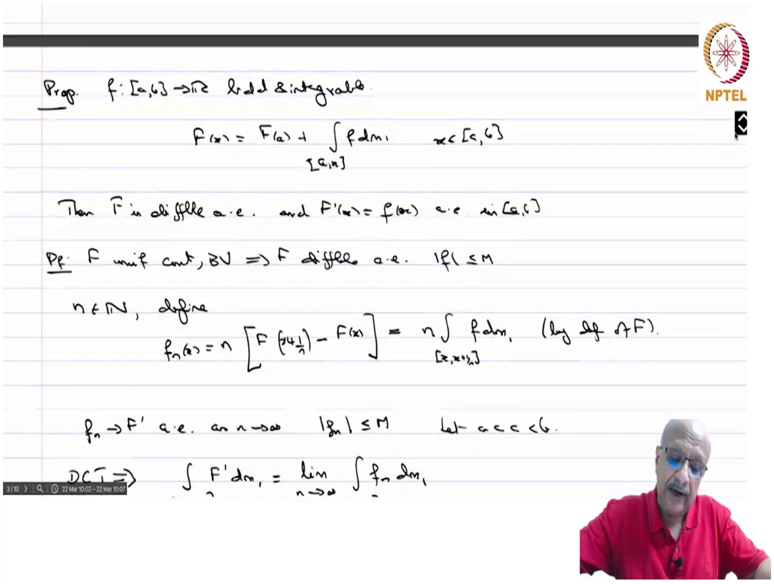
$$\int_{[a,c]} F' \, dm_1 = \lim_{n \to \infty} \int_{[a,c]} f_n dm_1 = \lim_{n \to \infty} n\left[\int_{[a,c]} (F(x + 1/n) - F(x)) \, dm_1\right]$$

$$= \lim_{n \to \infty} n\left[\int_{[a+1/n,c+1/n]} F(x)dm_1 - \int_{[a,c]} F(x)) \, dm_1\right]$$

$$= \lim_{n \to \infty} \left[n \int_{[c,\, c+1/n]} F(x)dm_1 - n \int_{[a,\, a+1/n]} F(x) \, dm_1\right]$$

(Refer Slide Time: 13:27)





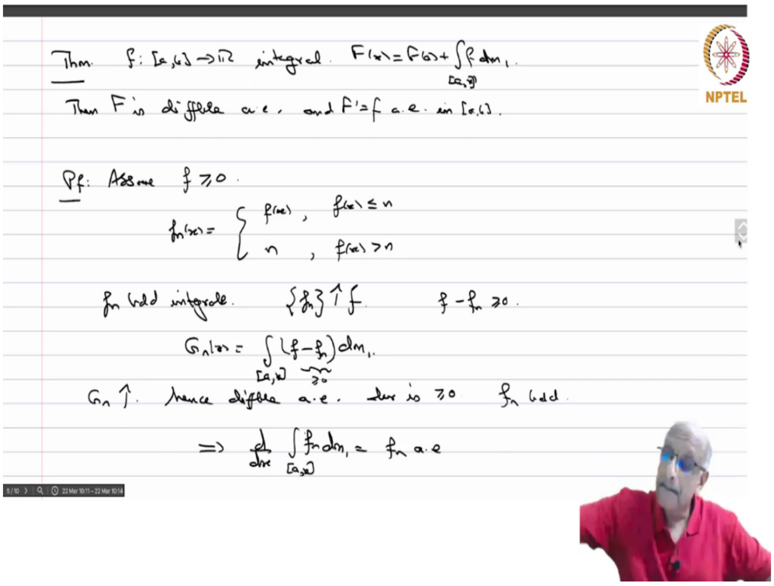
Now, F is uniformly continuous. So, given $\varepsilon > 0$ there exists a $n \in \mathbb{N}$ such that for all $\forall n \geq N \; and \; x \in [a,\, b]$ we have $|F(x + 1/n) - F(x)| < \varepsilon$.

So, if $\forall n \geq N \; and \; x \in [a,\, b]$ I am putting open b because I am doing x plus 1 by n end so, it has to be inside there.

You have that $|n|$ of integral x x plus 1 by n f dm 1 minus F of x this equal to mod n times integral x to x plus 1 by n and they can also they did this remain integral because it is continuous and so on so, x to x plus 1 by n of Ft minus F of x because Fx is a constant dt and this is less than epsilon and therefore, when you bring it out you get n gets canceled, so, this is less than epsilon for n began to n.

So, if you now take the limit here therefore, we have integral a to c F dash dm 1 is the limit so, we have to we are just taking the limit so, that is equal to F of c minus F of a and that is equal to integral f dm 1 in a to c again by definition. So, you have integral a to c F dash minus f dm 1 equal to 0 for all c and this implies that F dash equal to f almost everywhere by the previous proposition. So, that proves that theorem. So, now, we want to get rid of the fact we said if it is a bounded and integrable function. So, we know we want to say the same thing without the boundedness hypothesis.

(Refer Slide Time: 17:47)

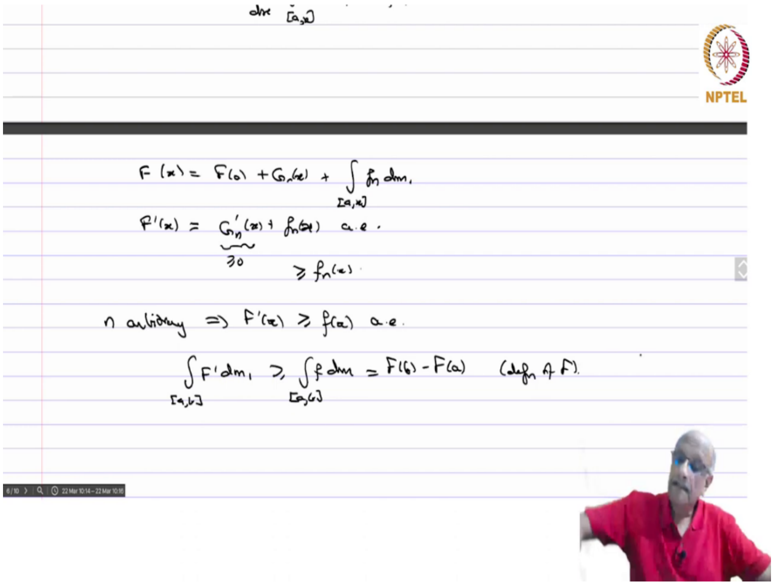


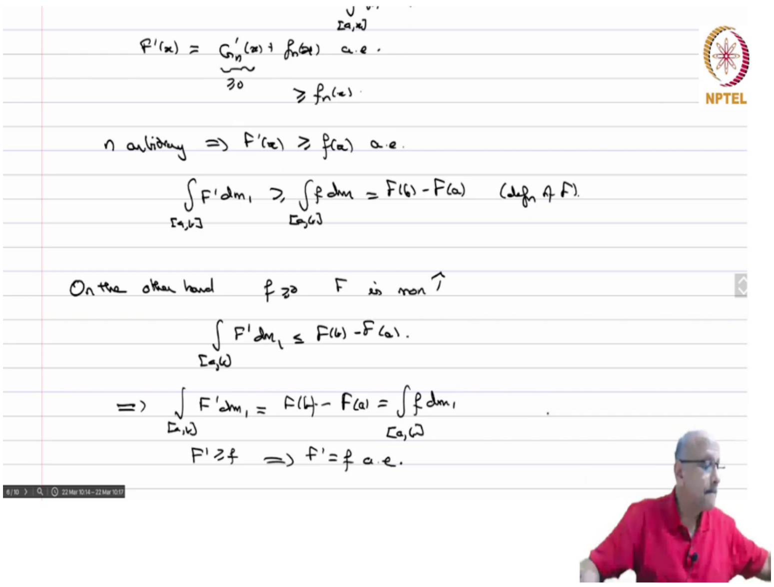
So, theorem f from ab to R integrable so, this is no boundedness assumption anymore capital F of x equals capital F of a plus integral a to x f dm 1 indefinite integral then F is differentiable almost everywhere and F dash equal to f almost. So, this is like the fundamental theorem of calculus. So, we have this in ab.

Proof. So, assume f is non-negative and define fn of x equals f of x if f of x is less than equal to n this trick we have done before n equal to n if f of x is bigger than n, So, we cut it off at the value n so, then fn is bounded integrable and fn increases to f. So, f minus fn is non-negative. Now, you define Gn of x equals integral a to x F minus fn dm 1. So, this is a non-negative function.

So, Gn is monotonic increasing and hence differentiable almost everywhere because it as a monotonic function we are we do not worry about what about this. This is if this was bounded then we know by the previous proposition is differentiable, but because it is non negative, we already know this is a differentiable because it is now a monotonic function and derivative is non-negative and fn is bounded this implies d by dx of integral a to x fn dm 1 equals fn almost everywhere.

(Refer Slide Time: 20:42)





Now I am going to write Fx equals a Fa plus Gn of x, Gn of x is F minus fn plus integral a to x fn dm 1. Then $F^{'}(x) = G_{n}^{'}(x) + f_{n}(x)$ which exists almost everywhere because absolutely continuous for I mean a uniformly continuous function is $G_{n}^{'}(x) + f_{n}(x)$ almost everywhere and $G_{n}^{'}(x) \geq 0$. So, this is greater or equal to 0 fn x.

Now n is arbitrary so, this implies that $F^{'}(x) \geq f(x)\, a.e.$ so,

$$\int_{[a,b]} F^{'} dm_{1} \geq \int_{[a,b]} f dm_{1} = F(b) - F(a).$$

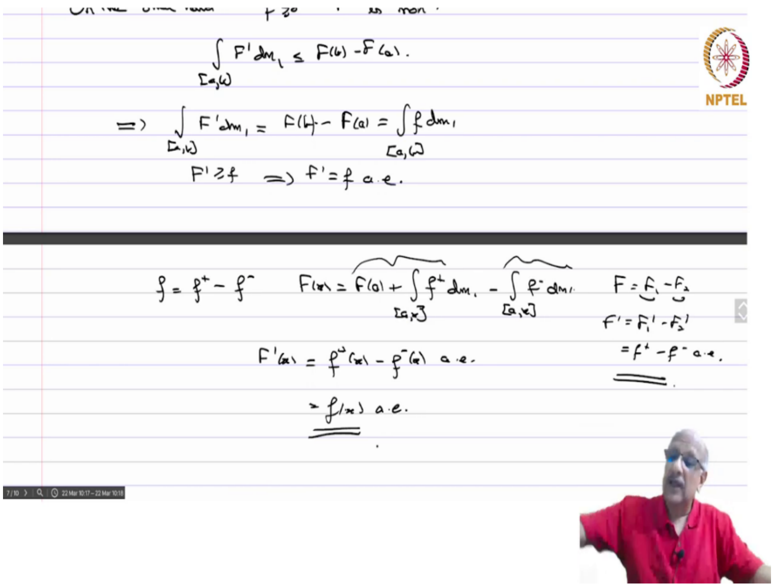
definition of f.

On the other hand, f is non-negative, so, F is monotonically increasing and therefore,

$$\int_{[a,b]} F^{'} dm_1 \leq F(b) - F(a).$$

implies $\int_{[a,b]} F^{'} dm_1 = F(b) - F(a) = \int_{[a,b]} f dm_1$. because for the monotonic functions we know that this is true. and if $F^{'}$ is greater than equal to f and this implies that $F^{'} = f\ a.e.$ almost everywhere.

(Refer Slide Time: 23:29)



So, this proves it for non-negative functions. So, in general you take $f = f^{+} - f^{-}$ and then

$$F(x) = F(0) + \int_{[a,x]} f^{+} dm_1 - \int_{[a,x]} f^{-} dm_1$$

and then you get $F^{'}(x) = f^{+}(x) - f^{-}(x)\ a.e.$

$$= f\ a.e.$$

You can write how do you do that you can put this as the F_1 and this function is F_2. So, $F = F_1 - F_2$ and then each one involves a non-negative function. So, $F' = F_1' - F_2'$ and $F_1 = F_1^+ - F_1^-$ almost everywhere by the first step. So, this proves that.

So, we have concluded that if we have an integrable function and then you define that indefinite integral then it is differentiable almost everywhere and the derivative is equal to F dash. So, the next time we will see when function which has to be definitely bounded variation and uniformly continuous. So we want to know if, when such a function is needed, what is the necessary and sufficient condition for it to be written as the indefinite integral. And obviously, it is almost an indefinite integral of its own derivative almost. So with that we will stop.