Measure and Integration
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Lecture-46
7.5 – Functions of bounded variation

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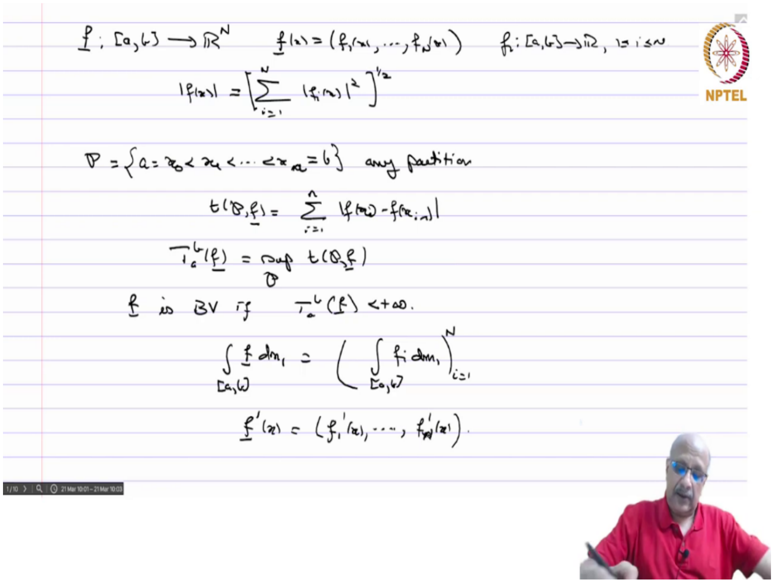

Functions of bounded variation:

We now continue with **Functions of bounded variation**. So, now we were looking at real valued functions. So, now consider f a vector valued map from $f: [a, b] \rightarrow \mathbb{R}^N$. So,

$$f(x) = \left(f_1(x), f_2(x), \ldots, f_N(x)\right).$$

So, each one $f_i: [a, b] \rightarrow \mathbb{R}$, $i \leq i \leq N$ and then $|f(x)| = \left[\sum_{i=1}^{N} |f_i(x)|^2\right]^{1/2}$ this vector so, this is the usual Euclidean norm.

and we say that if you have P any partition we say $P = \left\{a = x_0 < x_1 < \ldots < x_n = b\right\}$

$$t(P, f) = \sum_{i=1}^{N} |f_i(x) - f\ (x_{i-1})|$$

where mod of the vector is given by the Euclidean norm in this way and therefore, now you can say the $T^{b}_{a}(f) = sup_{P}\, t(P, f)$ f is now a vector.

So, this is how an f is BV bounded variation if $T^{b}_{a}(f) <+ \infty$. Now, given f you can define

$$\int_{[a,b]} f\, dx_1 = \left(\int_{[a,b]} f_i\, dx_1 \right)_{i=1}^{N}$$

so, this is n tupple. Similarly, $f'(x) = \left(f'_1(x), f_2'(x), ..., f_N'(x)\right)$. f dashed at any x is the gradient so, you have $f'_1(x)$ it is not the gradient this is $f'_N(x)$ this is derivative component choice. So, this is the derivative of the $f_N(x)$.

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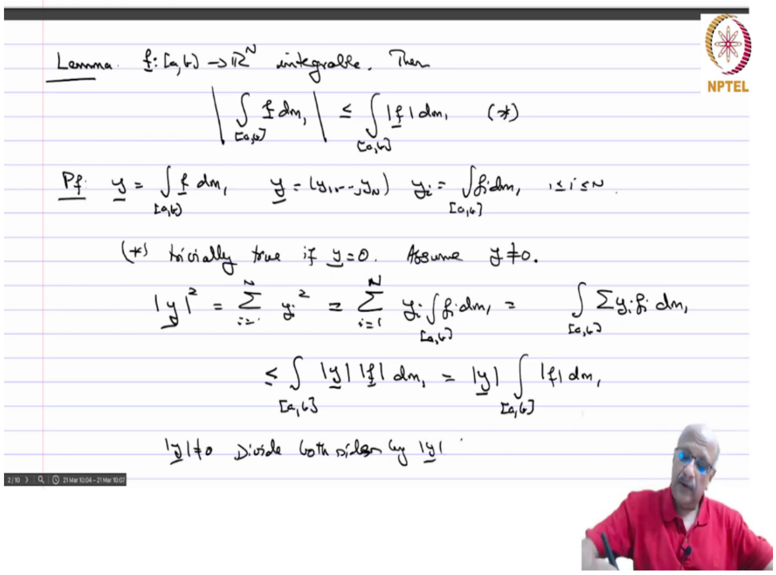

So, now, we have the usual estimation lemma which is all important so,

Lemma: Let $f: [a, b] \to \mathbb{R}^N$ integrable. Then integral

$$\left| \int_{[a,b]} f \ dm_1 \right| \leq \int_{[a,b]} |f| \ dm_1$$

remember this vector and its modulus is the usual Euclidean norm is less than or equal to mod f again Euclidean norm dm_1 over [a, b]. So, this is the usual theorem which you expect to have. So, let us see how we prove this.

Proof: So, let us take $y = \int_{[a,b]} f \, dm_1$, $y = \left(y_1, y_2, \ldots, y_N\right)$ and $y_i = \int_{[a,b]} f_i \, dm_1$, $1 \leq i \leq N$. So, now star trivially true if $y = 0$ so, assume $y \neq 0$, what is

$$|y|^2 = \sum_{i=1}^{N} y_i^2 = \sum_{i=1}^{N} y_i \int_{[a,b]} f_i \, dm_1 = \int_{[a,b]} \sum y_i f_i \, dm_1$$

and then you do not need to put the modulus because it is a square of a real number, and that is equal to sigma i equals 1 to n. I am going to now write it as yi integral fi dm 1.

But that is equal to since yi is now a number i equals 1 to n integral of ab sorry sigma, yi fi dm 1 and now, you apply the Cauchy Schwarz inequality

$$\leq \int_{[a,b]} |y||f|\, dm_1 = |y| \int_{[a,b]} |f|\, dm_1.$$

Then that is equal to mod y is again a number it comes out of the integral ab mod f dm 1. So, mod y is not 0. So, divide both sides by mod y to get star.

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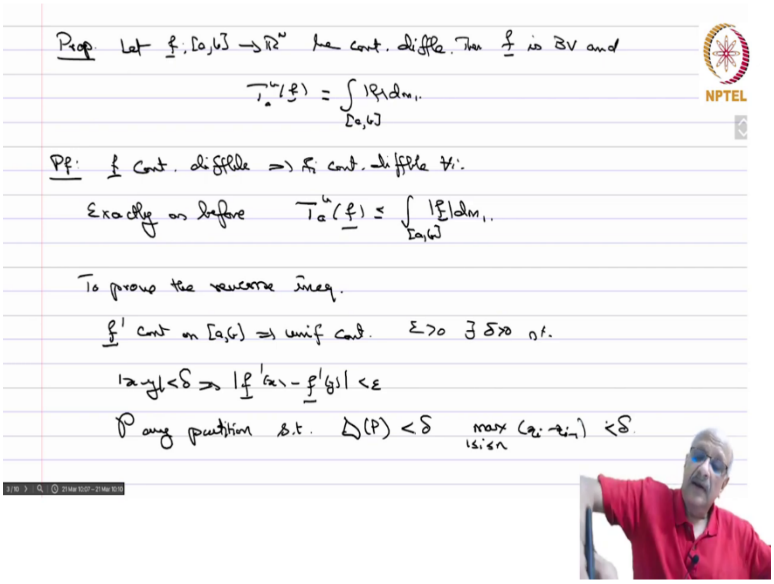
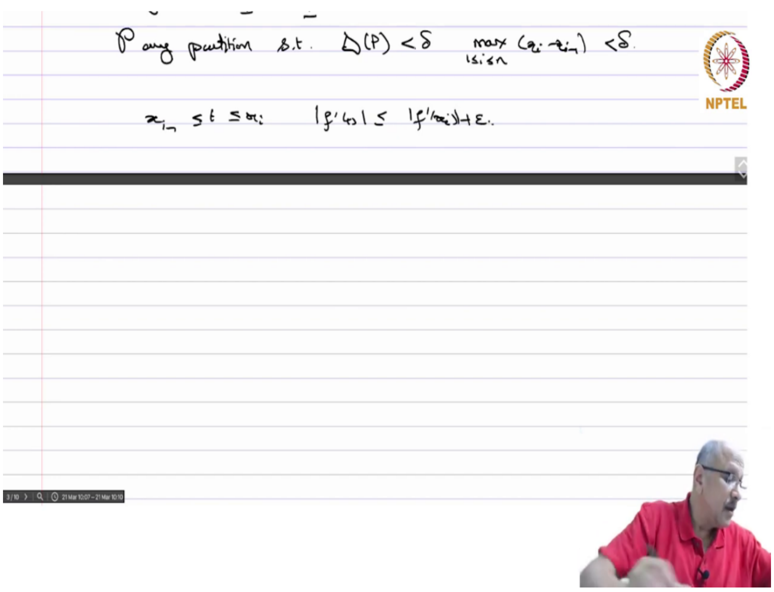




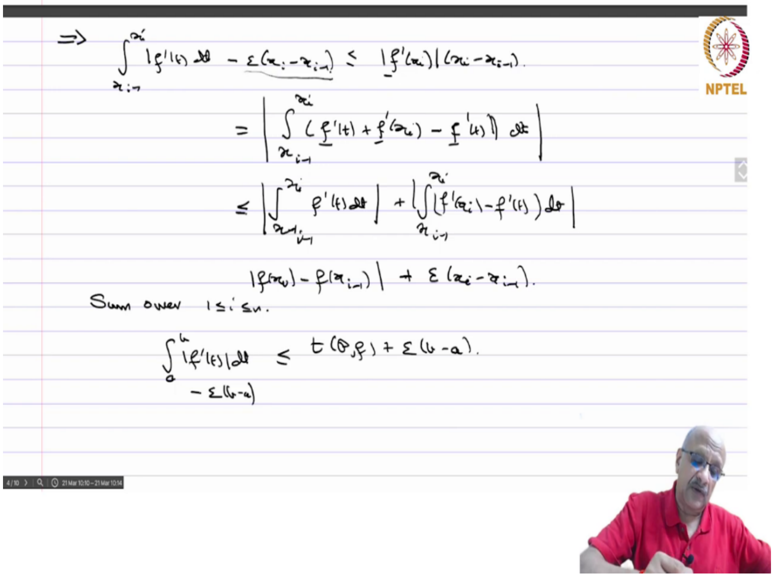
So, now we are going to imitate propositions which we did earlier. So,

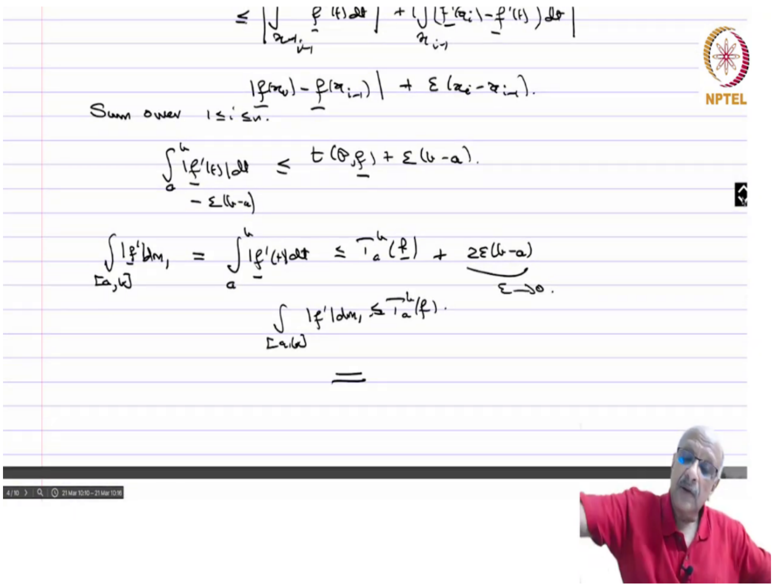
Proposition: Let $f: [a, b] \rightarrow \mathbb{R}^N$ be continuously differentiable that means, its derivatives for each i f_i is continuously differentiable. Then f is BV and $T_a^{\ b}(f) = \int_{[a,b]} |f|\, dm_1$.

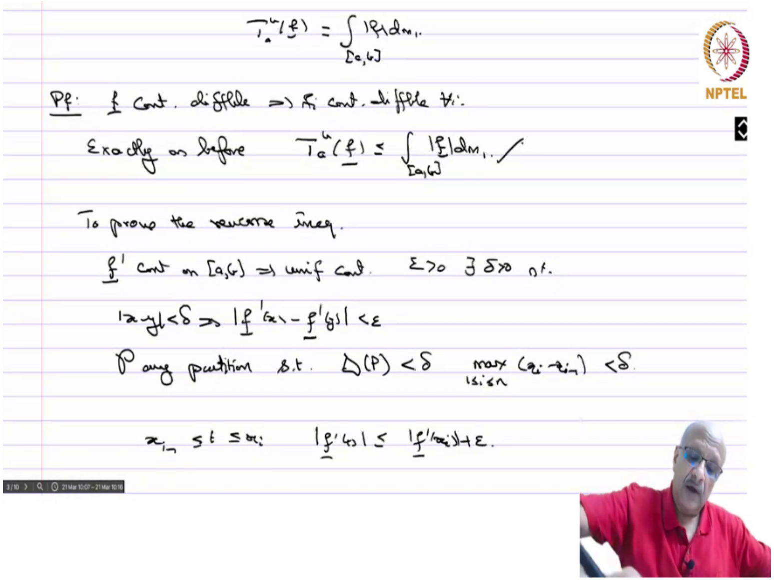
Proof: $f: [a, b] \rightarrow \mathbb{R}^N$ continuously differentiable that means, f_i continuously differentiable for each i, so, exactly as before so, you look up the previous proof, you get the $T_a^{\ b}(f) = \int_{[a,b]} |f|\, dm_1$.

So, this is already done for each you have to do it component wise and then it is exactly like before. Now, to prove the reverse inequality. So, now, what do you do you take so, f' continuous on ab implies uniformly continuous therefore, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f'(x) - f'(y)| < \varepsilon$. So, P any partition such that $\Delta(P) < \delta, \Rightarrow max_{1 \le i \le n}(x_i - x_j) < \delta$.

So, if $x_{i-1} \le t \le x_i$, $then\, |f_i(t)| \le |f'(x_i)| + \varepsilon$. (Refer Slide Time: 10:00)







So, that implies the $\int_{x_{i-1}}^{x_i} |f'(t)|\, dt - \varepsilon(x_i - x_{i-1}) \leq |f'(x_i)|(x_i - x_{i-1})$.

So I am just integrated I brought the epsilon to the side and integrated it and therefore, they $|f'(x_i)|(x_i - x_{i-1})$ and that is

$$= \left| \int_{x_{i-1}}^{x_i} \left(f'(t) + f'(x_i) - f'(t) \right) dt \right| \leq \left| \int_{x_{i-1}}^{x_i} f'(t) dt \right| + \left| \int_{x_{i-1}}^{x_i} \left(f'(x_i) - f'(t) \right) dt \right|.$$

Now, we can break this this is $\leq |f(x_i) - f(x_{i-1})| + \varepsilon(x_i - x_{i-1})$

But F is continuously differentiable and this integral here xi minus one to x f dash t dt is nothing but mod f of xi minus f of xi minus 1 plus here the if you take the modulus inside in the second integral that is less than epsilon in this interval we know and therefore, that is less than $\varepsilon(x_i - x_{i-1})$.

So, now you sum over all $1 \leq i \leq n$ so, you get integral

$$\int_{[a,b]} |f(t)^{'}| dt - \varepsilon(a - b) \leq t(P, f) + \varepsilon(b - a).$$

you will just get epsilon times b minus a and here also your minus epsilon times b minus a in the first because of the summation of the step.

In other words,

$$\int_{[a,b]} |f(t)'| dm_1 = \int_{[a,b]} |f(t)'| dt \leq T^b_a(f) + 2\varepsilon(b - a),$$

because it is continuously differentiable continuous map and therefore this riemann integrable etcetera etcetera etcetera and therefore, this is now you can let $\varepsilon \to 0$ because epsilon is arbitrarily chosen and so, you get the

$$\int_{[a,b]} |f(t)'| dm_1 \leq T^b_a(f)$$

and you have the other reverse inequality also and therefore, you have equality. So, this proves that theorem.

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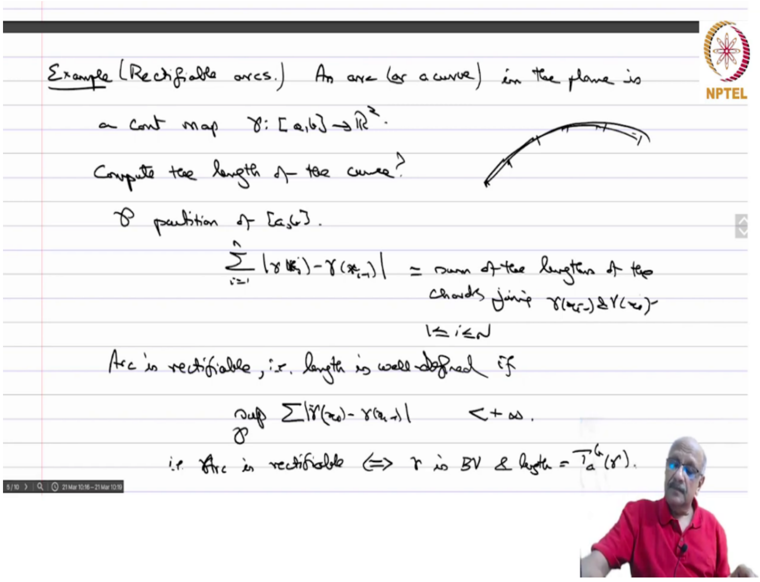

So, now let us consider so,

Example (Rectifiable arcs): An blocks so, an arc on a curve in the plane is a continuous map $\gamma: [a, b] \to \mathbb{R}$. So, how do you compute the length of what do we mean by the length of the curve. So, what we do is take P partition of [a,b] and then you take

$$\sum_{i=1}^{n} \left|\gamma(x_i) - \gamma(x_{i-1})\right| =$$

sum of the lengths of the chords called joining $\gamma(x_i)$ and $\gamma(x_{i-1})$, $1 \leq i \leq n.$

So, you have a curve like this for each so, you have $\gamma(x_1)$, $\gamma(x_2)$, $\gamma(x_3)$ you mark off the points and then you join these by straight lines and then some of the length and you say arc is rectifiable that is length is well defined if

$$sup_P \sum |\gamma(x_i) - \gamma(x_{i-1})| <+ \infty$$

in other words that is arc is rectifiable if and only if γ is BV and length is precisely $T^b_a(\gamma)$.

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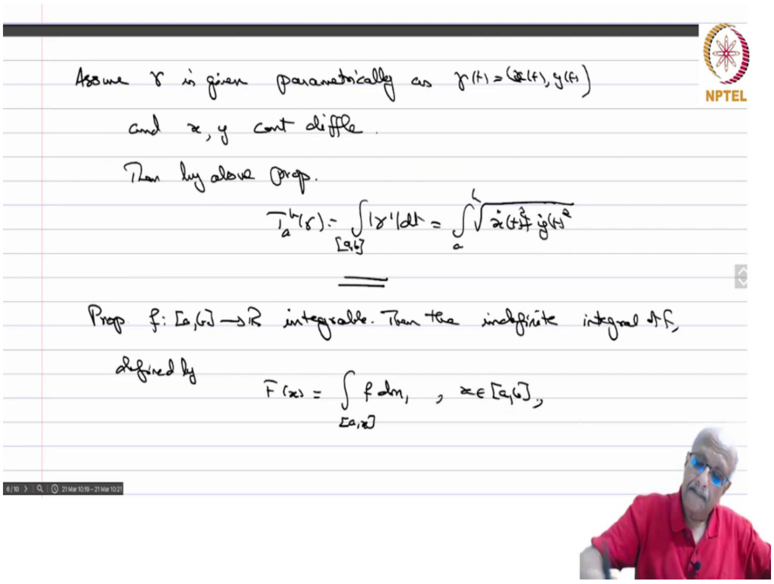

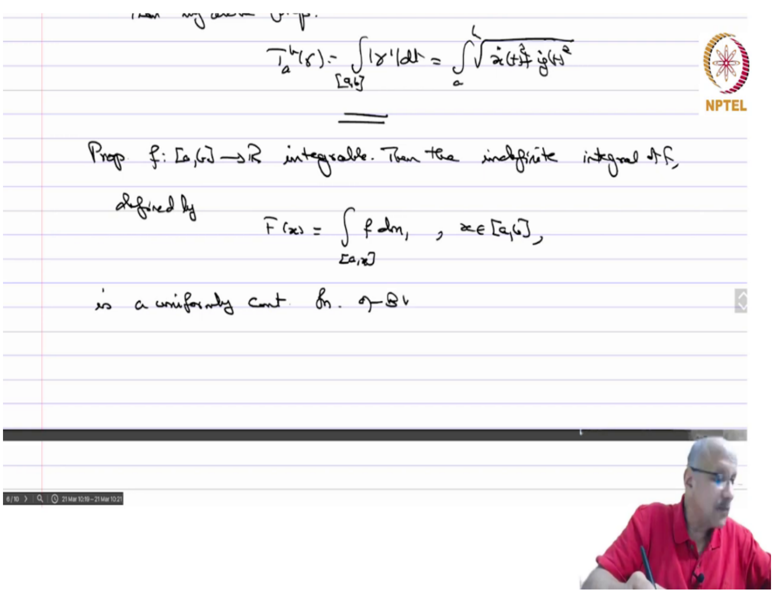

So, the length of a curve is nothing but the total variation of the curve and the function has to be a bounded variation. So, now, let us take assume gamma is given parametrically as gamma t equals xt y fi and xy continuously differentiable then by above proposition

$$T^{b}{}_{b}(\gamma) = \int_{[a,b]} |\gamma^{'}|\,dt = \int_{a}^{b} \sqrt{x^{\cdot}(t)^{2} + y^{\cdot}(t)^{2}}\,dt$$

and this is exactly the formula which we have studied in the original calculus courses for the length of a curve which is differentiable.

So, this is the cover that formula in this case we have through something more general. Now, we come back to the l valued functions. So,

Proposition: Let $f: [a, b] \to \mathbb{R}$ integrable then the indefinite integral of f defined by capital

$$F(x) = \int_{[a,x]} f \, dm_1, \quad x \in [a, b],$$

is a uniformly continuous function of bounded variation.

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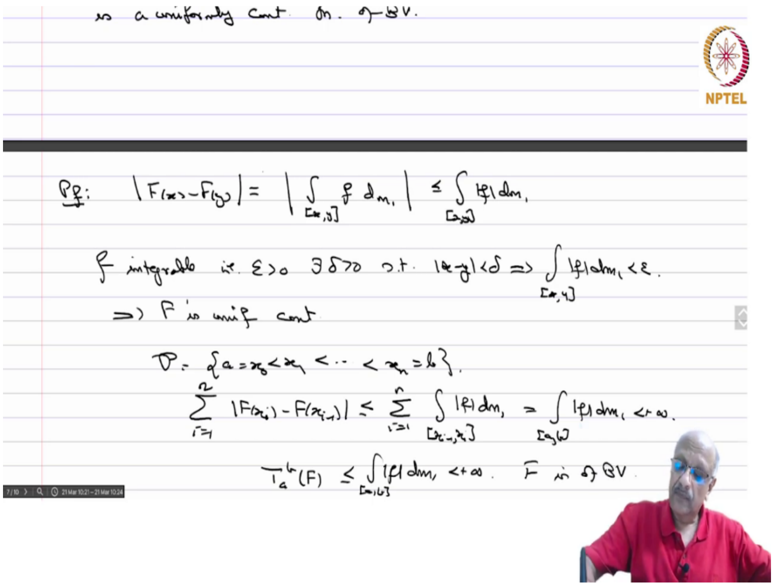

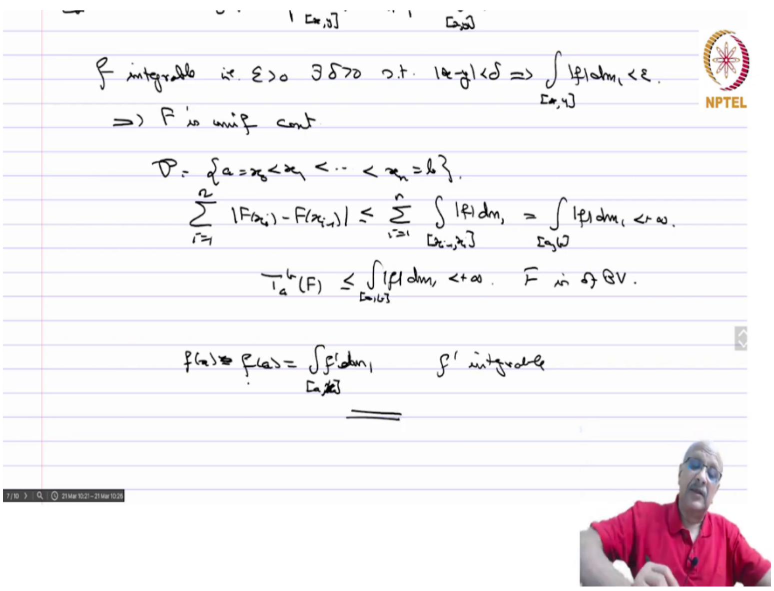

Proof, $F(x) - F(y)| = \left| \int_{[x, y]} f \, dm_1 \right| \leq \int_{[x,y]} |f| \, dm_1$. Now f is integrable that is given $\varepsilon > 0$, $\exists \delta > 0$ $s.t.$ $|x - y| < \delta \Rightarrow \int_{[x,y]} |f| \, dm_1 < \varepsilon$. So, this is the what we call the absolute continuity of the indefinite integral we have proved this already for the integrable functions and therefore, we have so, this implies that capital F is uniformly continuous.

Now, let $P = \left\{a = x_0 < x_1 < \ . \ . \ . \ < x_n = b\right\}$, partition of $[a, b]$ then you have

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \sum_{i=1}^{n} \int_{[x_i, x_{i-1}]} |f| \, dm_1 = \int_{[a, b]} |f| \, dm_1 <+ \infty.$$

$$T_a^{\ b}(F) \leq \int_{[a, b]} |f| \, dm_1 <+ \infty$$

So, F is of bounded variation.

So, if you want for instance we were asking the question when can you write a function

$$f(x) - f(a) = \int_{[a, x]} f' \, dm_1.$$

When f is differentiable and we said for instance since the cantor function you cannot do it and the reason is because if you have an indefinite integral which a to X this indefinite integral even if f' is integrable then the original function is uniformly continuous and of bounded variation.

So, that already places a set of restrictions on functions for which the theorem can be true. Therefore, if you have functions is differentiable almost everywhere then if you want if you have a hope for the fundamental theorem of calculus to be true, it has to be uniformly continuous and a bounded variation in the next section a little later we will give you a complete set of necessary and sufficient conditions which will tell you when you can actually do this even this is not enough but this at least necessary condition. So, we will continue to study the indefinite integral in the next session.