Measure and Integration
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Lecture No-45
Functions of bounded variation

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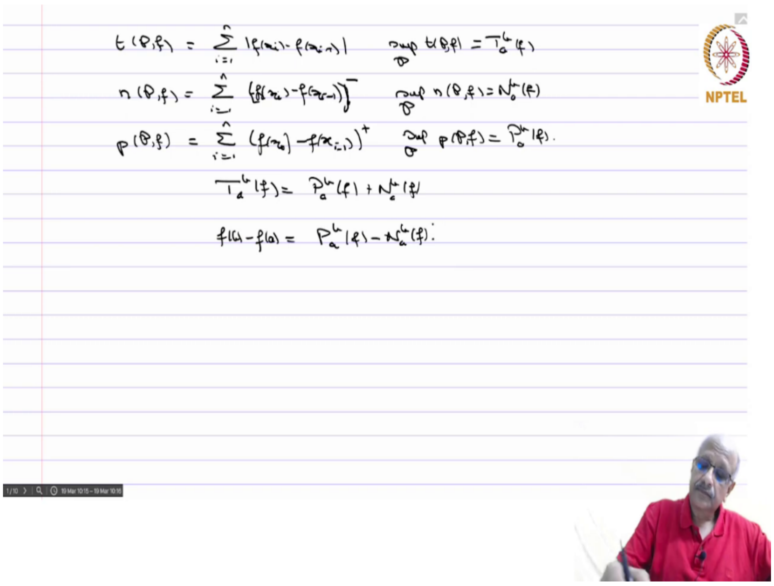

So, we were looking at Functions of Bounded Variations. So, we had

$$t(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \;\; ; \;\; \sup_P t(P,f) = T^b_a(f).$$

$$n(P,f) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^- \;\; ; \;\; \sup_P n(P,f) = N^b_a(f)$$

$$p(P,f) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ \;\; ; \;\; \sup_P p(P,f) = P^b_a(f)$$

$$T^b_a(f) = P^b_a(f) + N^b_a(f),\; f(b) - f(a) = P^b_a(f) - N^b_a(f).$$

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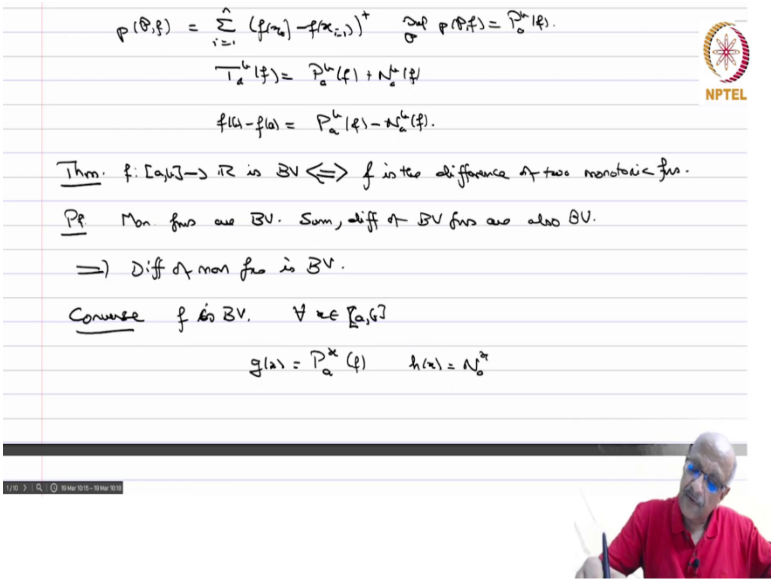

So, now, this will help us to characterize functions of bounded variation. So, we have the following:

Theorem: $f: [a, b] \to \mathbb{R}$ is BV if and only if f is the difference of two monotonic functions. So, this is another easy and nice characterization, no supremum etcetera if f is can be written as a difference of 2 monotonic functions then it is a bounded variation and vice versa if f is of BV then it can be written in that fashion.

Now, monotonic functions are BV some differences of BV functions are also BV. So, the difference of monotonic functions is of bounded variation. So, now, we have to prove the converse. So, let us take f is BV now, for every x in ab, you define

$$g(x) = P^{x}_{a}(f),\ h(x) = N^{x}_{a}(f).$$

g of x equals P ax of f and h of x equals n ax of f then we are taking after all we are what are you doing when you take a partition from a to x and then you have another point y which is beyond that.

Now, if you want to take the P ax and N ax, you have to take the partitions here plus some more partitions here and then take the sum and then the supremum. So, consequently, given any partition if you of ay or so it is obvious from this that g and h are monotonically increasing.

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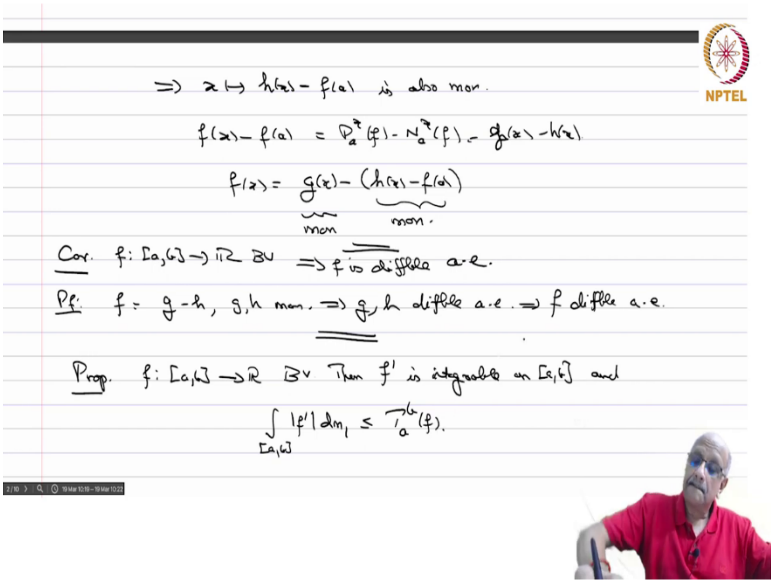

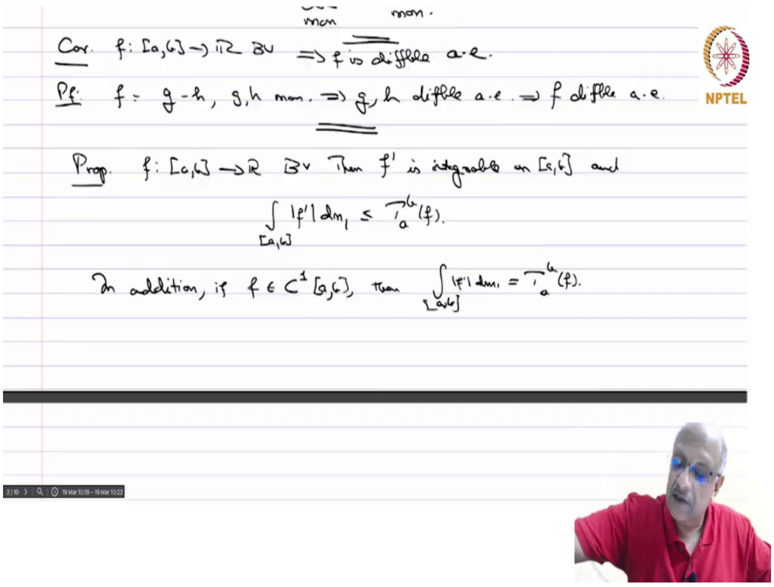

So, in particular $x \to h(x) - f(a)$ is also monotonic. Now, we saw that

$$f(x) - f(a) = P^{x}_{a}(f) - N^{x}_{a}(f) = g(x) - h(x).$$

Therefore, $f(x) = g(x) - h(x) + f(a)$ and this is monotonic and this is also monotonic therefore, every function of bounded variation can be written as the difference of 2 monotonic functions.

Corollary: $f: [a, b] \to \mathbb{R}$ is BV implies f is differentiable almost everywhere.

proof: f=g-h, g, h monotonic this implies g that g and h are differentiable almost everywhere and this implies that f is differentiable almost everywhere so, this is very important.

So, bounded variation functions are another class of functions which are differentiable almost everywhere but they are really coming from the class of monotonic functions because every function of bounded variation is necessarily the difference of 2 monotonic functions.

Proposition: $f: [a, b] \to \mathbb{R}$ is BV. Then f' is integrable on $[a, b]$ and

$$\int_{[a,b]} |f'| dm_1 \leq T^b_a(f).$$

In addition, if $f \in C^1[a, b]$, then $\int_{[a,b]} |f'| dm_1 \leq T^b_a(f)$.

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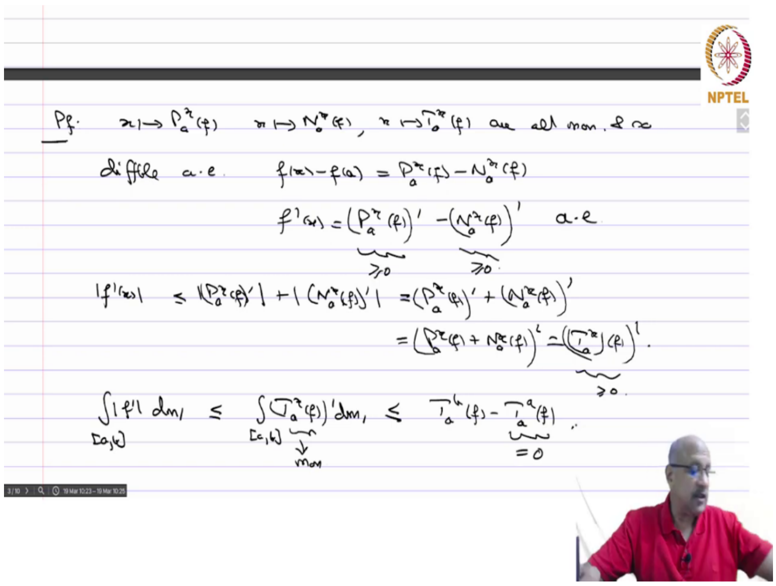

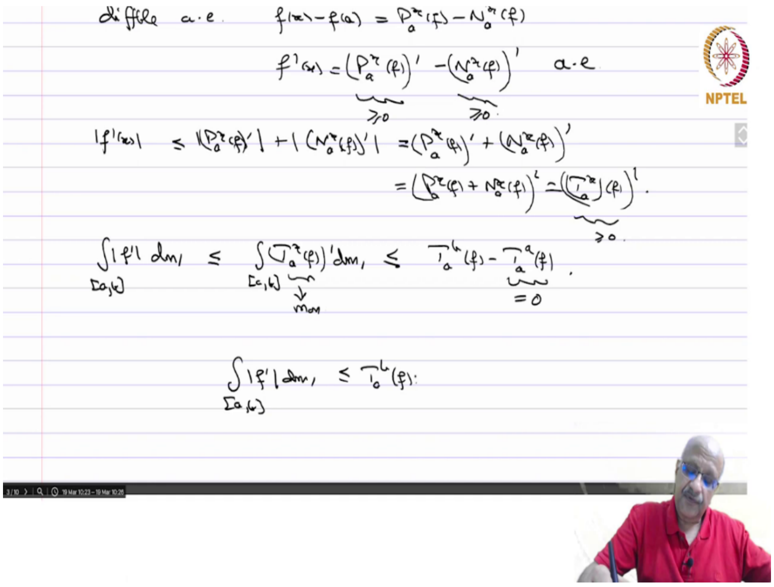

proof: so, $x \to P^x_a(f)$, $x \to P^x_a(f)$, $x \to T^x_a(f)$ are all monotonic as we saw and so, differentiable almost everywhere and you also have $f(x) - f(a) = P^x_a(f) - N^x_a(f)$

and therefore, $f'(x) = (P^x_a(f))' - (N^x_a(f))'$ almost everywhere and these are all non-negative because they are the derivatives of monotonically increasing functions.

Therefore,

$$|f'(x)| \le |(P^x_a(f))'| + |(N^x_a(f))'| = (P^x_a(f))' + (N^x_a(f))' = (P^x_a(f) + N^x_a(f))'$$

$$= (T^x_a(f))'$$

Therefore, $\int_{[a,b]} |f'| dm_1 \le \int_{[a,b]} (T^x_a(f))' dm_1 \le T^b_a(f) - T^a_a(f) = T^b_a(f)$.

integral mod f dash dm 1 over ab is less than or equal to mod f dash this T ax f dash.

So, we have this inequality.

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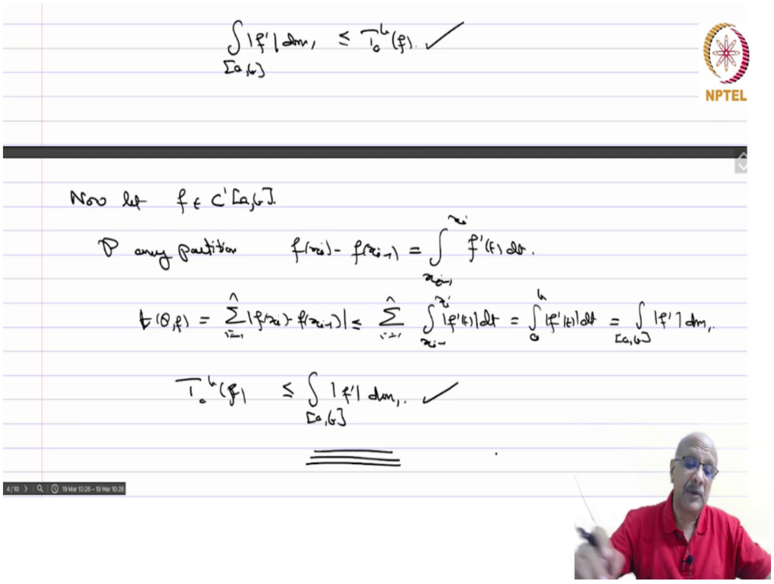

Now, we are assuming now, let $f \in C^1[a, b]$, so, it is a continuously differentiable function. So, f dash is continuous and therefore, you have if P any partition, we have

$$f(x_i) - f(x_{i-1}) = \int_{x_{i-1}}^{x_i} f'(t)dt.$$

So, $t(P, f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f'(t)|dt = \int_{b}^{a} |f'(t)|dt = \int_{[a,b]} |f'|dm_1.$

So, $T^b_{\ a}(f) \leq \int_{[a,b]} |f'|dm_1.$

So, you have one side inequality here another side equal to here and that completes the proof.

So, now, we will look at this enough for us for the moment but it is interesting also to know the functions of bounded variation which are vector valued that means taking values in Rn we are taking f of ab to R next we will look at f of ab to Rn. So, those you can define the same way as functions of bounded variation and prove similar results and it leads to something quite interesting. And so we will take it up next time.