Measure and Integration
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Lecture No-44
Functions of Bounded Variation

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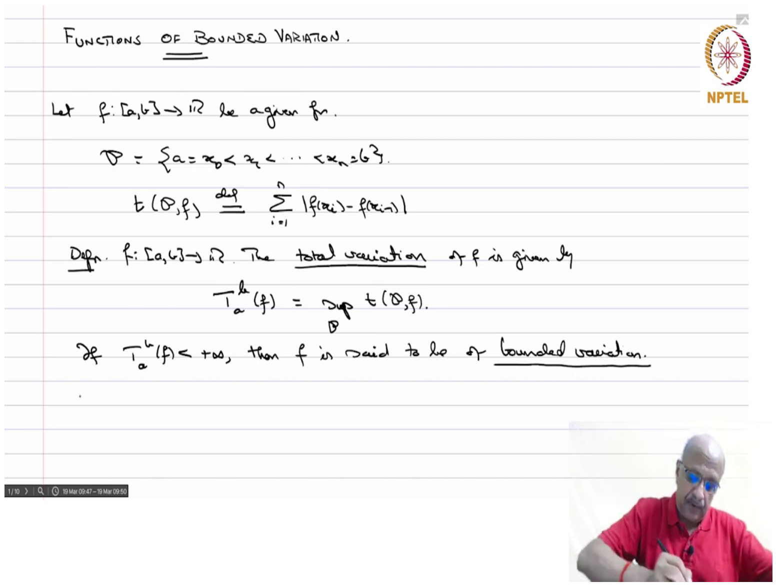

So, we will now investigate another class of the functions which will be differentiable almost everywhere. These are very important classes called **functions of bounded variation**. You might have seen these already in your analysis course, anyway.

So, let $f: [a, b] \to \mathbb{R}$ be a given function. Consider a partition

$$P = \{a = x_0 < x_1 < \ldots\ldots < x_n = b\}$$

And you define $t(P, f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

Definition: $f: [a, b] \to \mathbb{R}$. The total variation of f is given by

$$T^b_a(f) = \sup_P t(P, f).$$

If $T^b_a(f) < \infty$, then f is said to be of bounded variation. So, functions of bounded variations are functions whose total variation is finite or bonded.

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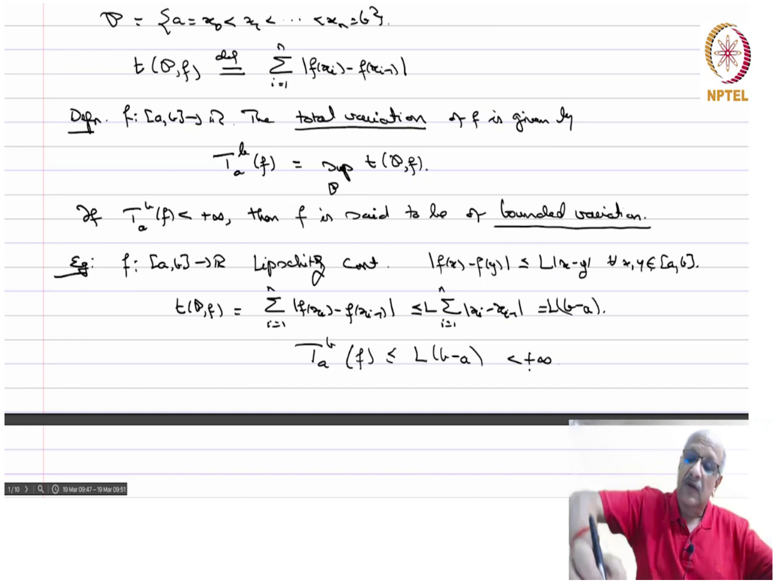

Examples: $f: [a, b] \rightarrow \mathbb{R}$ Lipschitz continuous. It means

$$|f(x) - f(y)| \leq L|x - y|, \forall\, x, y \in [a, b].$$

Now, this I sub-bounded variation because if you take any partition p so,

$$t(P, f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq L \sum_{i=1}^{n} |x_i - x_{i-1}| = L(b - a).$$

$$\Rightarrow T^{b}_{a}(f) \leq L(b - a) <+ \infty.$$

So, therefore, you have that Lipschitz continuous function is of course, of bounded variation.

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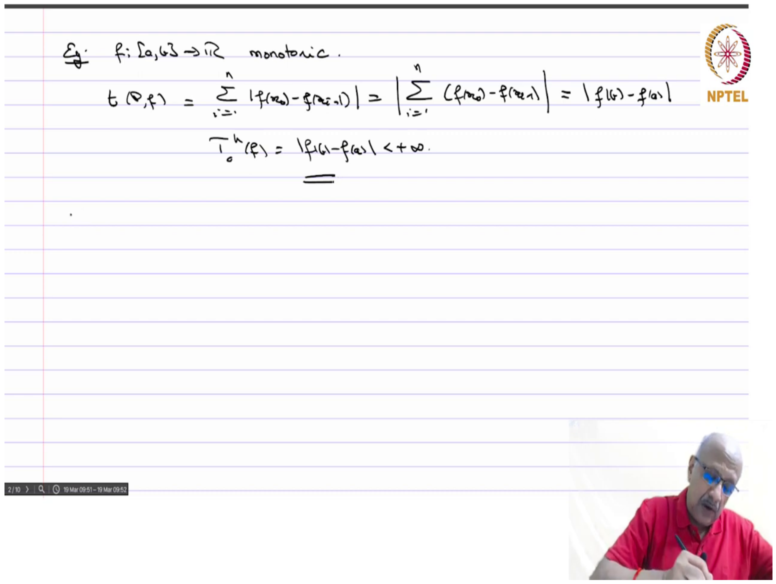

Example: $f: [a, b] \rightarrow \mathbb{R}$ monotonic. It could be increasing or monotonic decreasing. So, if you take any P partition then

$$t(P, f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq |\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))| = |f(b) - f(a)|.$$

Therefore, $T^b_a(f) = |f(b) - f(a)| <+ \infty$.

Therefore, monotonic function is also a function of bounded variation.

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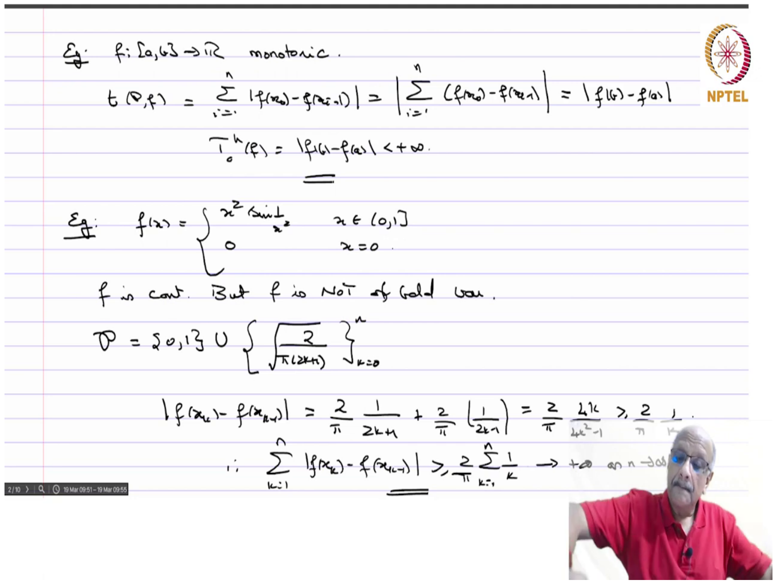

Example: So, you now take fx equals x square sin 1 by x square if for x belonging to 0, 1 and 0 for x equal to 0. So, then f is a continuous function f is continuous, but f is not of bounded variation. So, this is an example of a function which is not of bounded variation. So, to see this let us take a partition p which is equal to the 2 points 0 and 1 union square root of 2 by pi times 2k plus 1 k equal 0 to n.

So, these are all points in 0, 1. So, you consider this partition. So, now, if you take mod f of xk minus f of xk minus 1 this will be equal to 2 by pi of 1 by 2k plus 1 plus 2 by pi 1 by 2k minus 1. Because sin of these things of the reciprocal is odd multiples of pi by 2 so, it is always 1. So, it is only the denominator which contributes.

Now, this is equal to 2 by pi of 4k by 4k square minus 1 and that is greater than equal to 2 by pi. Now, 4k by 4k square minus 1 4k square minus 1 is less than 4k square. So, the fraction is bigger than equal to 4k by 4k squared which is 1 over k. Therefore, the sigma mod f of xk minus f of xk minus 1 is greater than 2 by pi sigma 1 by k. So, this k equals 1 to n. So, if you increase n this is a divergent series and therefore, this goes to plus infinity as n tends to infinity. And therefore, this is not of bounded variation.

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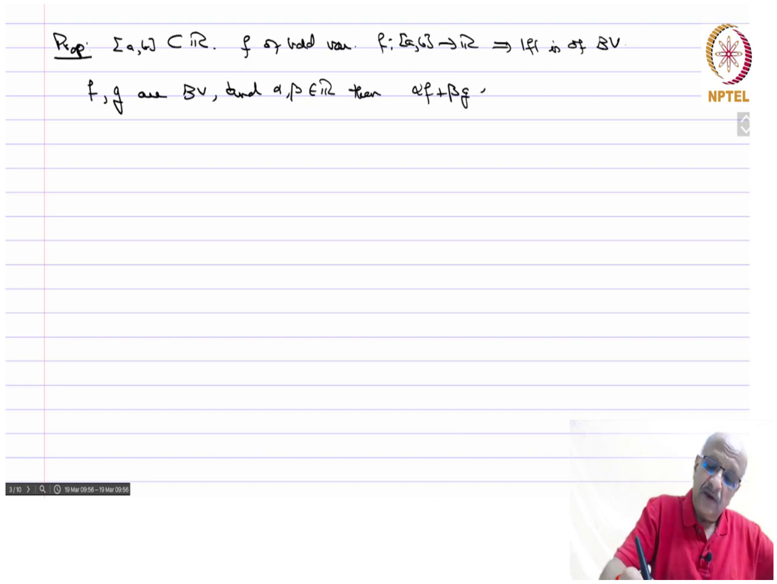

Proposition: $[a, b] \subset \mathbb{R}$ and f of bounded variation f:$[a, b] \subset \mathbb{R}$ of course implies |f| is of bounded variation. I will say BV for bounded variation. Then, if f and g are BV then and alpha beta in R then $\alpha f + \beta g$ is BV.

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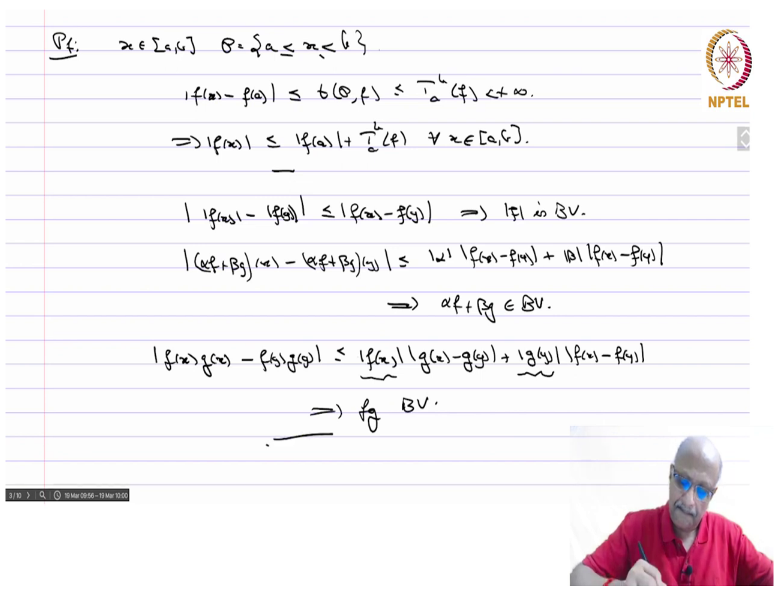

proof: So, if $x \in [a, b]$. So, you simply consider the partition $P = \{a \le x_1 < b\}$. Then

$$|f(x) - f(y)| \le t(P, f) \le T^b{}_a(f) <+ \infty.$$

$$\Rightarrow |f(x)| \le |f(a)| + T^b{}_a(f),\ \forall\, x \in [a, b].$$

Therefore, you have $||f(x)| - |f(y)|| \leq |f(x) - f(y)| \Rightarrow |f|$ is BV.

$$|(\alpha f + \beta g)(x) - (\alpha f + \beta g)(y)| \leq |\alpha||f(x) - f(y)| + |\beta||f(x) - f(y)|$$

$$\Rightarrow \alpha f + \beta g \text{ is BV.}$$

So, $|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|$.

$$\Rightarrow fg \text{ is BV.}$$

Just apply the definition and you will get it immediately. So, this is the thing.

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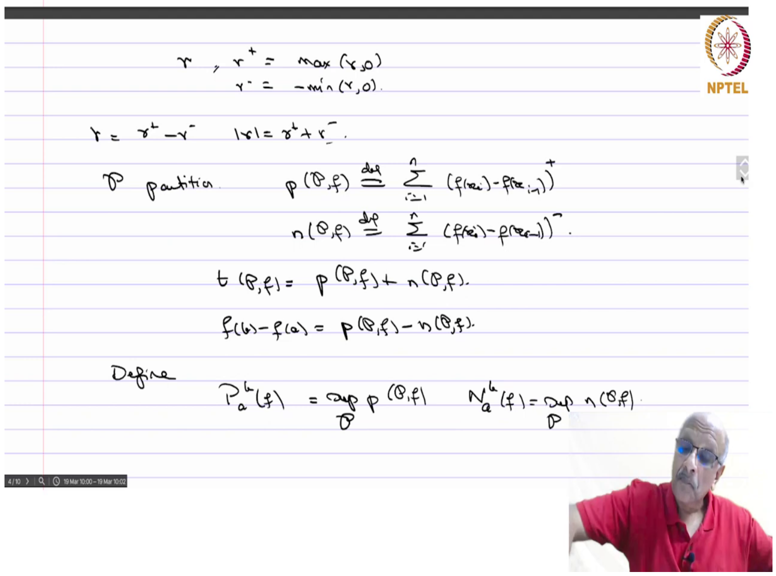

So, now, given a real number r we write r plus equals max of r and 0 and r minus equals minus min of r and 0. So, in other words you have just like you write the positive and negative parts if if r is positive then r plus is r, if r is negative then r minus r is equal to minus r minus. So, in general r equals or plus minus r minus and mod r equals r plus plus r minus.

So, now p partition and you define then you have p of p, f we define as sigma i equals 1 to n fxi minus f of xi minus 1 plus and then n of p f as sigma i equals 1 to n fx i minus f of xi minus 1 minus. So, then you have t p, f is equal to p of p, f plus n of p, f. And then fb minus fa equals p of p, f minus n of p, f straightforward calculations. Now define capital P a, b of f is supremum P of p, f taken over all partitions and N a, b of f equals supremum over n of p, f taken overall partitions

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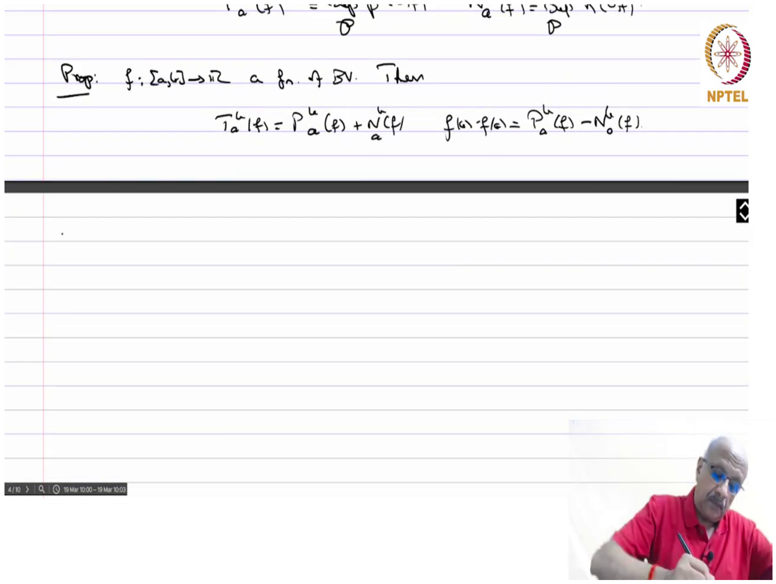

Proposition: $f: [a, b] \rightarrow \mathbb{R}$ a function of BV, then

$$T^b{}_a(f) = P^b{}_a(f) + N^b{}_a(f),\ f(b) - f(a) = P^b{}_a(f) - N^b{}_a(f).$$

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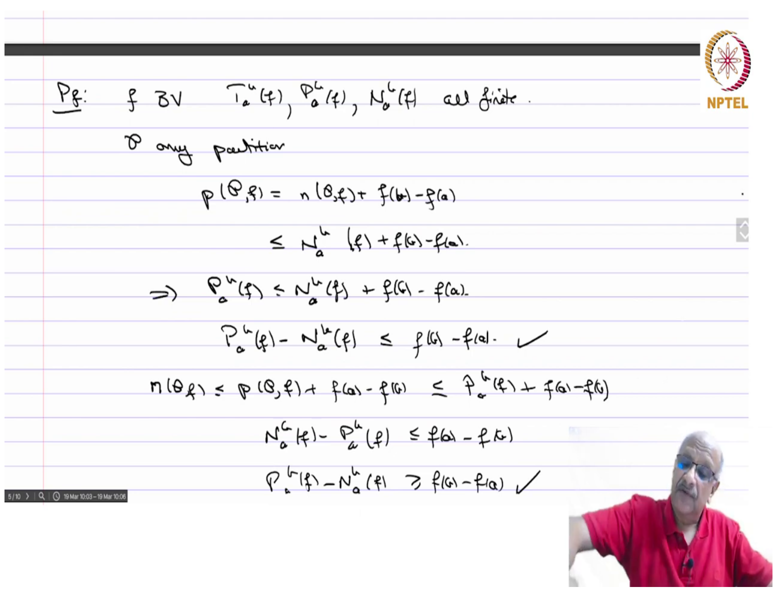

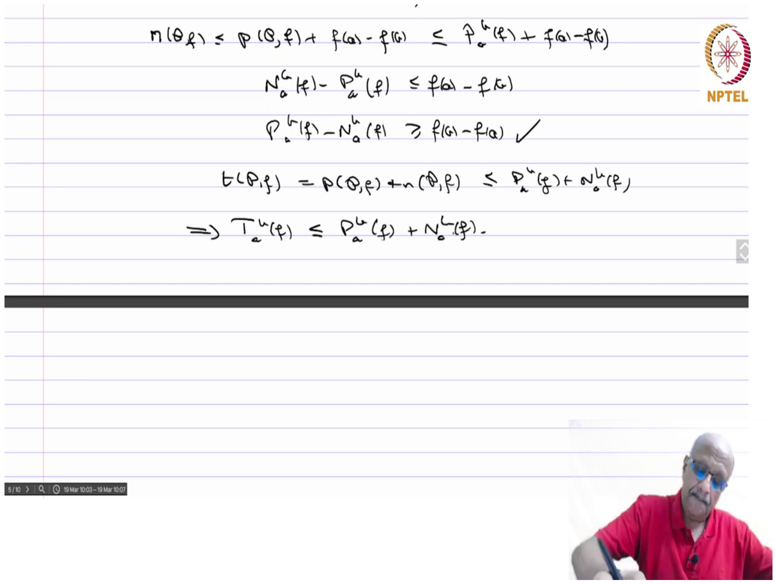

proof. So, f is BV. So, that means $T^{b}_{a}(f)$, $P^{b}_{a}(f)$, $N^{b}_{a}(f)$ all finite. So, P any partition you have

$$P(P, f) = n(P, f) + f(b) - f(a) \leq N^{b}_{a}(f) + f(b) - f(a).$$

$$\Rightarrow P^{b}_{a}(f) - N^{b}_{a}(f) \leq f(b) - f(a).$$

Now, you use the same relationship again and now you write

$$N(P, f) \leq P(P, f) + f(a) - f(b) \leq P^{b}_{a}(f) + f(a) - f(b).$$

$$\Rightarrow N^{b}_{a}(f) - P^{b}_{a}(f) \leq f(a) - f(b).$$

$$\Rightarrow P^{b}_{a}(f) - N^{b}_{a}(f) \geq f(b) - f(a).$$

And now that it will be less than equal to captain P a, b of f plus fa minus fb and from that if you take supremum again you have NA b of f minus P a, b of f is less than equal to fa minus fb and that will give you P a, b of f minus N a b of f is greater than equal to fb minus fa and so, you have the 2 (())(19:14) inequalities and therefore, you have these 2 are equal P a, b minus N a, b equals fb minus fa and that proves the statement here. Now for the you have the t of p f for any partition is equal to p of p, f plus n of p, f and therefore, you get immediately the T a, b of f this is less equal to P a, b or f plus N a, b of f just a supremum. So, that is less than equal to P a, b of f plus N a, b of f.

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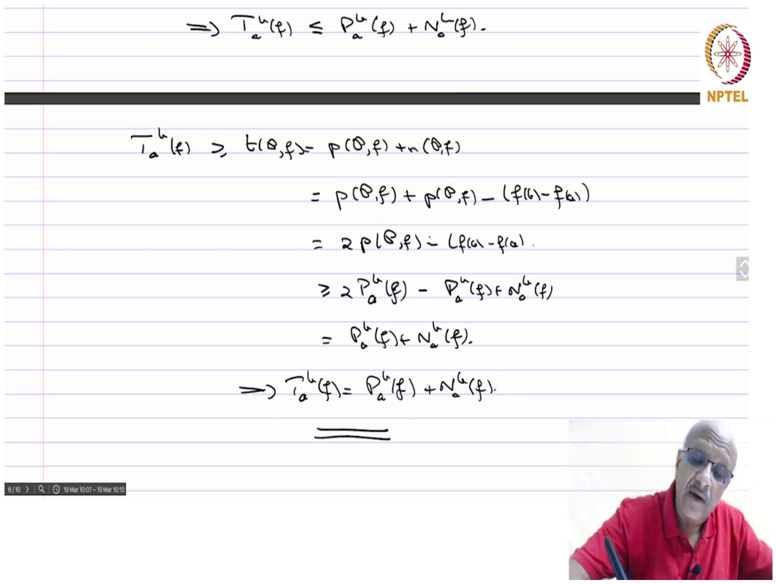

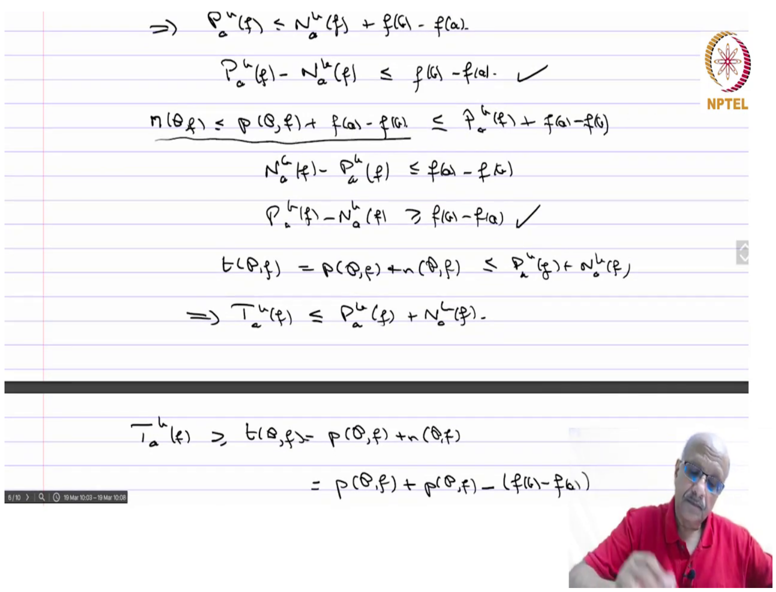

Now, we want to prove the reverse inequality. So, for that you take T a, b of f is greater equal to t of p, f, which is equal to p of p, f plus n of p, f. Now, this n of p, f I will write as equal to p of p f, n of p, f is nothing but p of p, f minus fb minus fa. We just wrote it recently here instead of fa minus fb I am writing this, so, that is equal to 2 times p of p, f minus or minus fb plus fa. Now let me keep it minus fb minus fa.

So, now, if I took the supremum this is bigger than equal to 2 times P a, b of f, I am just taking the supremum and minus fb minus fa I have already I am going to use the equation there this 1 and therefore, I am going to write it as minus P a, b f plus N a, b f and that will give you a P a, b f plus N a, b f. And so, you have the reverse inequality also and therefore, you have the conclusion. So, this implies that T a, b f equals P a, b f plus N a, b f. So, this will helped us to make a very nice characterization of functions of bounded variation. So, we will see that in the next session.