Measure and Integration
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Lecture No-43
Monotonic Functions

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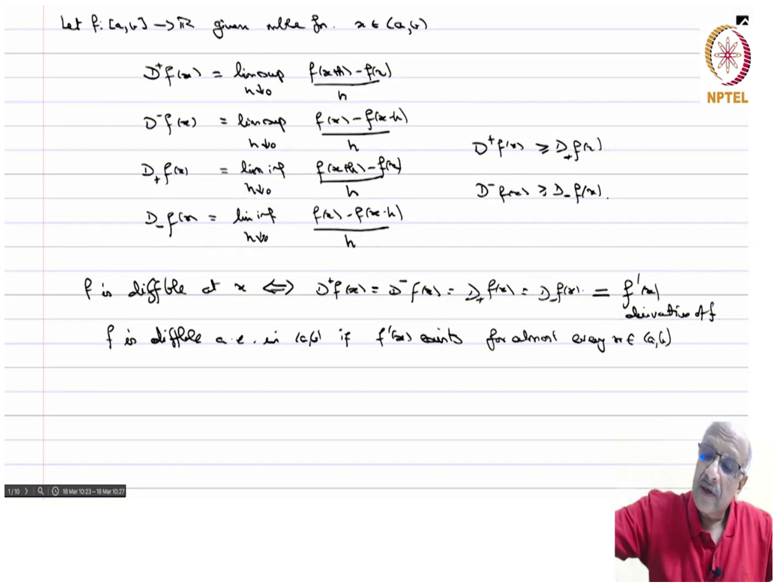

We will now show that monotonic function on any finite interval is almost everywhere differentiable. So let $f:[a,b]\to\mathbb{R}$ given measurable function. So, if $x\in(a,b)$, we can define the following 4 quantities: they are all well-defined, because, as you can see, this is nothing but

$$D^+f(x) = \lim_{h\downarrow 0} \sup \frac{f(x+h)-f(x)}{h}$$

$$D^-f(x) = \lim_{h\downarrow 0} \sup \frac{f(x)-f(x+h)}{h}$$

$$D_+f(x) = \lim_{h\downarrow 0} \inf \frac{f(x+h)-f(x)}{h}$$

$$D_-f(x) = \lim_{h\downarrow 0} \inf \frac{f(x)-f(x+h)}{h}.$$

So then you for instance, you have some obvious inequalities, $D^+f(x) \geq D_+f(x), D^-f(x) \geq D_-f(x)$.

Now, f is differentiable at x if and only if $D^{+}f(x) = D_{+}f(x) = D^{-}f(x) = D_{-}f(x) = f'(x)$ and that is obvious (())(02:54).

So f is differentiable almost everywhere in (a, b) if f dash takes exists x in a, b, f dash x exists at almost or rather for almost every x in a, b that means except on the set of measures 0 is this, and what is f dash x? This common value is called f dash x derivative of f.

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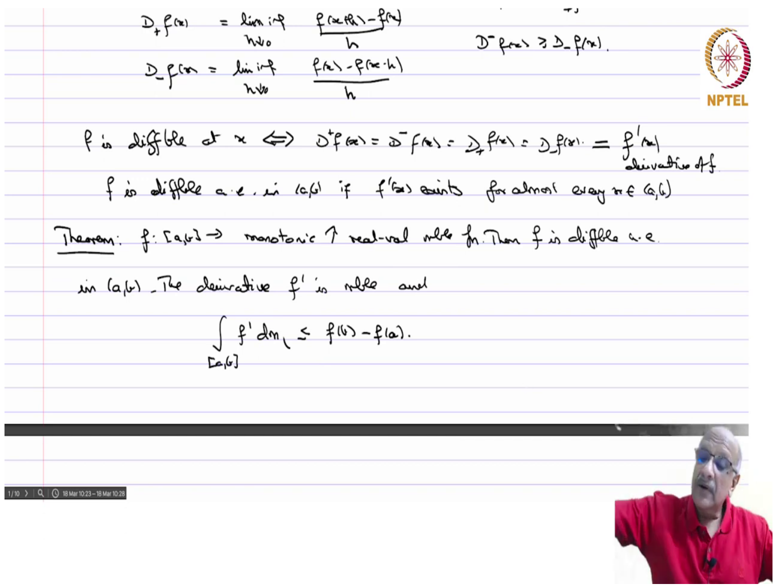

So, now, we sketch the proof of this important theorem, I say sketch proof is more or less complete, but there will be little points which need to be checked, which can be done easily it is a time consuming process to prove the entire thing in full detail and therefore, I will give almost all the steps necessary and so, I just call it a sketch of a proof.

Theorem: $f: [a, b] \to \mathbb{R}$ monotonically increasing function, increasing real valued measurable function. Then, f is differentiable almost everywhere in (a, b). The derivative f' is measurable and $\int_{[a,b]} f' dm_1 \leq f(b) - f(a)$.

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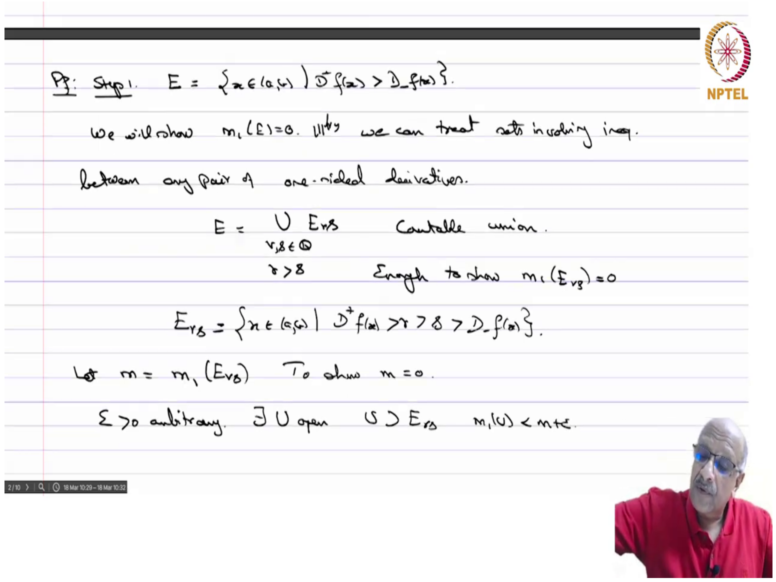

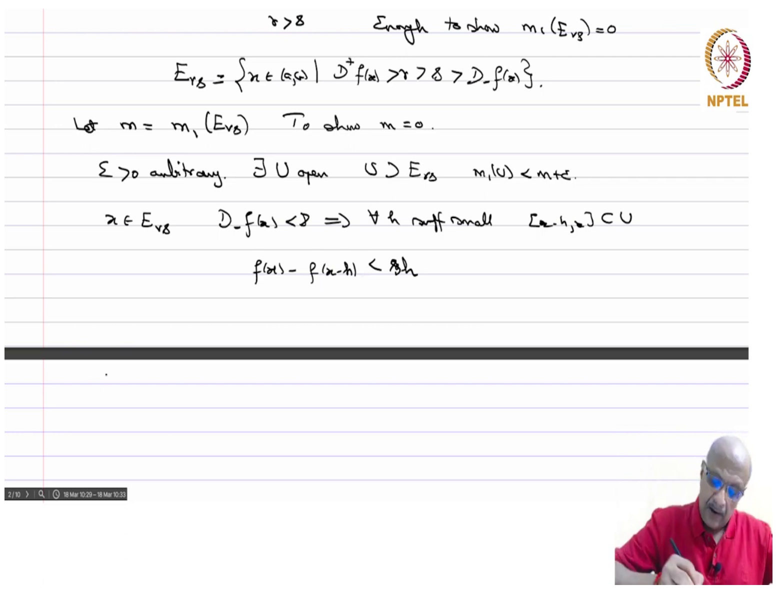

proof: **step 1.** So, you consider the set $E = \{x \in (a, b) \colon D^{+}f(x) > D\_f(x)\}$

So, we will show $m_1(E) = 0$. Now similarly we can treat sets involving inequality between any pair of 1 sided derivative.So, we can write

$$E = \cup_{r,s\in\mathbb{Q},\, r>s} E_{rs}.$$

Now, this is countable because rational is rational is countable and therefore, you have, so, enough to show $m_1(E_{rs}) = 0$. So, $E_{rs} = \{x \in (a,b) \colon D^+f(x) > r > s > D\_f(x)\}$.

So, let $m = m_1(E_{rs})$. To show m=0. So, let epsilon greater than 0 arbitrarily there exists U open, $U \supset E_{rs}$ and $m_1(U) < \epsilon + m$. So, let $x \in E_{rs}$. So, $D\_f(x) < s \Rightarrow$ for all h sufficiently small, $[x - h, x] \subset U$ and $f(x) - f(x-h) < sh$.

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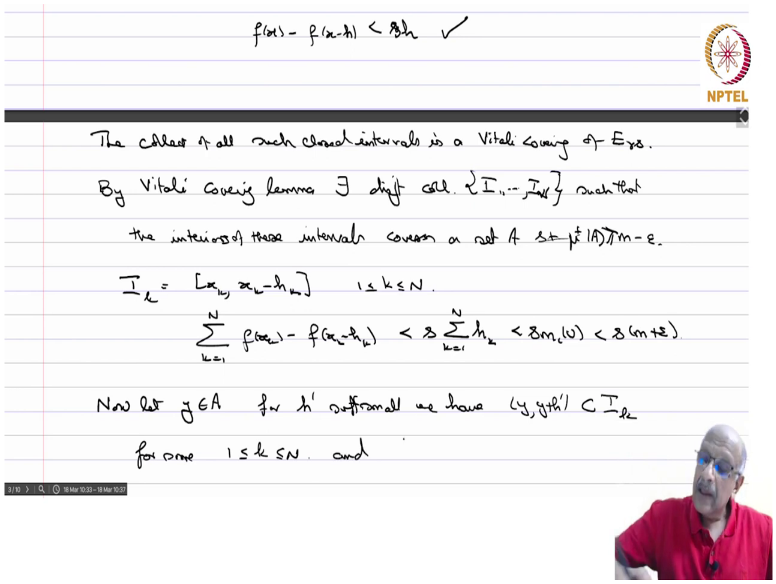

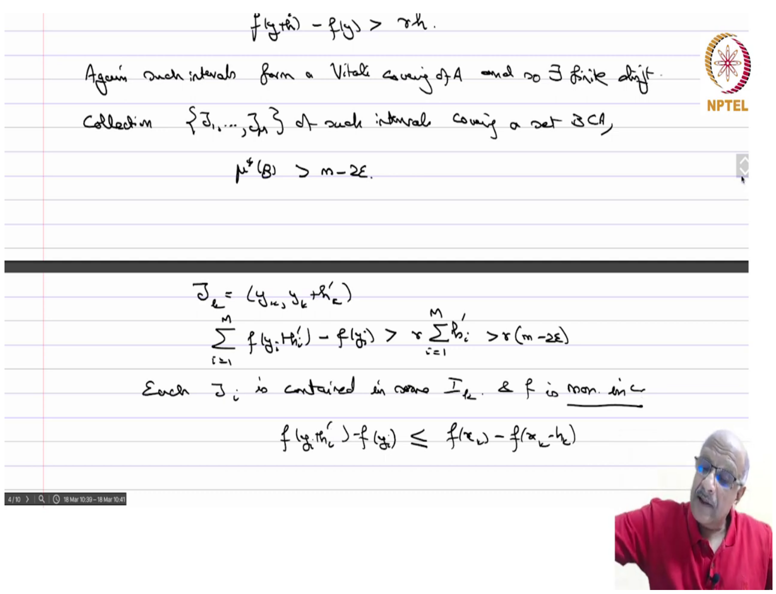

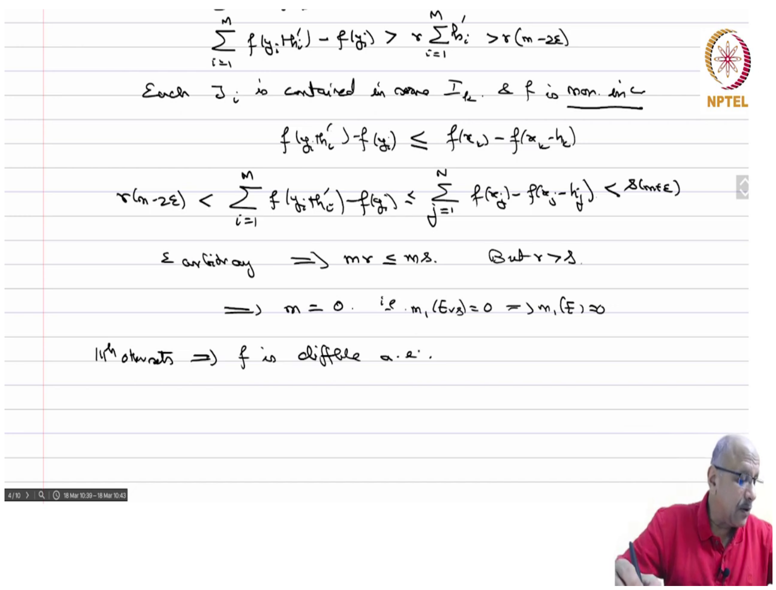

So, the collection of all such closed intervals is Vitali covering because given any x I can find the sufficiently small s like this of E_{rs}. So, by Vitali covering lemma there exists a disjoint collection I_1,..., I_n such that the interiors of these intervals covers a set A of measure of such that

$$\mu^*(A) > m - \epsilon.$$

Now, what does this? Vitali covering lemma says you can cover it by disjoint intervals and the portion which is uncovered is less than epsilon. So, if the portion uncovered is less than epsilon then the portion covered must be bigger than m minus epsilon because m is a measure of that set and as an interior of these intervals because I am going to further work with derivative and so on. And it does not matter as we said whether we had closed intervals or open intervals the endpoints contribute nothing to the measure.

Now, you take Ik equals xk xk minus hk one less than equal to k less than equal to N then what do you know sigma k equals 1 to n f of x k minus f of xk minus hk is coming from here is therefore, less than s times sigma k equals 1 to N hk, but these are disjoint intervals all contained in U and therefore, this is less than s times measure of U which is s less than s times m plus epsilon.

Now, let y be belong to A A is set which is covered by all the I1, I2 In's. Now there for h dash sufficiently small we have y, y plus h dash contained in U is in fact contain in some Ik for some 1 less than equal to k less than equal to n and it is in Ers. Remember Ers means D plus is bigger than r, D minus is less than s. So, and you have f of y plus h minus fy is greater than r times h, h dash y plus h dash minus fy greater than this.

So, again such intervals form a Vitali covering of A and so, there exists a finite collection J1 Jm of such intervals finite disjoint collection of such intervals covering as set B contained in A and mu star B is greater than m minus 2 epsilon. So, if you take Jk is equal to yk yk plus hk dash, then you have sigma i equals 1 to m f of y i plus Hi dash minus f of yi is greater than r times sigma i equals 1 to m hi dash and that is bigger than r times m minus 2 epsilon. Because all these cover B and B itself as bigger measure than this and therefore, this is measured is…

Now, each Ji is contained in some Ik and f is monotonically increasing therefore, f of yi plus h dash Hi dash minus f of yi. So, there is a difference of 2 points values and 2 points inside the bigger one. So, that is f of x k minus f of x k minus h k. Now, this interval contains this interval and therefore, this has to be bigger.

So, adding all these things for the each term here you have a term there and therefore, you have sigma i equals 1 to m f of yi plus Hi dash minus f of yi is less than equal to sigma j equals 1 to N f of x j minus f of x j minus hj and this is less than s into m plus epsilon and this one is bigger than r into m minus 2 epsilon. Now, epsilon arbitrarily implies mr is less than equal to sn ms. But, r is bigger than s. So, this implies m equal to 0. So, that is mu of Ers equal to 0 sorry m1 implies m1 of E is 0 and from there we can deduce and that will imply finally, similarly, other sets and this implies f is differentiable almost everywhere.

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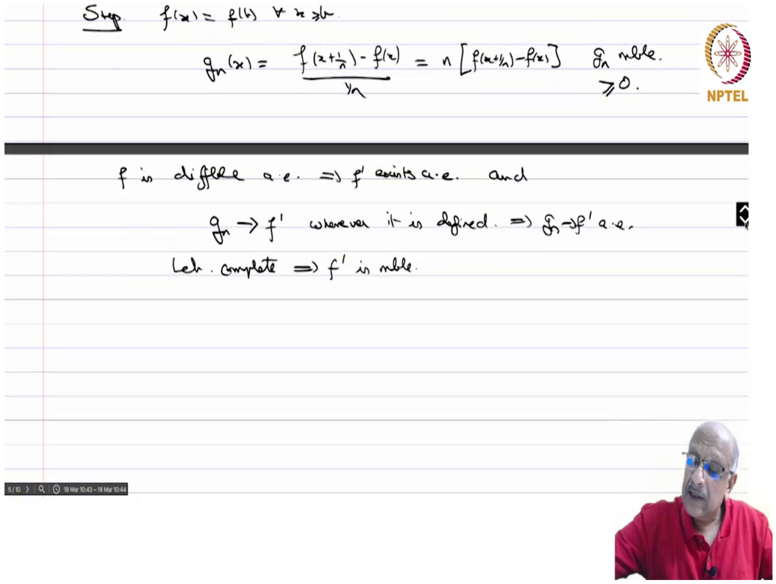

So, now step 2, we have to show the inequality for that integral there. So, you take fx equal to fb for all x bigger than equal to b. And define gn of x, we need to do that only to define the following function gn of x is f of x plus 1 by n minus f of x by 1 by n. So, I will write this as n times f of x plus 1 by n minus f of x.

So, this is clearly gn is measurable and because it is f it is monotonically increasing this also greater than or equal to 0. So, f is differentiable almost everywhere that means f dash exists almost everywhere and gn must therefore, converge to f dash wherever it is defined. So, that means gn converges to f dash almost everywhere.

Now, lebesgue measure is complete. This implies that f dash is measurable. So, we have seen an example of incomplete in the case of incomplete measure spaces then if you have a sequence of measurable functions, then the limit need not be measurable, but if the space is complete there is no problem we have seen that also and therefore, you have that if you have you have a sequence of measurable functions converging to a function, so, the function is itself measurable.

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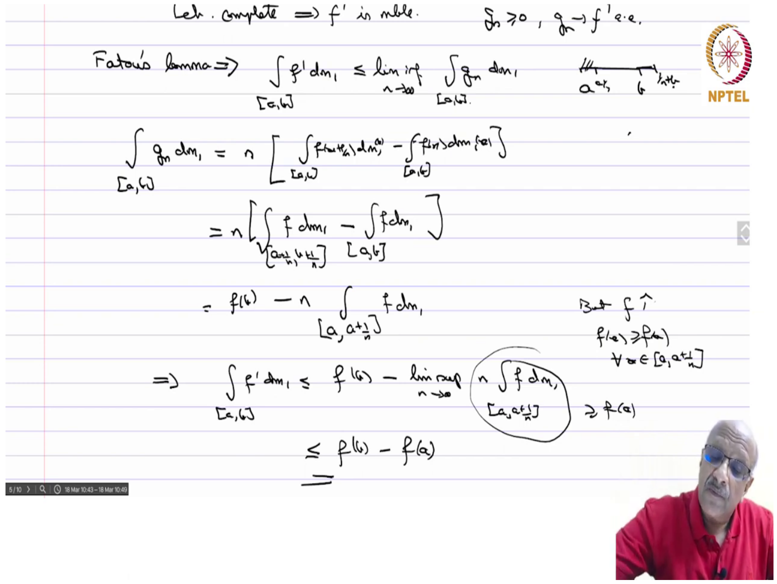

So, by Fatou's lemma because all the gn are non-negative. So, gn non-negative gn goes to f dash almost everywhere and therefore by Fatou's lemma you have that integral ab f dash dm1 is less than equal to lim inf n tending to infinity integral gn d m1 of over ab. So, that is compute integral a b gn dm1, what is gn? Gn is f of x plus 1 by n minus f of x by 1 by n.

So, this will be n times integral over a b f of x plus 1 by n d m1 minus integral over a, b f of x dm1 next let me write dm1 and we have already done this exercise this is equal to n times integral a plus 1 by n, b plus 1 by n, f dm1 minus integral a b, f dm1. So, you have a, b a plus 1 by n b plus 1 by n and therefore, if you take the difference of these 2 sets, this is equal to O so, this portion is gone this portion a is there and then you only have the middle portion here.

Now, in this portion the function is fixed this portion goes in this portion the function is fixed and therefore, that is equal to fb. So, you will just get fb minus n times integral a to a plus 1 by n f dm1. So, the Fatou's lemma result now pumps integral a, b f dash dm1 is less than or equal to fb lim inf of this, so fb minus lim sup n tending to infinity n times integral a to a plus 1 by n, f dm1, but f is monotonic increasing so fx is greater equal to fa for all x in a a plus 1 by n.

So, this integral can be bounded below by f of a since this n here f of a into 1 by n into n and therefore, lim sup will be minus. So, this is less than or equal to fb minus fa. Because this quantity is bigger than or equal to f of a and therefore, lim sup will be bigger than fa minus lim sup will be less than minus fa. And that completes the proof of this theorem. So this completes the proof that all monotonic functions are differentiable almost everywhere. So, next time we will see another class of functions, which are differentiable almost everywhere which come out of monotonic functions.