Measure and Integration
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Lecture No-42
Vitali Covering Lemma

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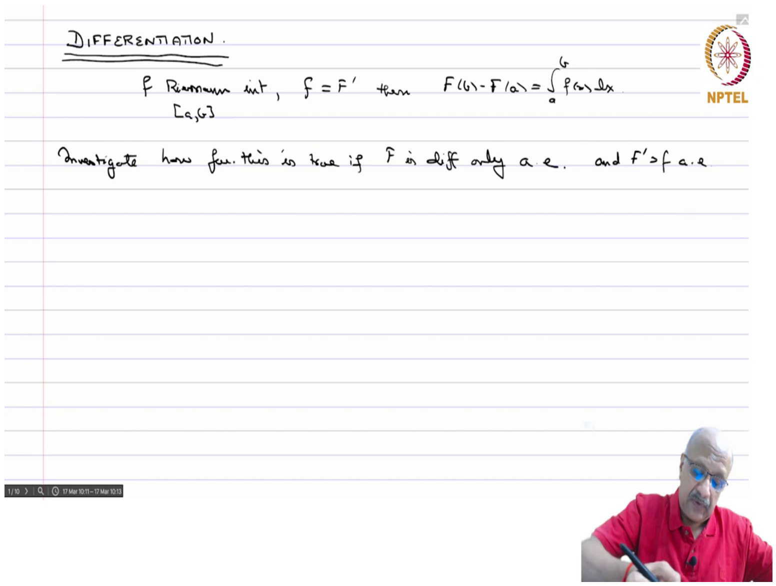

We now start a new chapter. So, this is differentiation. So, one of the important features of differential integral calculus is that differentiation and integration really have 2 sides of the same coin, more precisely the fundamental theorem states that if you have f is a Riemann integrable function, which is the derivative of a function capital F.

So, F Riemann integrable $f = F'$, then $F(b) - F(a) = \int_a^b f(x)dx$.

So, we would like to investigate how far this goes. So, investigate how far this is true, if F is differentiable only almost everywhere and $F' = f$ almost everywhere. So, we want this and we want to know if you can have the same (())(01:57) set.

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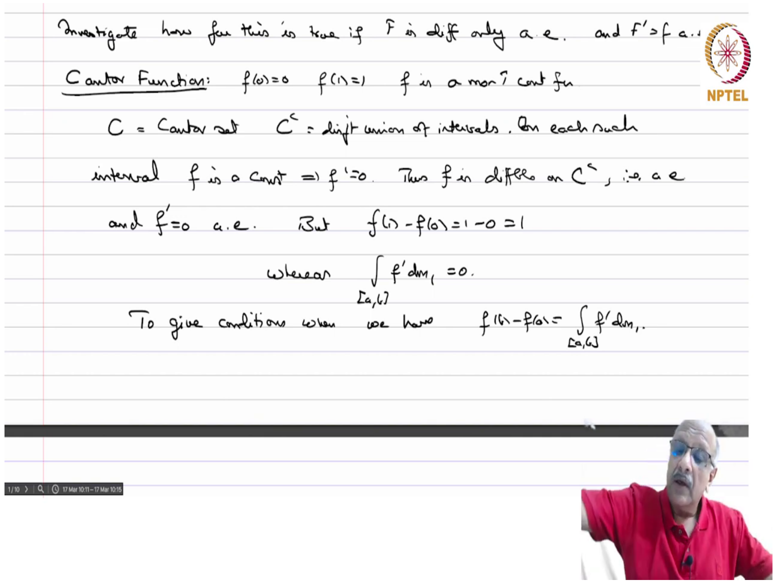

Cantor function. So, $f(0) = 0$, $f(1) = 1$, f is a monotonically increasing continuous function. Now, if C= the cantor set, then C^c = the disjoint union of intervals if you remember the construction and on each interval on each such interval, if you remember the construction of the cantor function f is a constant $f' = 0$. This f is differentiable on C^c that is C is of measure 0. It is almost everywhere and $f' = 0$ almost everywhere.

But in $f(1) - f(0) = 1 - 0 = 1$, whereas, $\int_{[a,b]} f' dm_1 = 0$.

So, these two are not equal. So, the fundamental theorem of calculus one has to be a little careful when we want to do. So, we want to investigate to give conditions when we have $f(b) - f(a) = \int_a^b f'(x)dx$. So, when can we write this? So, we want to see, so, we will start looking at various classes of differentiable functions which are differentiable almost everywhere.

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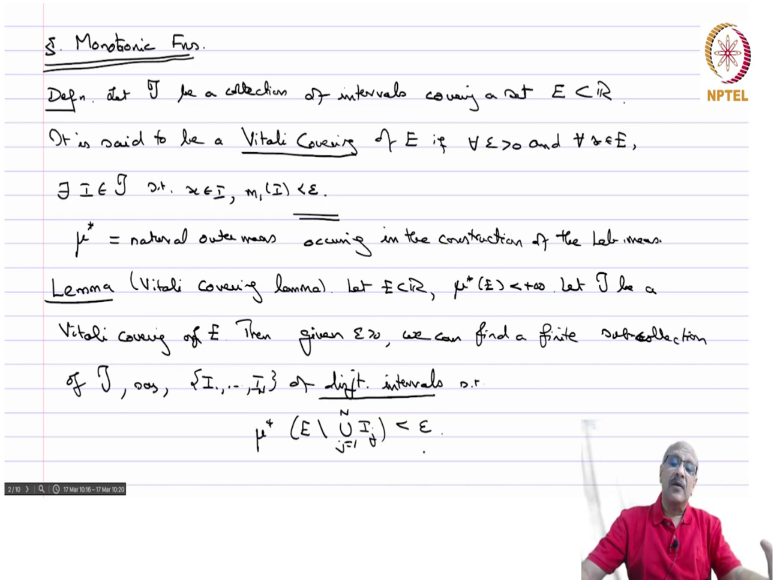

Monotonic functions.

Definition: Let I be a collection of intervals covering a set $E \subset \mathbb{R}$. It is said to be a Vitali covering of E if for every $\epsilon > 0$ and for every x in E, there exists $I \in I$ such that $x \in I$ and $m_1(I) < \epsilon$.

Lemma: (Vitali covering lemma). Let $E \subset \mathbb{R}$, $\mu * (E) < \infty$. Let I be the Vitali covering of E. Then given $\epsilon > 0$, we can find a finite subcollection of I, say $\{I_1, I_2, ..., I_N\}$, of disjoint intervals such that $\mu * (E \setminus \cup_{j=1}^{N} I_j) < \epsilon$.

So, you are having a Vitali covering of a set of finite outer measure then you can extract a disjoint collection of intervals such that their union covers almost completely the given set namely almost covers it means given any epsilon it will cover E except for a set of measures less than epsilon. So, this is the Vitali covering lemma.

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Proof: Observe that the intervals can be open, closed or half-open.

By adding or removing end-pts, meas does not change.

WLOG we assume all intervals in $\mathcal{J}$ are closed.

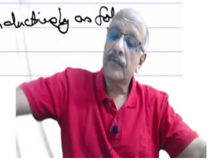
$\mu^*(E) < +\infty \Rightarrow \exists\ U$ open $U \supset E$, $\mu^*(U) (= m(U)) < +\infty$.

Further, since we are dealing with a Vitali covering, WLOG $\forall I \in \mathcal{J}$,

we have $I \subset U$.

Let us choose $I_1 \in \mathcal{J}$ arbitrarily.

Having chosen $I_1, \ldots, I_n$ disjt, we choose I_{n+1} inductively as foll



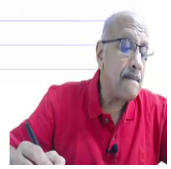
Having chosen $I_1, \ldots, I_n$ disjt, we choose I_{n+1} inductively as foll

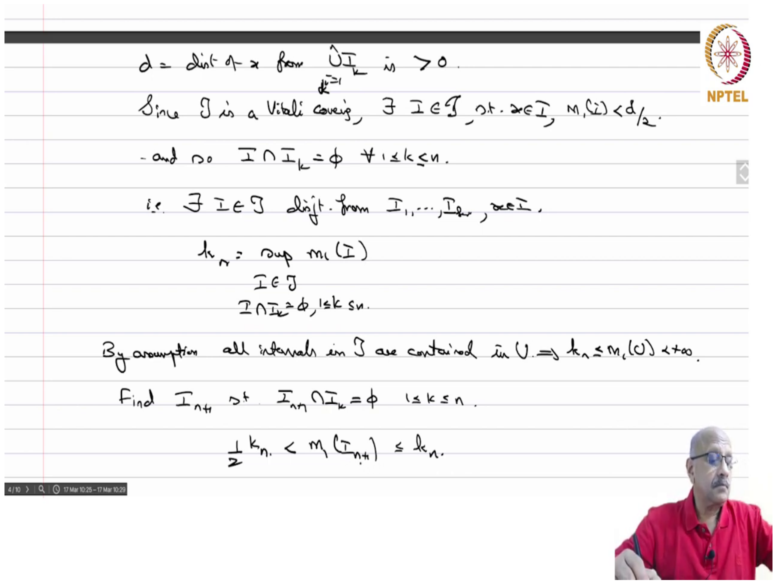
- $E \subset \bigcup_{k=1}^{n} I_k$, process stops
- If not, let $x \in E \setminus \underbrace{\bigcup_{k=1}^{n} I_k}_{\text{closed}}$.

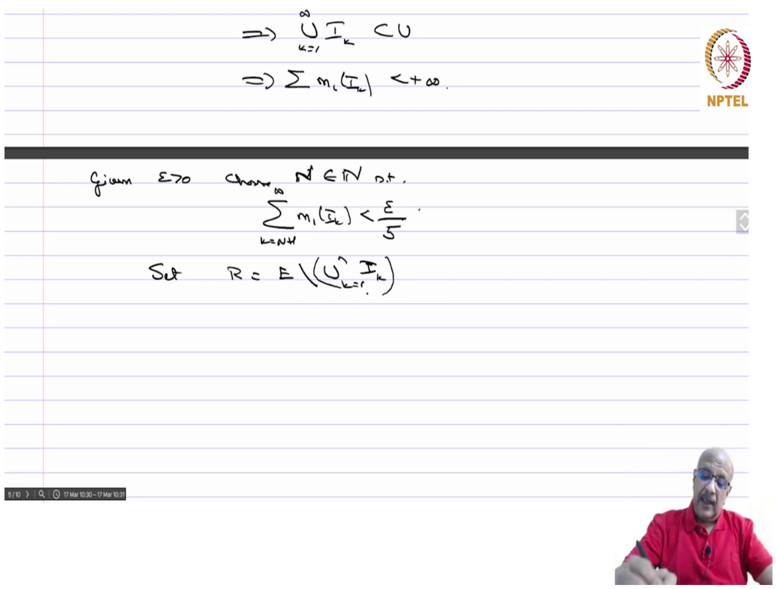
$d =$ dist of x from $\bigcup_{k=1}^{n} I_k$ is > 0.

Since $\mathcal{J}$ is a Vitali covering, $\exists\ I \in \mathcal{J}$ s.t. $x \in I$, $m(I) < d/2$.

- and so $I \cap I_k = \phi \quad \forall\ 1 \le k \le n$.







proof: Observe that the intervals can be open closed or half open by adding or removing endpoints measure does not change. So, nothing changes as far as a measure of this concern and therefore without loss of generality, we assume all intervals in I are closed. Now,

$$\mu * (E) < \infty \Rightarrow \exists\, U \supset E,\ s.t.\ \mu * (U)\,(= m_1(U)) < \infty.$$

Further since, we are dealing with a Vitali covering that means only small intervals count this without loss of generality for every I in I, we have $I \subset U$. Now, let us choose $I_1 \in I$ arbitrarily. Having chosen $I_1, I_2, \ldots, I_N$ disjoint, we choose I_{N+1} inductively as follows:

- If $E \subset \cup_{k=1}^{N} I_k$, then the process stops.
- If not let $x \in E \setminus \cup_{k=1}^{N} I_k$.

So, the distance $d =$ distance of x from $\cup_{k=1}^{N} I_k$ is positive. So, since I is a Vitali covering that exists I in I such that x belongs to I and $m_1(I) < \frac{d}{2}$. So, by and of course and so, $I \cap I_k = \phi$, for all $1 \leq k \leq N$. So, the conclusion that is given there exists I in I disjoined from I1 to Ik and x belongs to I. So, now you said kn to be the supremum of m1 of I, I belong to I, I intersection Ik is empty 1 less than equal to k less than equal to n. So, by assumption all intervals in I are contained in U. So, this implies that kn is less than equal to m1 of U which is finite.

So, kn is a finite quantity we have taken supremum over something so, we want to know that it is finite. Now, find I_{N+1} such that $I_{N+1} \cap I_k = \phi,\ 1 \leq k \leq N \ and \frac{k_N}{2} \leq m_1(I_{N+1}) \leq k_N$.

This implies that $\cup_{k=1}^{\infty} I_k \subset U \Rightarrow \sum_{k=1}^{\infty} m_1(I_k) < \infty$.

So, $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} m_1(I_k) < \frac{\epsilon}{5}$.

Set $R = E \setminus \cup_{k=1}^{N} I_k$.

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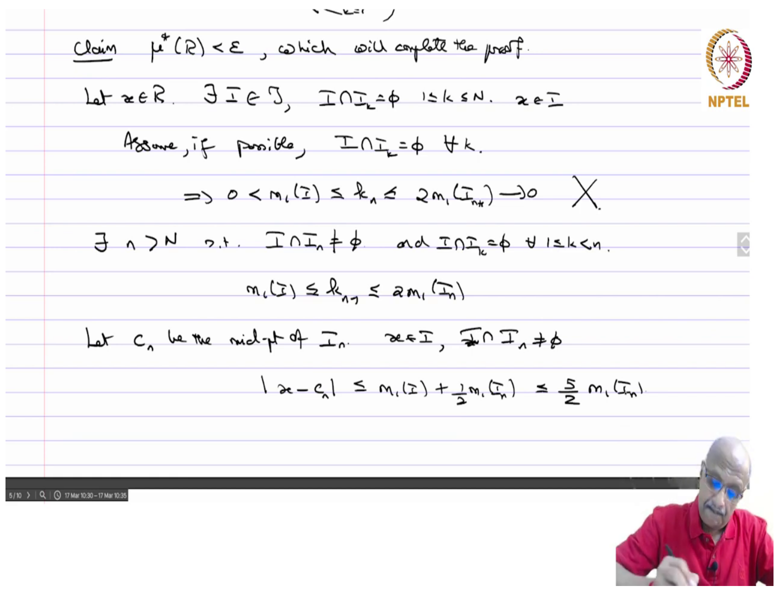

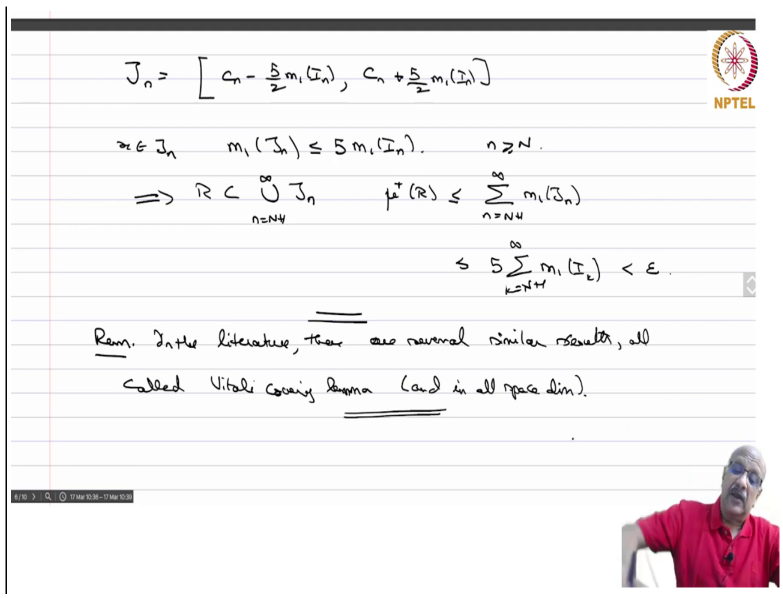

So, claim: $\mu * (R) < \epsilon$, which will complete the proof.

So, let $x \in R$, so, that exists I in I, $I \cap I_k = \phi$, $1 \le k \le N$, $x \in I$. Assume if possible, $I \cap I_k = \phi$ for all k, we will get a contradiction.

So, this will not be possible, why is it so? Then this will imply that

$$0 < m_1(I) \leq k_n \leq 2m_1(I_{n_k}) \to 0,$$

and therefore, you have a contradiction. So, there exists $n \geq N$ such that $I \cap I_n \neq \phi,\ I \cap I_k = \phi,\ \forall\, 1 \leq k < n$. Now $m_1(I) \leq k_{n-1} \leq 2m_1(I_n)$. So, let c_n be the midpoint of I_n. So, $x \in I,\ I \cap I_n \neq \phi$. Therefore,

$$|x - c_n| \leq m_1(I) + \frac{1}{2}m_1(I_n) \leq \frac{5}{2}m_1(I_n).$$

Now, you said $J_n = [c_n - \frac{5}{2}m_1(I_n),\ c_n + \frac{5}{2}m_1(I_n)]$. Then $x \in J_n$ and $m_1(J_n) \leq 5m_1(I_n),\ n \geq N$. So, this implies that $R \subset \cup_{n=N+1}^{\infty} J_n$. So,

$$\mu * (R) \leq \sum_{n=N+1}^{\infty} m_1(J_n) \leq 5 \sum_{k=N+1}^{\infty} m_1(I_k) < \epsilon.$$

That completes the proof.

So, this is a fairly complicated lemma, but anyway there are no deep ideas in it. Just a question of selecting the definition of supremum summation of series (())(26:24) but it is the several.

Remark: in the literature there are several similar results called Vitali covering lemma and in all space dimensions not necessarily R you can have an $\mathbb{R}^2$, $\mathbb{R}^n$ and so on.

And the basic idea is that if you have a set of finite outer measure which is covered by means of basic open sets which are of arbitrarily small size then you can extract a disjoint finite set from them. Such that in a union almost covers E in the sense that the complement of the uncovered portion can be as small as measure (())(27:42) So, that is called the Vitali covering lemma. Now we will use this to show that monotonic functions are differentiable almost everywhere. So, that will be the next thing we will do.