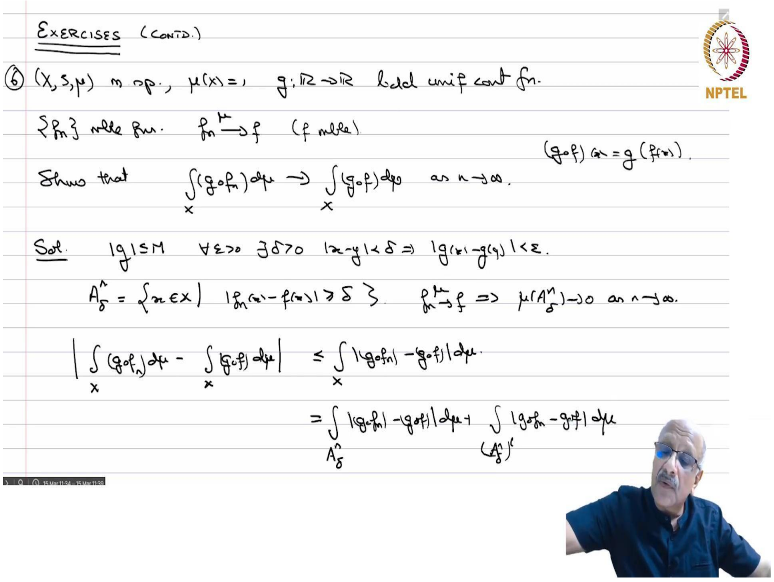
Measure and Integration
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No-41

(Refer Slide Time: 0:16)



So, we continue the exercises, we now do two problems, which illustrate once more the principle of divide and rule which I explained when proving the Weierstrass theorem. So, there are some integrals which you want to estimate. So, it is a good idea to sometimes split the integral over a set and its complement. 1 of the sets, we know that the measure is small, we do not know too much about the function. On the other hand, we know that the function is small, but we do not know too much about the measure. So, these two complementary things will compensate for each other and we will be able to control the integral and estimate it nicely.

So, this is the idea which we want to do. So, here are the two exercises which I want to do. (6) (X, S, μ) measure space, $\mu(X) = 1$, $g: \mathbb{R} \to \mathbb{R}$ bounded uniformly continuous function, f_n measurable functions, f_n converges to f in measure (f measurable). Show that

$$\int_X g \circ f_n d\mu \to \int_X g \circ f d\mu \; as \; n \to \infty.$$

Solution: So, g is bounded, so, we can say $|g| \leq M$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. So, now you take

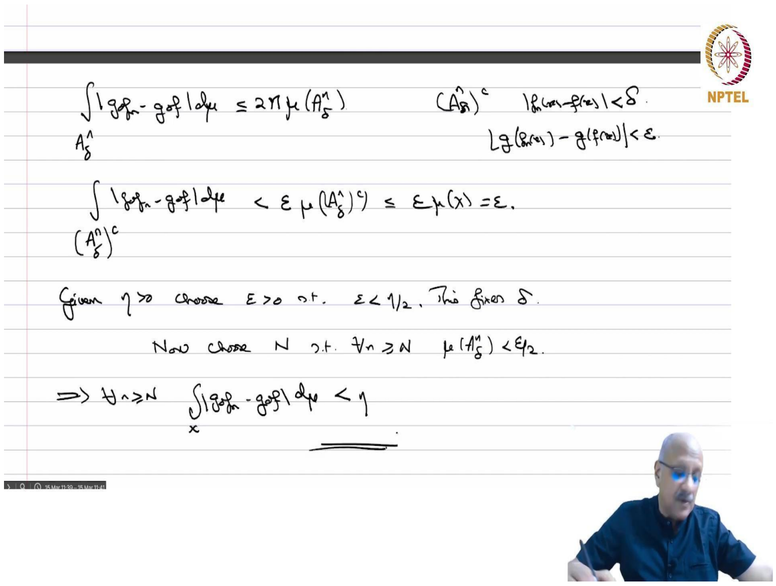
$$A^{\delta}_{n} = \{x \in X : |f_{n}(x) - f(x)| \geq \delta\}.$$

So, then we know that $f_{n} \to f$ in measure. So, that will imply that $\mu(A^{\delta}_{n}) \to 0$. So, now, we are looking at

$$|\int_{X} g \circ f_{n} d\mu - \int_{X} g \circ f d\mu| \leq \int_{X} |g \circ f_{n} - g \circ f| d\mu$$

$$= \int_{A^{\delta}_{n}} |g \circ f_{n} - g \circ f| d\mu + \int_{(A^{\delta}_{n})^{c}} |g \circ f_{n} - g \circ f| d\mu$$

(Refer Slide Time: 4:43)



So, $\int_{A^{\delta}_{n}} |g \circ f_{n} - g \circ f| d\mu \leq 2\pi\mu(A^{\delta}_{n})$ and

$$\int_{(A^{\delta}_{n})^{c}} |g \circ f_{n} - g \circ f| d\mu \leq \epsilon\mu((A^{\delta}_{n})^{c}) \leq \epsilon\mu(X) = \epsilon.$$

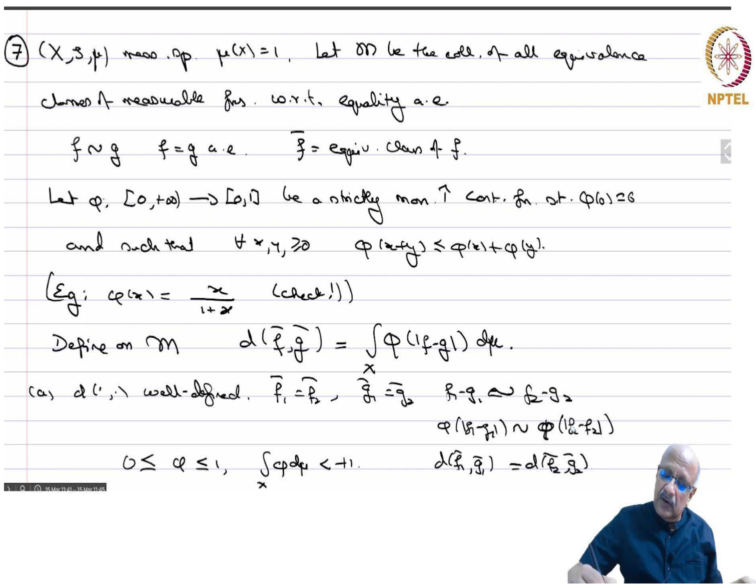
So, given $\eta > 0$ choose $\epsilon > 0$ such that $\epsilon < \frac{\eta}{2}$. This fixes the delta. Now, choose capital N such that for all $n \geq N$, $\mu(A^{\delta}_{n}) < \frac{\epsilon}{2}$.

And therefore, this implies for all $n \geq N$,

$$\int_X |g \circ f_n - g \circ f| d\mu < \eta.$$

That proves the theorem.

(Refer Slide Time: 7:13)



(7) (X, S, μ) measure space, $\mu(X) = 1$. Llet M be the collection of all equivalence classes of measurable functions with respect to equality almost everywhere so, you say $f \sim g$ if $f = g$ a.e. and the set of all equivalence classes $\overline{f}$ = equivalence classes of f. Let ϕ: $[0, \infty] \rightarrow [0, 1]$ be a strictly monotonically increasing continuous function such that $\phi(0) = 0$ and such that for all $x, y \geq 0$, $\phi(x + y) \leq \phi(x) + \phi(y)$.

So, example, $\phi(x) = \frac{x}{x+1}$ (check!) So, now a define on M,

$$d(\overline{f}, \overline{g}) = \int_X \phi(|f - g|) dx.$$

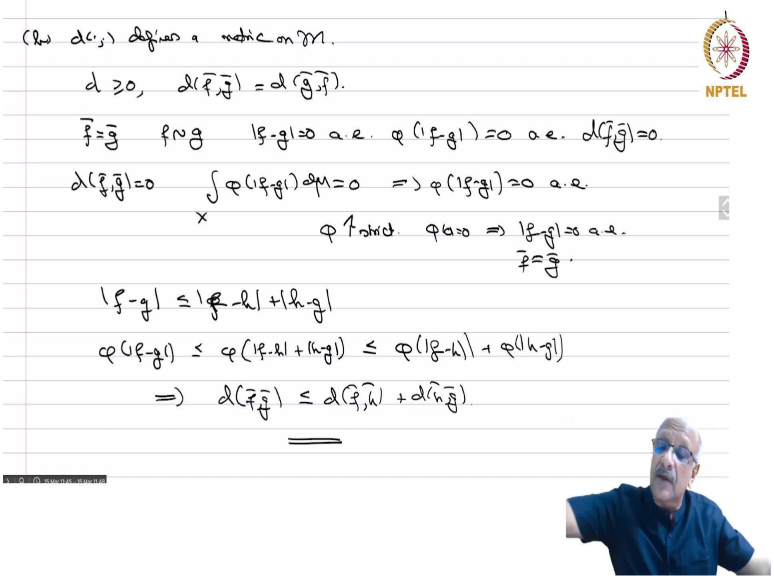
(a) d is well defined because if you change it does not matter. If $\overline{f_1} = \overline{f_2}$, $\overline{g_1} = \overline{g_2}$, then

$$f_1 - g_1 \sim f_2 - g_2, \ \phi(|f_1 - g_1|) \sim \phi(|f_2 - g_2|).$$

That means they are equal almost everywhere and therefore, $d(\overline{f_1}, \overline{g_1}) = d(\overline{f_2}, \overline{g_2})$.

So, it does not matter by which representative you calculate and therefore, this is well defined and also $0 \leq \phi \leq 1$, $\int_X \phi < 1$. So, it is rarely defined so, so, this is well defined.

(Refer Slide Time: 11:11)



(b) d(.) defines a metric on M. So, of course, $d \geq 0, d(\overline{f}, \overline{g}) = d(\overline{g}, \overline{f})$. Now, assume $\overline{f} = \overline{g}$ bar that means $f \sim g$, so, $|f - g| = 0$ almost everywhere, that means, $\phi(|f - g|) = 0$ almost everywhere therefore, $d(\overline{f}, \overline{g}) = 0$.

Conversely, if $d(\overline{f}, \overline{g}) = 0$, that means $\int_X \phi(|\overline{f} - \overline{g}|)d\mu = 0 \Rightarrow \phi(|\overline{f} - \overline{g}|) = 0$ $a.e.$

ϕ is strictly monotonic increasing and $\phi(0) = 0 \Rightarrow |\overline{f} - \overline{g}| = 0 \Rightarrow \overline{f} = \overline{g}$ $a.e.$

So, that is also true then the triangle inequality, so, you have

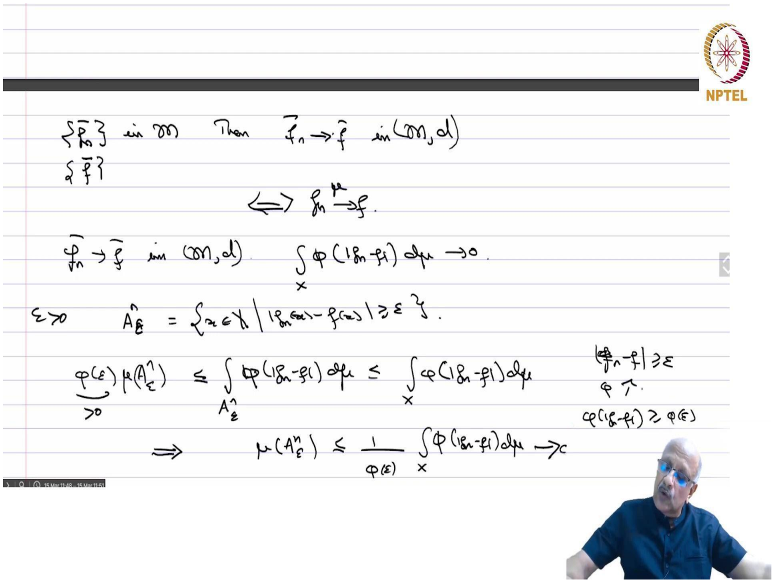
$$|f - g| \leq |f - h| + |h - g|$$

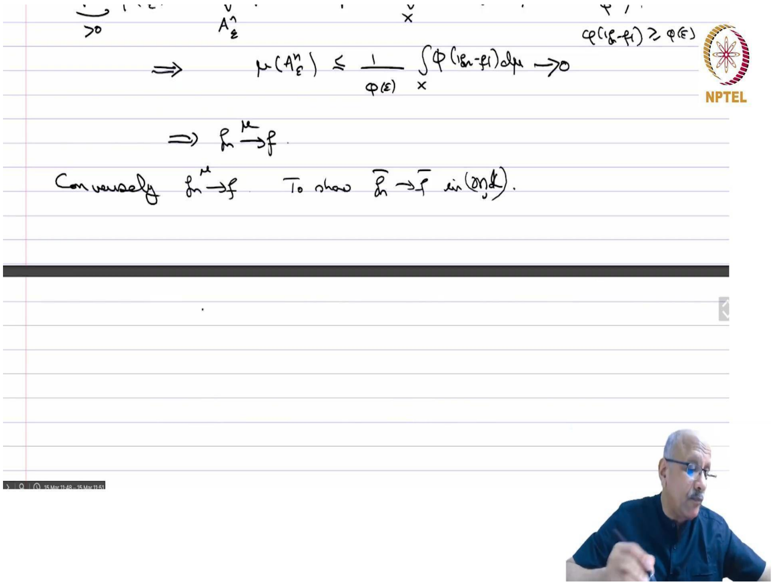
and therefore, $\phi(|f - g|) \leq \phi(|f - h| + |h - g|) \leq \phi(|f - h|) + \phi(|h - g|)$.

$$\Rightarrow d(f, g) \leq d(f, h) + d(h, g).$$

So, therefore, d defines a metric.

(Refer Slide Time: 13:42)





So, now is the interesting part of this exercise namely $\{\overline{f}_n\}\ in\ M$, then $\overline{f}_n \rightarrow \overline{f}(\in M)$ in (M,d) if and only if $\overline{f}_n \rightarrow \overline{f}$ in measure.

It does not matter which representative we take because we saw that if two functions differ almost everywhere, then they converge to the same function in measures and so it does not matter. So, $\overline{f}_n \rightarrow \overline{f}$ in measure, so, the convergence in measure can be thought of as a metric convergence in some suitable topology in some function space.

So, this is a very interesting property. So, let us assume that $\overline{f}_n \rightarrow \overline{f}$ in (M,d). That means

$$\int_X \phi(|\overline{f}_n \to \overline{f}|)d\mu \to 0.$$

Now, you take $\epsilon > 0$, $A^n_\epsilon = \{x \in X: |f_n(x) - f(x)| \geq \epsilon\}$. So,

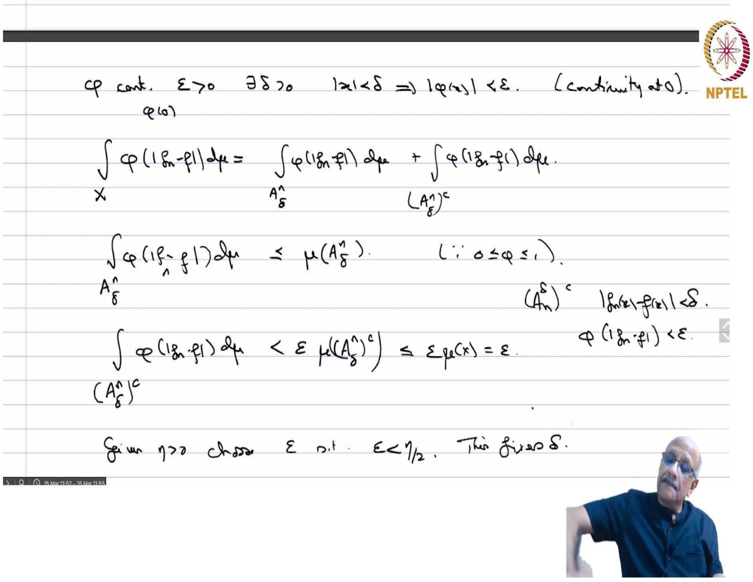
$$\phi(\epsilon)\mu(A^n_\epsilon) \leq \int_{A^n_\epsilon} \phi(|\overline{f}_n \to \overline{f}|)d\mu \leq \int_X \phi(|\overline{f}_n \to \overline{f}|)d\mu$$

$$\Rightarrow \mu(A^n_\epsilon) \leq \frac{1}{\phi(\epsilon)}\int_X \phi(|\overline{f}_n \to \overline{f}|)d\mu \to 0.$$

Therefore, you have that $\overline{f}_n \to \overline{f}$ in measure.

Now, conversely let $\overline{f}_n \to \overline{f}$ in measure. To show $\overline{f}_n \to \overline{f}$ in (M,d).

(Refer Slide Time: 17:15)

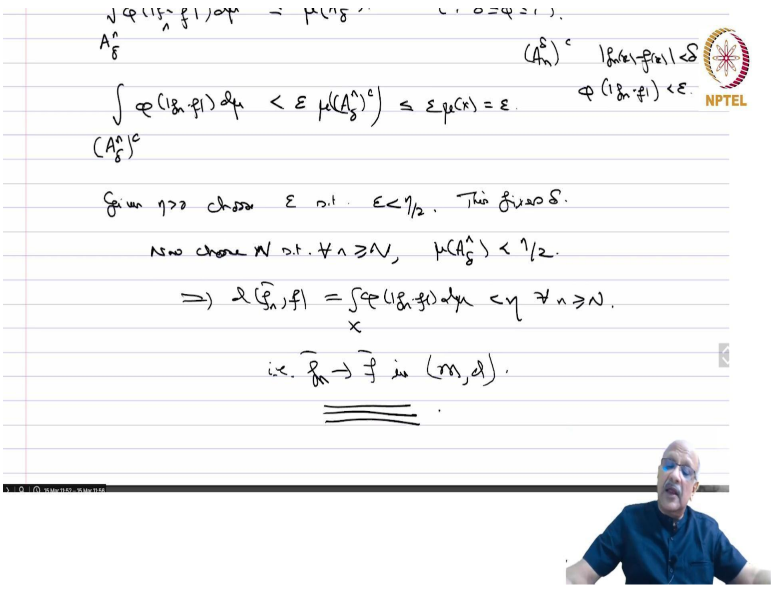


So, ϕ is continuous. So, given any epsilon positive there exists a delta positive, set mod x less than delta implies mod phi of x is less than epsilon. So, because it is continuous at 0 and phi of 0 is 0 and therefore, we are using continuity at 0. So, now, let us take the integral over x phi of mod fn minus f d mu.

So, again we are now going to use the epsilon delta. I mean the divide and rule, so, this is equal to the integral An delta phi of mod fn minus f d mu plus integral An delta complement phi of mod fn minus f d mu. So, the first one, the integral An delta phi of mod fn minus f d mu phi is between 0 and 1. So, this is just less than mu of An delta.

Now, what about the complement integral An delta complement phi of mod fn minus f d mu. So, on a delta complement we have mod fn x minus f x is less than delta. So, that means the phi of mod fn minus f is less than epsilon. So, the phi of mod fn minus f is less than epsilon so this is less than epsilon times mu of An delta complement. We do not know anything about it this less than mu of x that is equal to epsilon, so, given the eta positive choose epsilon such that epsilon is less than eta by 2 this fixes the delta.

(Refer Slide Time: 20:19)



Now, choose n such that for all n greater or equal to capital N, you have mu of An delta is less than eta by 2 and that will tell you the d of fn bar f which equals the integral over x mod phi of mod fn minus f d mu is then less than eta for all n greater than n and that is exactly saying that is fn bar converges to f bar in m. So, with that we complete the exercises and next time we will start on the new topic.